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THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 21

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 Newton Street, Kingsway, W.C.2;
AND PUBLISHED FOR THE SOCIETY BY
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.4.

1923

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PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON PLANE CURVES OF DEGREE $2n$ WITH TANGENTS HAVING BI-n-POINT CONTACT

By Harold Hilton.

[Received July 27th, 1921.—Read November 17th, 1921.]

1. The interesting properties of bitangents of a plane quartic curve suggest that it is worth while to see if there are analogous properties of curves of higher degree. The configuration formed by a quartic and a bitangent is a particular case of the configuration formed by a curve of degree $2n$ (a "$2n$-ic") and a tangent which has n-point contact with it at two distinct points. Such a tangent will be called a "bi-n-point tangent"; and it is to properties of such bi-n-point tangents that this paper is devoted. If $n > 2$, a $2n$-ic with a bi-n-point tangent is specialized, and is not the most general $2n$-ic possible.

2. We first consider some unicursal $2n$-ics with bi-n-point tangents. The simplest such curves are those with a multiple point of order $2n-1$ [a "$(2n-1)$-ple point"].

First of all take the case when the $2n$-ic has at the $(2n-1)$-ple point a single superlinear branch of order $2n-1$.

Suppose the curve has a bi-n-point tangent touching at E_1, E_2. Take a triangle of reference ABC such that C is the $(2n-1)$-ple point, AC the tangent at C to the superlinear branch, and (AB, E_1E_2) is a harmonic

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range. Then it will be seen at once that with suitable homogeneous coordinates the equation of the curve is

$$zy^{2n-1} = (x^2+ay^2)^n. \qquad \text{(i)}$$

There are two inflexions other than E_1, E_2 lying on

$$(2n-1)x^2+ay^2 = 0.$$

Many properties of (i) depend on the fact that it can be projected into a curve with BC as axis of symmetry. We note also that the lines joining C to the two points of contact (other than E_1, E_2) of the tangents to the curve from any point on AB form an involution whose double rays pass through the inflexions. The lines joining C to the points of contact of the bitangents belong to the same involution.

Any two such curves with the same superlinear branch and tangent thereat, and also the same bi-n-point tangent and points of contact, are in plane perspective.

3. Consider now the $2n$-ic with a $(2n-1)$-ple point C, and two real bi-n-point tangents touching at E_1 and E_2, F_1 and F_2, and meeting at D. Let $(AD,\ F_1F_2)$ and $(BD,\ E_1E_2)$ be harmonic ranges. Choose homogeneous coordinates with triangle of reference ABC such that D is $(1, 1, -1)$. Let the lines CE_1, CE_2 be

$$(x-y)^2+ax^2 = 0,$$

and let CF_1, CF_2 be

$$(x-y)^2+by^2 = 0.$$

Using the fact that when we substitute $-x$ for z in the equation of the curve, we get

$$\{(x-y)^2+ax^2\}^n = 0,$$

and similarly when we substitute $-y$ for z, we readily see that the equation of the curve becomes

$$b^n(z+y)\{(x-y)^2+ax^2\}^n - a^n(z+x)\{(x-y)^2+by^2\}^n = (x-y)(z+x)(z+y)\,u_{2n-2},$$

after multiplication by the factor $x-y$; where u_{2n-2} is homogeneous of degree $2n-2$ in x, y, z.

Using now the fact that C is a $(2n-1)$-ple point, we have $u_{2n-2} \equiv 0$, and the equation becomes

$$b^n(z+y)\{(x-y)^2+ax^2\}^n = a^n(z+x)\{(x-y)^2+by^2\}^n. \qquad \text{(i)}$$

From (i) the following facts are easily deduced. If n is odd, only one of the tangents at C is real

$$(b-a+ab)x+(a-b+ab)y = 0. \tag{ii}$$

If n is even, there is another pair of real tangents at C, provided

$$ab\,(2a+2b+ab) < 0.$$

The lines CE_1 and CE_2, CF_1 and CF_2, the tangent (ii) and CD, pairs of other tangents at C, the lines joining C to pairs of intersections of the curve with any line through D, all form an involution pencil. Its double rays are the lines joining C to the points of contact of the two tangents to the curve from D other than the bi-n-point tangents. Their equation is

$$ax^2-(a+b+ab)\,xy+by^2 = 0. \tag{iii}$$

The points of contact of the tangents from any other point $(1, 1, 2nk-1)$ of CD lie on a 4-ic

$$(z+x)(z+y)\,\{ax^2-(a+b+ab)\,xy+by^2\}$$
$$+k\,\{(x-y)^2+ax^2\}\,\{(x-y)^2+by^2\} = 0,$$

with a biflecnode at C, at which (iii) are the tangents, and passing through E_1, E_2, F_1, F_2.

The point H conjugate to C for all the conics through E_1, E_2, F_1, F_2, namely,

$$abz^2+a(x+z)^2+b(y+z)^2+t(x+z)(y+z) = 0,$$

is the point on (ii) such that DH, DC are harmonic conjugates with respect to the bi-n-point tangents. One of these conics has DAB as self-conjugate triangle.

4. Consider now the case of any unicursal $2n$-ic with three real bi-n-point tangents. It is readily shown that these tangents are not concurrent, and we may therefore take them as sides of the triangle of reference. Hence any point on the curve can be expressed in terms of a parameter t in the form

$$x : y : z = u^n : v^n : w^n, \tag{i}$$

where $u \equiv p_0t^2+2p_1t+p_2$, $v \equiv q_0t^2+2q_1t+q_2$, $w \equiv r_0t^2+2r_1t+r_2$.

If the conic obtained by elimination of t from

$$x : y : z = u : v : w, \tag{ii}$$

is

$$\phi(x, y, z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0, \tag{iii}$$

the $2n$-ic is

$$ax^{2/n}+by^{2/n}+cz^{2/n}+2fy^{1/n}z^{1/n}+2gz^{1/n}x^{1/n}+2hx^{1/n}y^{1/n}=0. \qquad \text{(iv)}$$

The equation (iv) is therefore the most general equation of a unicursal $2n$-ic with three real bi-n-point tangents.

If a unicursal $2n$-ic has three real bi-n-point tangents, their six points of contact lie on a conic.

For, if the intersections of (iii) with the sides of the triangle of reference are

$$(0,\ y_1,\ z_1),\ (0,\ y_1',\ z_1'),\ (x_2,\ 0,\ z_2),\ (x_2',\ 0,\ z_2'),\ (x_3,\ y_3,\ 0),\ (x_3',\ y_3',\ 0), \qquad \text{(v)}$$

then, since $y_3y_3'/x_3x_3'=a/b$, &c.,

$$y_1y_1'z_2z_2'x_3x_3'=z_1z_1'x_2x_2'y_3y_3'. \qquad \text{(vi)}$$

Now the intersections of (iv) with the sides of the triangle of reference are

$$(0,\ y_1^n,\ z_1^n),\ (0,\ y_1'^n,\ z_1'^n),\ \text{\&c.}, \qquad \text{(vii)}$$

and the condition that these should lie on a conic is

$$(y_1y_1'z_2z_2'x_3x_3')^n=(z_1z_1'x_2x_2'y_3y_3')^n; \qquad \text{(viii)}$$

which is an immediate deduction from (vi).

If three real bi-n-point tangents of a unicursal $2n$-ic form a triangle ABC, the remaining tangents to the $2n$-ic from A, B, C touch a conic and meet the opposite sides in six points lying on a conic.

If $y=mx$ is a tangent from C to (iii), then $y=m^nx$ is a tangent from C to (iv). Hence, if the tangents from A, B, C to (iii) meet BC, CA, AB in the points (v), the tangents from A, B, C to (iv) meet BC, CA, AB in the points (vii). But the theorem is well known for $n=1$, so that the proof follows as for the previous theorem.

In the case $n=2$, we have:—

"The remaining tangents from the vertices of a triangle formed by three bitangents of a trinodal quartic touch a conic."

To find the double points of (iv), we notice that, if the conics

$$\phi(\epsilon_1x,\ \epsilon_2y,\ \epsilon_3z)=0 \quad\text{and}\quad \phi(\eta_1x,\ \eta_2y,\ \eta_3z)=0,$$

where ϵ_1, ϵ_2, ϵ_3, η_1, η_2, η_3 are n-th roots of unity, meet at a point (X, Y, Z)

not on a side of the triangle of reference, then (X^n, Y^n, Z^n) is a double point of (iv).

For, if in (ii) $t = t_1$ gives

$$x : y : z = X : Y : Z$$

for the conic $\epsilon_1 x : \epsilon_2 y : \epsilon_3 z = u : v : w$,

and $t = t_2$ gives $x : y : z = X : Y : Z$

for the conic $\eta_1 x : \eta_2 y : \eta_3 z = u : v : w$,

then both $t = t_1$ and $t = t_2$ give

$$X^n : Y^n : Z^n = u^n : v^n : w^n.$$

The result applies in general, where

$$\phi(x, y, z) = 0$$

is any unicursal curve, and not necessarily a conic. Also the intersections of any two given unicursal $2n$-ics with the same three bi-n-point tangents can be found in this way.

We see that the curve (iv) can have no real crunode, when n is odd.

As an illustration, consider the intersections of the conics

$$\phi(x, y, z) = 0 \quad \text{and} \quad \phi(-x, y, z) = 0.$$

Two of them are on $x = 0$, and the other two are

$$[\pm(2fgh-bg^2-ch^2)^{\frac{1}{2}}, \quad a^{\frac{1}{2}}g, \quad -a^{\frac{1}{2}}h].$$

Hence, when $n = 2p$,

$$[(2fgh-bg^2-ch^2)^p, \quad a^p g^{2p}, \quad a^p h^{2p}],$$

and the two similar points are three nodes of (iv).

Putting $p = 1$, we get all three nodes of the quartic obtained by taking $n = 2$ in (iv).

It is easy to draw unicursal $2n$-ics with three real bi-n-point tangents. For, if the point with Cartesian coordinates (x, y) traces out a conic, (x^n, y^n) traces out such a $2n$-ic.

If $\lambda x + \mu y + \nu z = 0$ is an inflexional tangent of (iv),

$$\lambda x^n + \mu y^n + \nu z^n = 0 \tag{ix}$$

osculates (iii). It is readily proved that in this case the point of contact with (iii) lies on

$$\Delta xyz = (n-1)(ax+hy+gz)(hx+by+fz)(gx+fy+cz).^{*}$$

Hence (iv) has six inflexions whose parameters are given by

$$\Delta uvw = (n-1)(au+hv+gw)(hu+bv+fw)(gu+fv+cw),$$

in addition to the points of contact of the bi-n-point tangents.

If $n = 2$, we find that when (ix) osculates (iii),

$$(A\lambda+B\mu+C\nu)^2 = 3\Delta(a\mu\nu+b\nu\lambda+c\lambda\mu),$$

which is the tangential equation of the conic touched by the six inflexional tangents of the quartic.

A unicursal $2n$-ic cannot have more than three bi-n-point tangents, if $n > 5$; for the existence of a fourth bi-n-point tangent would imply more inflexions than are consistent with Plücker's equations.

To determine whether, when $n = 2, 3, 4, 5$, a unicursal $2n$-ic exists with more than three bi-n-point tangents, we find $\lambda : \mu : \nu$ such that (ix) has bi-n-point contact with the conic (iii). Then $\lambda x+\mu y+\nu z = 0$ has bi-n-point contact with (iv).

For $n = 2$, we get at once

$$\lambda : \mu : \nu = F/f : G/g : H/h,$$

so that the fourth bitangent of (iv) when $n = 2$, is

$$Fx/f+Gy/g+Hz/h = 0.$$

Its points of contact lie on the conics

$$(ax+by+cz)^2 = 4(f^2yz+g^2zx+h^2xy), \quad ghx^{\frac{1}{2}}+hfy^{\frac{1}{2}}+fgz^{\frac{1}{2}} = 0,$$

the former being the conic through the points of contact of the four bitangents. We may verify at once that for $n = 2$ (iv) is

$$ghx^{\frac{1}{2}}+hfy^{\frac{1}{2}}+fgz^{\frac{1}{2}}+(ghFx+hfGy+fgHz)^{\frac{1}{2}} = 0.$$

If n is even, $fgh \neq 0$, for otherwise (iv) has a double point on a side of the triangle of reference. There would be no loss of generality in taking f, g, h as all equal to unity.

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* $\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, and A, B, ... are the co-factors of a, b, ... in Δ.

For $n=3$, we find that there are values of $\lambda : \mu : \nu$ making (ix) have bi-n-point contact with (iii), namely,

$$-a^3gh : 4h^3(4gh-3af) : 4g^3(4gh-3af),$$

provided $$abg^2 = ach^2 = 2gh(2gh-af).$$

If the unicursal 6-ic has more than four bi-3-point tangents it has six, and by suitable choice of homogeneous coordinates its equation becomes

$$(\surd 5-2)(x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}})+(x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}})^2 = 0\,;$$

and it has bi-3-point tangents

$$x = 0, \quad (\surd 5+3)\,x = 2\,(x+y+z), \text{ \&c.}$$

Similarly for $n=4$, $n=5$ unicursal n-ics exist with equations

$$2x^{\frac{1}{2}}+2y^{\frac{1}{2}}+z^{\frac{1}{2}}+y^{\frac{1}{4}}z^{\frac{1}{4}}+z^{\frac{1}{4}}x^{\frac{1}{4}}+3x^{\frac{1}{4}}y^{\frac{1}{4}} = 0,$$

$$x^{\frac{2}{5}}+y^{\frac{2}{5}}+z^{\frac{2}{5}}+y^{\frac{1}{5}}z^{\frac{1}{5}}+z^{\frac{1}{5}}x^{\frac{1}{5}}+x^{\frac{1}{5}}y^{\frac{1}{5}} = 0,$$

having for bi-n-point tangents

$$x = 0, \quad y = 0, \quad z = 0, \quad x+y+z = 0.$$

5. We consider now the general $2n$-ic with bi-n-point tangents.

First take the case in which the bi-n-point tangents are all concurrent at O. Suppose the curve projected so that O bisects the distances between the points of contact of each of two bi-n-point tangents. Take these tangents as axes of reference. Then if their points of contact are given by

$$y = (1+ax^2)^n = 0, \qquad x = (1+by^2)^n = 0,$$

the equation of the curve evidently becomes

$$xy\,f_{2n-2}(x, y)+(1+ax^2)^n+(1+by^2)^n-1 = 0, \tag{i}$$

where f_{2n-2} is of degree $2n-2$ in x and y.

Suppose that there is another bi-n-point tangent through O, namely $y = tx$. Then

$$tx^2 f_{2n-2}(x, tx)+(1+ax^2)^n+(1+bt^2x^2)^n-1$$

is an n-th power. This is only possible, if it is the n-th power of an expression of the form $1+cx^2$. Hence O bisects the distance between the points of contact of $y = tx$, so that, if a $2n$-ic has three concurrent bi-n-point tangents, their six points of contact lie on a conic.

Suppose now that the $2n$-ic has four bi-n-point tangents concurrent at O. If we take two of them as axes of reference, the equation of the curve is (i); and, if we then change the axes of reference to the other two, the equation takes the form

$$O = \{1+a(y+\alpha x)^2\}^n + \{1+b(y+\beta x)^2\}^n - 1$$
$$+n(y+\alpha x)(y+\beta x)(k+lx+my+px^2+\ldots).$$

Remembering that the four bi-n-point tangents are now

$$x = 0, \quad y = 0, \quad y+\alpha x = 0, \quad y+\beta x = 0,$$

we see that their points of contact are given by

$$x = \{1+(a+b+k)\,y^2\}^n = 0, \quad y = \{1+(a\alpha^2+b\beta^2+k\alpha\beta)\,x^2\}^n = 0,$$
$$y+\alpha x = \{1+b(\beta-\alpha)^2\,x^2\}^n = 0, \quad y+\beta x = \{1+a(\alpha-\beta)^2\,x^2\}^n = 0.$$

These points all lie on

$$a(y+\alpha x)^2+b(y+\beta x)^2+k(y+\alpha x)(y+\beta x)+1 = 0.$$

Hence the points of contact of any four concurrent bi-n-point tangents lie on a conic.

If there are other concurrent bi-n-point tangents, we see by applying this result to each set of four concurrent bi-n-point tangents in turn that:—

The points of contact of any number of concurrent bi-n-point tangents all lie on a conic.

6. Suppose now a $2n$-ic has three real bi-n-point tangents, which are not concurrent. Take them as the sides of the triangle of reference. Let the points of contact of $x = 0$, $y = 0$ be given by

$$(cz^2+2fzy+by^2)^n = 0, \quad (cz^2+2gzx+ax^2)^n = 0.$$

It will be at once proved that the equation of the curve is of the form

$$xy\,u_{2n-2}+(cz^2+2gzx+ax^2)^n+(cz^2+2fzx+by^2)^n-c^nz^{2n} = 0. \qquad \text{(i)}$$

[u_r, v_r, $f_r(x, y, z)$, ... denote homogeneous polynomials of degree r in x, y, z]; and since $z = 0$ is also a bi-n-point tangent, we must get an n-th power equated to zero when we put $z = 0$ in (i). It follows that equation (i) must be of the form

$$xyz\,u_{2n-3}+(cz^2+2gzx+ax^2)^n+(cz^2+2fzy+by^2)^n+(ax^2+2hxy\pm by^2)^n$$
$$-a^nx^{2n}-b^ny^{2n}-c^nz^{2n}=0. \qquad \text{(ii)}$$

In the ambiguity of sign the plus must be taken when n is odd, and either sign when n is even.

If the plus sign is taken, we may write equation (ii) in the form

$$xyz\,v_{2n-3}+(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)^n=0. \qquad \text{(iii)}$$

We see that the six points of contact of the three bi-n-point tangents $x=0$, $y=0$, $z=0$ lie on a conic. Three such bi-n-point tangents are called "syzygetic." Any three real bi-n-point tangents are syzygetic, if n is odd.

If the minus sign is taken, P_1, P_2, Q_1, Q_2, R_1, R' lie on a conic, where P_1 and P_2, Q_1 and Q_2, R_1 and R_2 are the points of contact of the sides BC, CA, AB of the triangle of reference, and (AB, R_2R') is a harmonic range. The three bi-n-point tangents are called "asyzygetic" in this case.

A $2n$-ic having n-point contact with BC at P_1 and P_2, with CA at Q_1 and Q_2, with AB at R_1, and having at least $(n-1)$-point contact with AB at R_2 or R' has BC, CA, AB as bi-n-point tangents by the extension of Carnot's theorem. Hence to be given three bi-n-point tangents of a $2n$-ic and their points of contact is equivalent to $6n-1$ conditions (not $6n$).

Suppose now that the $2n$-ic has four real bi-n-point tangents, no three of which are concurrent, and that three of them are syzygetic. By a suitable choice of homogeneous coordinates we may make these three have equations $x=0$, $y=0$, $x+y+z=0$, while the fourth is $z=0$. Then, by (iii), the equation of the $2n$-ic is of the form

$$xy(x+y+z)f_{2n-3}(x, y, z)+(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)^n=0,$$

and it meets $z=0$ where

$$xy\,(x+y)f_{2n-3}(x, y, 0)+(ax^2+2hxy+by^2)^n=0.$$

The left-hand side of this is an n-th power and is therefore

$$(ax^2+2hxy+by^2)^n \quad \text{or} \quad (ax^2+2hxy-by^2)^n,$$

the latter being only possible, if n is even.

In the former case the equation of the curve is

$$xyz(x+y+z)u_{2n-4}+(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)^n=0,$$

and the points of contact of all four bi-n-point tangents lie on a conic; *i.e.* the four bi-n-point tangents are "syzygetic." The same holds if the three given syzygetic tangents are concurrent.

This argument can be at once extended, and we obtain the result:—

If a $2n$*-ic has* r *real bi-*n*-point tangents whose equations are*

$$l_1x+m_1y+n_1z=0,\ \ldots,\ l_rx+m_ry+n_rz=0,$$

then, if n *is odd, or if* n *is even and the points of contact of five of the tangents lie on a conic, the equation of the* $2n$*-ic is*

$$(l_1x+m_1y+n_1z)\ldots(l_rx+m_ry+n_rz)\,u_{2n-r}+u_2^n=0; \tag{iv}$$

and the points of contact of all the tangents lie on the conic $u_2=0$. *There cannot be more than* n *real bi-*n*-point tangents in this case.*

Whether the bi-n-point tangents are real or not, the equation of the $2n$-ic is (iv), if the points of contact of the tangents are known to lie on a conic.

7. Suppose a $2n$-ic has four bi-n-point tangents p, q, r, s. Let the points of contact of p be P_1 and P_2, &c. Project S_1 and S_2 into the circular points at infinity and let p, q, r form the triangle ABC. Then p, q, r are syzygetic or asyzygetic according as

$$(AR_1\,.\,AR_2\,.\,BP_1\,.\,BP_2\,.\,CQ_1\,.\,CQ_2)\div(AQ_1\,.\,AQ_2\,.\,BR_1\,.\,BR_2\,.\,CP_1\,.\,CP_2)$$
$$\text{is } +1 \text{ or } -1.$$

Also q, r, s are syzygetic or asyzygetic according as

$$(AQ_1\,.\,AQ_2)\div(AR_1\,.\,AR_2)\ \text{ is } +1 \text{ or } -1.$$

It follows that of the four possible trios

$$qrs,\quad prs,\quad pqs,\quad pqr,$$

formed from p, q, r, s, either

(i) each trio is asyzygetic,

(ii) two trios are syzygetic and two asyzygetic,

or (iii) each trio is syzygetic.

In this third case the fact that the common chords of each of the circles $Q_1Q_2R_1R_2$, $R_1R_2P_1P_2$, $P_1P_2Q_1Q_2$ and the conic $P_1P_2Q_1Q_2R_1R_2$ are equally inclined to the axes of this conic, proves that $P_1P_2Q_1Q_2R_1R_2S_1S_2$

all lie on the same conic. We have already shown that this third case is the only one possible when n is odd. That, when n is even, each of the three cases can actually occur follows from the well-known case $n = 2$.

The property of four bi-n-point tangents just noticed may be applied to discover what possible trios of syzygetic bi-n-point tangents can exist when the $2n$-ic has five bi-n-point tangents p, q, r, s, t. Quite straightforward but rather lengthy work shows that the only cases consistent with the above property of the four bi-n-point tangents are

(i) all ten trios asyzygetic,

(ii) all ten trios syzygetic,

(iii) all but pst, qst, rst asyzygetic,

(iv) all but pst, qst, rst syzygetic,

(v) all but prt, pst, qrt, qst asyzygetic,

(vi) all but prt, pst, qrt, qst syzygetic,

(vii) pqr, qrs, rst, stp, tpq asyzygetic, and psq, sqt, qtr, trp, rps syzygetic.

But of these cases (ii) is only possible when all ten points of contact lie on a conic, and is the only case which can occur when n is odd; while (iv) is never possible, for it requires that two distinct conics should pass, one through $P_1P_2Q_1Q_2R_1R_2S_1S_2$ and one through $P_1P_2Q_1Q_2R_1R_2T_1T_2$. In case (vi) $P_1P_2Q_1Q_2R_1R_2S_1S_2$ lie on a conic, and t meets p and q, r and s in an involution to which T_1 and T_2 belong.

8. Non-projective properties of the general $2n$-ic are usually modified, if the curve has the line at infinity as bi-n-point tangent. In this section we give a few illustrations. Considerations of space limit us to the enunciations. For the methods of proof the reader may refer to Ch. xi of the author's *Plane Algebraic Curves*, and to a paper in the *American Journal of Mathematics*, Vol. 39 (1917), p. 86.

The class of the curve will be taken as m. The number of its inflexions is ι including the $n-2$ inflexions at each point of n-point contact. We must distinguish:—

Case 1, in which the line at infinity has bi-n-point contact at the circular points ω and ω'.

Case 2, in which the line at infinity is a bi-n-point tangent whose points of contact X and Y are not ω and ω'. We shall suppose X' and Y' the harmonic conjugates of X and Y with respect to ω and ω'.

First consider the *evolute*, whose degree, class, and number of inflexions are n', m', ι'.

In Case 1 $\omega\omega'$ has n-point contact with the evolute at ω and ω' We have

$$n' = \iota+4, \quad m' = m, \quad \iota' = 2(n-2).$$

To illustrate, notice that the evolute of a three-cusped hypocycloid is also a three-cusped hypocycloid ($n = 2$, $m = 3$, $\iota = 0$).

In Case 2 $\omega\omega'$ has $(n+1)$-point contact with the evolute at X' and Y'. We have

$$n' = \iota+6, \quad m' = m+2, \quad \iota' = 2(n-1).$$

If the curve has r other bi-n-point tangents, the evolute has $2rn$ $(n-2)$-ple points on $\omega\omega'$ each having a single superlinear branch of order $n-2$. The values of n', m', ι' just given are unaltered.

Next consider the *pedal* with respect to any point.

It is in general of degree $2(m-n+1)$ and has an $(m-2n+2)$-ple point at ω and ω'. In Case 1 one branch at ω (and ω') has n-point contact with $\omega\omega'$. In Case 2 $\omega\omega'$ has $(n-1)$-point contact with the pedal at two points, but does not touch it at ω or ω'.

Lastly, consider the *orthoptic locus*.

In Case 1 the $m-2n+1$ tangents from ω to the $2n$-ic other than $\omega\omega'$ give $\frac{1}{2}(m-2n+1)(m-2n)$ branches of the locus through ω not touching $\omega\omega'$, and $m-2n+1$ branches of the locus having n-point contact with $\omega\omega'$ at ω. In addition $\omega\omega'$ has $\frac{1}{4}\{2n-1+(-1)^n\}$-point contact with the locus at two points other than ω and ω'. The degree of the locus is

$$(m-2n)m+m+n-\tfrac{1}{2}\{1-(-1)^n\}.$$

For instance, for $n = 3$, $m = 5$ the orthoptic locus is a conic.*

In Case 2 the locus has $\frac{1}{2}(m-2n+2)(m-2n+1)$-ple points at ω, ω' and also $m-2n+2$ linear branches having $(n-1)$-point contact with $\omega\omega'$ at each of X', Y'. Its degree is

$$(m-1)(m-2n+2).$$

But if $(X, Y, \omega\omega')$ is a harmonic range, so that $X \equiv Y'$ and $Y \equiv X'$, the locus has $\frac{1}{2}(m-2n+2)(m-2n+1)$-ple points at ω and ω', $m-2n+1$

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* This case was overlooked in the paper quoted above. It should be added to those given in Ex. 16 on p. 174 of *Plane Algebraic Curves*: and the number of cases in which the locus is a cubic or quartic is greater than that given in the footnote on that page.

linear branches having $(n-1)$-point contact with $\omega\omega'$ at each of X and Y, and also another linear branch having $(n-1)$-point contact with $\omega\omega'$. Its degree is

$$(m-2n)m+m+n-1.$$

Throughout this section we have supposed that the curve is not specialised except by the fact that the line at infinity is a bi-n-point tangent. Further modifications will be necessary, if, for instance, other bi-n-point tangents pass through ω and ω'.

[Since this paper was written an article by T. Ota, *Tôhoku Math. Journal*, Vol. 19 (1921), pp. 69–88, has appeared dealing with similar problems. The overlapping is slight, and I have not thought it necessary to make any alteration in my own MS.]

NOTE ON THE RESULTANT OF A NUMBER OF POLYNOMIALS OF THE SAME DEGREE

By F. S. MACAULAY.

[Received May 17th, 1921.—Read June 9th, 1921.]

THE following note is an addendum to a paper ("Some Formulæ in Elimination") in the *Proceedings*, Ser. 1, Vol. 35 (1902), pp. 3–18. In that paper an expression was found, by Sylvester's dialytic method, for the resultant of n homogeneous polynomials $C_1, C_2, \ldots, C_n$, in the form of a quotient of a certain determinant by a certain minor of the same. The object of the present note is to find a simpler and more symmetrical form for the quotient when $C_1, C_2, \ldots, C_n$ are all of the same degree m.

Notation, etc.—A power product in general, $x_1^{l_1} x_2^{l_2} \ldots x_n^{l_n}$, will be denoted by ω.

A power product $x_1^{l_1} x_2^{l_2} \ldots x_n^{l_n}$ such that

$$l_1 \leqslant m-1,\ l_1+l_2 \leqslant 2(m-1),\ \ldots,\ l_1+l_2+\ldots+l_{n-1} \leqslant (n-1)(m-1)$$

will be denoted by Ω. See footnote.*

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* The number of products $x_1^{l_1} x_2^{l_2} \ldots x_n^{l_n}$ of degree t, such that

$$l_1 \leqslant a,\quad l_1+l_2 \leqslant a+(m-1),\quad l_1+l_2+l_3 \leqslant a+2(m-1),\quad \ldots,$$

$$l_1+l_2+\ldots+l_{r-1} \leqslant a+(r-2)(m-1) \leqslant t \leqslant a+(r-1)(m-1),\quad r \leqslant n,$$

is
$$(a+1) \sum_{p=r-1}^{n-1} (-1)^{n-p-1} \frac{1}{p} \binom{a+pm}{p-1} \binom{a+pm-p-t}{n-p-1},$$

where $\binom{t}{p}$ denotes the coefficient of x^p in $(1+x)^t$. I do not know of a simple proof. Three cases are applicable to the latter half of this note.

(i) The number of products Ω, as defined above, of degree $(n-1)(m-1)$ is

$$\frac{m}{n-1} \binom{mn-1}{n-2},$$

which is $1/n$ of the whole number of products of degree $mn-n+1$; so that, if $C_1, C_2, \ldots, C_n$ are multiplied by all products Ω of degree $(n-1)(m-1)$, the number of resulting polynomials

A power product which is reduced in $x_1, x_2, \ldots, x_p$, *i.e.* not divisible by any one of $x_1^m, x_2^m, \ldots, x_p^m$, will be denoted by $\omega^{(p)}$, or $\Omega^{(p)}$ if it be an Ω. Hence every Ω is an $\Omega^{(1)}$; and $\Omega^{(n-1)}$, $\Omega^{(n)}$ have the same meanings as $\omega^{(n-1)}$, $\omega^{(n)}$.

Suppose each of $C_1, C_2, \ldots, C_n$ is multiplied by every ω of degree $t-m$, and that the array of the coefficients of the resulting polynomials (of degree t) is written down, the columns (less numerous in general than the rows) corresponding to the several power products of degree t. The number of linearly independent rows (*i.e.* the rank of the array considered as a matrix of determinants) is the same for $C_1, C_2, \ldots, C_n$ as for $x_1^m, x_2^m, \ldots, x_n^m$, *i.e.* the rank is equal to the total number of products $x_1^m \omega, x_2^m \omega^{(1)}, \ldots, x_n^m \omega^{(n-1)}$ of degree t, or the number of columns headed by such products. Hence the rows corresponding to $C_1\omega, C_2\omega^{(1)}, \ldots, C_n\omega^{(n-1)}$ form a complete set of independent rows of the whole array. We denote by $D(n,t)$ the determinant whose rows correspond to $C_1\omega, C_2\omega^{(1)}, \ldots, C_n\omega^{(n-1)}$ and columns to $x_1^m\omega, x_2^m\omega^{(1)}, \ldots, x_n^m\omega^{(n-1)}$. The columns of the array not belonging to $D(n, t)$ are those corresponding to products $\omega^{(n)}$. $D(n, t)$ is a non-vanishing determinant (being equal to ± 1 when $C_1, C_2, \ldots, C_n$ are put equal to $x_1^m, x_2^m, \ldots, x_n^m$ respectively) and determines two sub-arrays of the whole array, viz. a vertical sub-array with the same columns as $D(n, t)$, and a horizontal sub-array with the same rows as $D(n, t)$. We denote the H.C.F. of all the determinants of the vertical sub-array by $R(n, t)$, and the quotient $D(n, t)/R(n, t)$ by $Q(n, t)$. It is a known property that this quotient $Q(n, t)$ is a common factor of all the determinants of the horizontal sub-array. It will be proved that $Q(n, t)$ is equal to a certain minor $\Delta(n, t)$ of $D(n, t)$; so that, as $R(n, t) = D(n, t)/Q(n, t)$, we shall have $R(n, t) = D(n, t)/\Delta(n, t)$.

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is equal to the number of power products of their degree $mn-n+1$. This curious fact must have some application, though it may not be a simple one.

(ii) The number of products $\omega_l\,\Omega$ of degree $l+(n-1)(m-1)$, where ω_l is any product of degree l, is

$$\frac{l+m}{n-1}\binom{l+mn-1}{n-2}.$$

(iii) The number of products $\omega_m\,\Omega$ of degree $(n-1)(m-1)$ is

$$\frac{2m}{n-1}\binom{mn+m-1}{n-2}-\frac{2m^2}{n-2}\binom{mn-1}{n-3}.$$

In (i), $a=m-1$, $p=n-1$; in (ii), $a=l+m-1$, $p=n-1$; in (iii), $a=2m-1$, $p=n-1$, $n-2$, $n-3$.

The horizontal sub-array is illustrated by the following diagram in which the determinant $D(n, t)$ is marked off by stars at the corners :—

	Cols. not in $D(n, t)$ $\omega^{(n)}$	Primary columns $(x_1^m, \dots, x_n^m)\,\omega^{(n-1)}$	Supplementary cols. $(x_1^m, \dots, x_{n-1}^m)^2\,\omega$
Primary rows, viz. rows representing $(C_1, C_2, \dots, C_n)\,\omega^{(n-1)}$		* Primary minor of $D(n, t)$	*
Supplementary rows $(C_1, \dots, C_{n-1})\,x_{n-1}^m\,\omega^{(n-2)}$ $(C_1, \dots, C_{n-2})\,x_{n-2}^m\,\omega^{(n-3)}$ ... ... $(C_1, C_2)\,x_2^m\,\omega^{(1)}$ $C_1\,x_1^m\,\omega$	Blank	*	Co-minor of primary minor denoted by $\Delta(n, t)$; the extraneous factor of $D(n, t)$, viz. $R(n,t) = D(n,t)/\Delta(n,t)$ *

The multipliers $\omega^{(p-1)}$ of C_p consist of all products $x_p^m\,\omega^{(p-1)}$, $x_{p+1}^m\,\omega^{(p)}$, ..., $x_{n-1}^m\,\omega^{(n-2)}$, $\omega^{(n-1)}$ of degree $t-m$, and this is the form in which they are shown in the diagram, of which only the last set $\omega^{(n-1)}$ contribute to the primary rows. When $t > n(m-1)$ there are no columns $\omega^{(n)}$, since the highest power product $\omega^{(n)}$ is $x_1^{m-1}x_2^{m-1}\dots x_n^{m-1}$. Also, when $t>n(m-1)$, $R(n, t)$ becomes the resultant R_n of $C_1, C_2, \dots, C_n$. The expression $R(n, t)$ may be called a nominal resultant when $t \leqslant n(m-1)$. The number of such nominal resultants for n variables is $(n-1)(m-1)$, viz. one for each degree t from m to $n(m-1)$. As regards $\Delta(n, t)$ see footnote.*

We have to prove that $\Delta(n, t) = D(n, t)/(R(n, t)$, or $Q(n, t)$. The quotient $Q(n, t)$ is shown in the paper (§ 5) to satisfy a reduction formula

$$Q(n, t) = Q(n-1, t) \dots Q(n-1, t-m+1)\,D(n-1, t-m) \dots D(n-1, m),$$

where $Q(r, t)$, $D(r, t)$ stand in the same relation to the r polynomials

$$(C_1, C_2, \dots, C_r)\; x_{r+1} = \dots = x_n = 0$$

as $Q(n, t)$, $D(n, t)$ to $(C_1, C_2, \dots, C_n)$. Also, as is proved below, $\Delta(n, t)$ satisfies the reduction formula

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* The $\Delta(n, t)$ of the diagram is apparently different from the $\Delta(n, t)$ of the paper (§ 6); but the fact that each of them is equal to $D(n, t)/R(n, t)$ shows that they are the same (on expansion) when the degrees of $C_1, C_2, \dots, C_n$ are all equal. They are independent of the elements in their rows and columns in respect to which they differ.

$$\Delta(n, t) = \Delta(n-1, t) \ldots \Delta(n-1, t-m+1) D(n-1, t-m) \ldots D(n-1, m),$$

where $D(r, t)$ is the same as before, but $\Delta(r, t)$ is related to the r polynomials

$$(C_1, \ldots, C_{r-1}, C_n)_{x_r = \ldots = x_{n-1} = 0}$$

as $\Delta(n, t)$ to $(C_1, C_2, \ldots, C_n)$. If we assume both formulæ, and continue the reduction, the D's throughout will have the same meaning in the one formula as in the other; and since $Q(1, t) = \Delta(1, t) = 1$, we shall have finally $Q(n, t) = \Delta(n, t)$. If it is not clear that the reduction can be carried to $Q(1, t)$, $\Delta(1, t)$, but only to $Q(2, t)$, $\Delta(2, t)$, then it can be verified that $Q(2, t) = \Delta(2, t) = a_1^{t-2m+1}$ (or 1 if $t < 2m$), where a_1 is the coefficient of x_1^m in C_1. The first step in the reduction of $Q(n, t)$ shows that it is dependent only on the coefficients of the polynomials

$$(C_1, C_2, \ldots, C_{n-1})_{x_n = 0}.$$

Hence the same is true of $\Delta(n, t)$. Hence also $\Delta(r, t)$ depends only on the coefficients of $(C_1, C_2, \ldots, C_{r-1})_{x_r = \ldots = x_n = 0}$; and it follows that $\Delta(r, t)$ will not be altered if C_n is replaced by C_r and $x_r = 0$ by $x_n = 0$, so that $\Delta(r, t)$ can be defined in the same way as $D(r, t)$ and $Q(r, t)$.

It remains to prove the formula for $\Delta(n, t)$. Arrange the power products of the supplementary columns (and the multipliers for the supplementary rows) in $m+1$ sets in order, 1st those not divisible by x_{n-1}, 2nd those divisible by x_{n-1} but not by x_{n-1}^2, and so on, and finally all those divisible by x_{n-1}^m. The determinant $\Delta(n, t)$ will then be found to break up into determinants in the diagonal, these being $\Delta(n-1, t)$, $\Delta(n-1, t-1)$, ..., $\Delta(n-1, t-m+1)$, and a final determinant of a different kind. This final determinant also breaks up into determinants in the diagonal if its columns (and the multipliers for its rows) are arranged in sets according to powers of x_n (instead of x_{n-1}) and is thus found to be $D(n-1, t-m) \ldots D(n-1, m)$. It would be useless to state the reasons in detail; the fact has to be verified by careful attention to both columns and rows, and to the definitions of $\Delta(n-1, t)$ and $D(n-1, t)$. It follows that $\Delta(n, t)$ is the extraneous factor of $D(n, t)$, viz.

$$R(n, t) = D(n, t)/\Delta(n, t).$$

This result is simpler than that of the paper, especially in respect to the fact that the primary minor is symmetrical in the coefficients of $C_1, C_2, \ldots, C_n$.

Since $$D(n, t) = R(n, t)\,\Delta(n, t),$$

and $\Delta(n, t) = 1$ when $t < 2m$, for there are then no supplementary rows, the formula for $\Delta(n, t)$ can be written

$$\Delta(n, t) = \prod_{p=0}^{t-2m} \Delta(n-1,\ t-p)\, R(n-1,\ t-m-p),$$

from which $\Delta(n, t)$ could be expressed as a product of powers of nominal and actual resultants R. I imagine that every nominal resultant is non-factorisable (irreducible) just as every actual resultant is.

We can replace $D(n, t)$ and $\Delta(n, t)$ in the formula for $R(n, t)$ by smaller determinants in the following way. Any power product whatever, and therefore the product $\omega_m \Omega$, where ω_m is of degree m and Ω a product of the kind defined above, is either an $\omega^{(n)}$ or else an $\omega^{(p-1)}$ and not an $\omega^{(p)}$; and, in the latter case, $\omega_m \Omega$ is an $x_p^m \Omega$; for it is divisible by x_p^m, and the quotient satisfies the definition of Ω. Hence the rows supplied by $(C_1, C_2, \ldots, C_n)\Omega$ to $D(n, t)$ have elements only in columns corresponding to $(x_1^m, x_2^m, \ldots, x_n^m)\Omega$, as $D(n, t)$ has no column corresponding to an $\omega^{(n)}$. The number of such rows is the whole number of polynomials $C_1\Omega$ (or $C_1\Omega^{(1)}$), $C_2\Omega^{(1)}, \ldots, C_n\Omega^{(n-1)}$, and the number of columns of $D(n, t)$ which they occupy is the same, viz. the whole number of products $x_1^m \Omega$, $x_2^m \Omega^{(1)}$, $\ldots$, $x_n^m \Omega^{(n-1)}$. Hence the rows and columns form a determinant $D'(n, t)$, which is a factor of $D(n, t)$, and has the same primary minor as $D(n, t)$. Also $R(n, t) = D'(n, t)/\Delta'(n, t)$, where $\Delta'(n, t)$ is the co-minor of the primary minor of $D'(n, t)$. This is illustrated by a diagram very similar to the former one.

	Cols. not in $D'(n, t)$ $\Omega^{(n)}$	Primary columns $(x_1^m, \ldots, x_n^m)\,\Omega^{(n-1)}$	Secondary columns $(x_1^m, x_2^m, \ldots, x_{n-1}^m)\,\Omega$ (Ω divisible by one of $x_2^m, x_3^m, \ldots, x_{n-1}^m$)
Primary rows $(C_1, C_2, \ldots, C_n)\,\Omega^{(n-1)}$		* Primary minor of $D'(n, t)$	*
Secondary rows $(C_1, \ldots, C_{n-1})\,\Omega^{(n-2)}$ $(C_1, \ldots, C_{n-2})\,\Omega^{(n-3)}$ … … … $(C_1, C_2)\,\Omega^{(1)}$ ($\Omega^{(n-2)}, \Omega^{(n-3)}, \ldots, \Omega^{(1)}$ divisible by $x_{n-1}^m, x_{n-2}^m, \ldots, x_2^m$ respectively)	Blank	*	Co-minor of primary minor $\Delta'(n, t)$; the extraneous factor, viz. $R(n, t) = D'(n, t)/\Delta'(n, t)$ *

The special property of $D'(n, t)$ and $\Delta'(n, t)$ is that, like $R(n, t)$, all reach a final form, viz. D'_n, Δ'_n, R_n, when $t > n(m-1)$; for, if

$$t = n(m-1)+p,$$

where $p \geqslant 1$, there is no $\Omega^{(n)}$, and every other Ω of the diagram is divisible by x_n^{p-1}. Also every $\Delta'(2, t)$ is equal to 1.

When $t \leqslant n(m-1)+1$, $\Delta'(n, t)$ satisfies the reduction formula

$$\Delta'(n, t) = \Delta'(n-1, t) \ldots \Delta'(n-1, t-m+1)\, D'(n-1, t-m) \ldots D'(n-1, m)$$

$$= \prod_{p=0}^{t-2m} \Delta'(n-1, t-p)\, R(n-1, t-m-p).$$

From the former of these two expressions for $\Delta'(n, t)$, we have

$$\Delta'_n = \Delta'^{m}_{n-1} D'[n-1, (n-1)(m-1)] \ldots D'(n-1, m)$$

$$= \prod_{r=1}^{n-2} \{D'[n-r, (n-r)(m-1)] \ldots D'(n-r, m)\}^{m^{r-1}}.$$

Thus Δ'_n contains no actual resultant R_{n-r} as a factor. Expressed as a product of nominal resultants

$$\Delta'_n = \prod_{p,\, r} \{R[n-r, (n-r)(m-1)-p]\}^{\frac{p+m}{r-1}\binom{p+rm-1}{r-2}}.$$

The question may be asked whether, for a value of $t \leqslant n(m-1)$, rows could be added to the array of the diagram so as to make it square, and such that its determinant may have the actual resultant R_n as a factor. The answer is that there is no way of doing this, when $t < (n-1)(m-1)$, so as to yield a determinant of less dimensions than D'_n. When, however, $t = (n-1)(m-1)$ the deficient rows are given by Sylvester's extension of Bezout's method [Salmon's *Higher Algebra*, Lesson IX; and *Cam. and Dub. Math. Jour.*, Vol. 7 (1852), p. 68], and have a surprisingly simple form. This method consists in expressing $C_1, C_2, \ldots, C_n$ in the form

$$C_1 = A_1 x_1^{a_1} + A_2 x_2^{a_2} + \ldots + A_n x_n^{a_n},$$

$$C_2 = B_1 x_1^{a_1} + B_2 x_2^{a_2} + \ldots + B_n x_n^{a_n}, \text{ etc.,}$$

where $a_1, a_2, \ldots, a_n$ are positive integers (excluding zero) whose sum is $m+n-1$. Then the polynomials $\Sigma \pm A_1 B_2 \ldots K_n$, which are of degree $(n-1)(m-1)$, supply the required rows, provided that only one such polynomial is taken for each set of values of $a_1, a_2, \ldots, a_n$. For example, if $C_1, C_2, \ldots, C_n$ are put equal to $x_1^m, x_2^m, \ldots, x_n^m$, the polynomials $\Sigma \pm A_1 B_2 \ldots K_n$ become the products $x_1^{m-a_1} x_2^{m-a_2} \ldots x_n^{m-a_n}$, which are the products $\Omega^{(n)}$ of degree $(n-1)(m-1)$. The coefficients of $\Sigma \pm A_1 B_2 \ldots K_n$ are of dimensions 1 in the coefficients of each of $C_1, C_2, \ldots, C_n$, *i.e.* of dimensions n altogether. Also $\Sigma \pm A_1 B_2 \ldots K_n$ can be changed to a polynomial F containing power products of the kind Ω only (without increasing the dimensions of the coefficients) by subtracting a polynomial of the form $X_1 C_1 + X_2 C_2 + \ldots + X_n C_n$, as proved below. Hence the polynomials F supply the needed rows for completing the array of the diagram into a square array, when $t = (n-1)(m-1)$. The determinant of the square array is $R_n \Delta'[n, (n-1)(m-1)]$, the extraneous factor $\Delta'[n, (n-1)(m-1)]$ being the common factor of the determinants of the array before completion. This factor $\Delta'[n, (n-1)(m-1)]$ is less than Δ'_n to the extent of the factor

$$\Delta'^{m}_{n-1} R[n-1, (n-1)(m-1)] \ldots R[n-1, (n-2)(m-1)],$$

and seems to be the least extraneous factor obtainable by the dialytic method.

The proof that $\Sigma \pm A_1 B_2 \ldots K_n$ can be changed to F, as stated above, is as follows. We assume the determinant $\Sigma \pm A_1 B_2 \ldots K_n$ to be symmetrical in the coefficients of $C_1, C_2, \ldots, C_n$, and denote it by its first row $|A_1, A_2, \ldots, A_n|$. Multiply it by $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, and write the result as

$$|A_1 x_1^{\alpha_1} + \ldots + A_n x_n^{\alpha_n}, \ldots, A_{n-1} x_{n-1}^{\alpha_{n-1}} + A_n x_n^{\alpha_n}, A_n x_n^{\alpha_n}|.$$

Expand this as a sum of determinants from the last column, viz.

$$\Sigma\, |A_1 x_1^{\alpha_1} + \ldots + A_n x_n^{\alpha_n}, \ldots, A_{n-1} x_{n-1}^{\alpha_{n-1}} + A_n x_n^{\alpha_n}, a|\, x_n^{\alpha_n} \omega,$$

where a is the coefficient of $x_n^{\alpha_n} \omega$ in $A_n x_n^{\alpha_n}$. Omit all of these determinants in which ω is divisible by $x_{n-1}^{\alpha_{n-1}}$. This may be done, since, if $\omega = x_{n-1}^{\alpha_{n-1}} \bar{\omega}$, the determinant is

$$|A_1, A_2, \ldots. A_{n-2}, C_1, a|\, x_1^{\alpha_1} \ldots x_n^{\alpha_n} \bar{\omega}, \quad \text{or} \quad x_1^{\alpha_1} \ldots x_n^{\alpha_n} \Sigma X_p C_p.$$

Now expand the remaining determinants

$$\Sigma\, |A_1 x_1^{\alpha_1} + \ldots + A_n x_n^{\alpha_n}, \ldots, A_{n-1} x_{n-1}^{\alpha_{n-1}} + A_n x_n^{\alpha_n}, a|\, x_n^{\alpha_n} \omega$$

from the last column but one, giving a sum

$$\Sigma\, |\ldots, A_{n-2} x_{n-2}^{\alpha_{n-2}} + A_{n-1} x_{n-1}^{\alpha_{n-1}} + A_n x_n^{\alpha_n}, a', a|\, x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \omega',$$

where ω' is of degree $\leqslant m-1$ in x_{n-1}; for all determinants in which the outside product is not divisible by $x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n}$ will cancel, since this is true on originally expanding from the last two columns, and all the omitted determinants were divisible by $x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n}$; and $A_{n-1} x_{n-1}^{\alpha_{n-1}} + A_n x_n^{\alpha_n}$ is of degree m, so that $x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \omega'$ is at most of degree $m+\alpha_{n-1}-1$ in x_{n-1}. We now omit from these every determinant in which ω' is divisible by $x_{n-2}^{\alpha_{n-2}}$, and expand the remaining determinants from the last column but two, giving a sum

$$\Sigma \mid \ldots,\ A_{n-3} x_{n-3}^{\alpha_{n-3}} + \ldots + A_n x_n^{\alpha_n},\ a'',\ a',\ a \mid x_{n-2}^{\alpha_{n-2}} x_{n-1}^{\alpha_{n-1}} x_n^{\alpha_n} \omega'',$$

where ω'' is of degree $\leqslant (m-1)$ in x_{n-2}, and of degree $\leqslant 2(m-1)$ in x_{n-2}, x_{n-1} combined; and we omit every determinant in which ω'' is divisible by $x_{n-3}^{\alpha_{n-3}}$. Proceeding in this way we arrive finally at a polynomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} F$, where all power products present in F are of the type Ω. The polynomial $\Sigma \pm A_1 B_2 \ldots K_n$ can be replaced by F, since it only differs from it by a polynomial $\Sigma X_p C_p$.

SUR LES SÉRIES ENTIÈRES A COEFFICIENTS ENTIERS

Par G. Pólya.

(*Communicated by* G. H. Hardy.)

[Received August 14th, 1921.—Read November 17th, 1921.]

1. En poursuivant les remarques de MM. Borel* et Fatou† sur les séries entières à coefficients entiers, je suis arrivé à quelques résultats,‡ mais je n'ai pu ni démontrer ni réfuter la supposition suivante : *Lorsqu'une série entière à coefficients entiers converge dans un cercle de rayon un, ou bien la fonction représentée est rationnelle ou bien son domaine d'existence est limité par le cercle de convergence.* J'ai dû me contenter d'énoncer ce théorème et de proposer aux mathématiciens de décider s'il est vrai ou non. Mon problème rencontra quelque intérêt§ et quelques années plus tard M. Carlson‖ le résolut par l'affirmative. Sa démonstration ajoute à la méthode développée par M. Borel et moi-même une nouvelle idée simple mais importante et susceptible, me semble-t-il, d'autres applications. En cherchant le fondement véritable sur lequel repose la démonstration de M. Carlson, je suis arrivé à généraliser le théorème dont je viens de parler.

Soit D un domaine simplement connexe situé dans le plan des z ne contenant que des points intérieurs et auquel appartienne le point $z = 0$. Il existe, comme on sait, une représentation conforme et biunivoque du domaine D sur un cercle qui fait correspondre le point $z = 0$ au centre du cercle. Autrement dit, il existe une fonction $\phi(z)$ analytique dans le domaine D, prenant des valeurs différentes en des points différents de D, prenant toutes les valeurs w satisfaisant à l'inégalité $|w| < \rho$, où ρ est une certaine constante et ne prenant que ces valeurs-là. Ajoutons la condition que

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* E. Borel, "Sur une application d'un théorème de M. Hadamard," *Bulletin des sciences math.* (2), Vol. 18 (1894), pp. 22–25.

† Fatou, "Sur les séries entières à coefficients entiers," *Comptes Rendus*, Paris, Vol. 138 (1904, 1), pp. 342–344.

‡ G. Pólya, "Über Potenzreihen mit ganzzahligen Koeffizienten," *Math. Annalen*, Vol. 77 (1916), pp. 497–513.

§ F. Hausdorff, "Zur Verteilung der fortsetzbaren Potenzreihen," *Math. Zeitschrift*, Vol. 4 (1919), pp. 98–103. G. Szegö, "Über Potenzreihen, deren Koeffizienten zahlentheoretische Funktionen sind," *Math. Zeitschrift*, Vol. 8 (1920), pp. 36–51.

‖ F. Carlson, "Ueber Potenzreihen mit ganzzahligen Koeffizienten," *Math. Zeitschrift*, Vol. 9 (1921), pp. 1–13.

$$\phi(0) = 0.$$

Supposons en outre que $$\phi'(0) = 1.$$

Alors la fonction $\phi(z)$ et la constante ρ sont *déterminées d'une manière univoque* comme on sait. J'appellerai ρ " le rayon de l'image de D."

Supposons que la fonction analytique $f(z)$ définie par la série entière

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots,$$

dont les coefficients $a_0, a_1, a_2, \ldots$ sont des nombres entiers, n'ait que des points singuliers isolés dans le domaine D et qu'elle y soit uniforme.

Si le rayon de l'image du domaine D est plus grand que l'unité, $f(z)$ est nécessairement une fonction rationnelle.

Si le rayon de l'image de D n'est pas supérieur à un, $f(z)$ peut être régulière en D sans être rationnelle.

Si le rayon en question est égal à l'unité et si la frontière de D satisfait à certaines conditions (par exemple, si elle est une courbe continue sans points multiples), la fonction $f(z)$ est ou bien rationnelle ou bien son domaine d'existence est limité par la frontière de D.

Dans cet énoncé le terme " nombre entier " signifie un nombre entier rationnel ou plus généralement un nombre entier appartenant à un corps quadratique imaginaire donné. L'énoncé ne suppose pas que les points singuliers de $f(z)$ dans le domaine sont en nombre fini, mais que ces points singuliers n'ont pas de point limite à l'intérieur de D. Des trois parties de l'énoncé c'est la première qui est la plus importante, la seconde en est un complément facile et la troisième n'est qu'un corollaire de la première.

Si la proposition en question a un certain attrait c'est qu'elle rapproche trois notions assez éloignées l'une de l'autre : la notion de la fonction rationnelle, celle du nombre entier et celle de la représentation conforme. Regardons attentivement par quelle voie ces trois éléments entrent dans la démonstration.

2. *Pour que la série entière*

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots \tag{1}$$

représente une fonction rationnelle, il faut et il suffit que la suite

$$a_0, \quad \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}, \quad \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix}, \quad \ldots,$$

ne contienne qu'un nombre fini de déterminants différents de zéro.

Ce théorème est dû à Kronecker.* Je n'aurai pas à utiliser le fait que le critère est nécessaire. Montrons rapidement qu'il est suffisant. Je pose

$$\begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \ldots & a_{n+r} \\ \vdots & \vdots & & \vdots \\ a_{n+r-1} & a_{n+r} & \ldots & a_{n+2r-2} \end{vmatrix} = A_n^{(r)}.$$

Le critère de Kronecker se rapporte à la suite des déterminants

$$A_0^{(1)}, \; A_0^{(2)}, \; A_0^{(3)}, \; A_0^{(4)}, \; \ldots .$$

Admettons que la condition posée soit remplie. Il existe donc un nombre N tel que

$$A_0^{(N)} = A_0^{(N+1)} = A_0^{(N+2)} = \ldots = 0.$$

Je fais usage de la relation connue

$$A_n^{(r)} A_{n+2}^{(r)} - A_n^{(r+1)} A_{n+2}^{(r-1)} = (A_{n+1}^{(r)})^2. \tag{2}$$

En admettant que l'on ait

$$A_n^{(N)} = A_n^{(N+1)} = \ldots = A_n^{(N+m)} = A_n^{(N+m+1)} = \ldots = 0 \tag{3}$$

on conclut de la relation (2), en y faisant $r = N, N+1, \ldots, N+m, \ldots$, que

$$A_{n+1}^{(N)} = A_{n+1}^{(N+1)} = \ldots = A_{n+1}^{(N+m)} = \ldots = 0.$$

Les égalités (3) étant supposées valables pour $n = 0$, seront valables pour $n = 1, 2, 3, \ldots$. On a, en particulier,

$$A_0^{(N)} = A_1^{(N)} = A_2^{(N)} = \ldots = 0,$$

ce qui entraîne, comme on sait bien,† que la série (1) est égale à une fonction rationnelle dont le dénominateur est d'un degré inférieur à N.

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* L. Kronecker, "Zur Theorie der Elimination einer Variablen aus zwei algebraischen Gleichungen," *Monatsberichte d. k. preuss. Akademie d. Wiss. zu Berlin* (1881), pp. 535–600. Voir pp. 566–567.

† Voir Borel, *l.c.* Le raisonnement repose sur la remarque suivante qu'on établit à

Pour compléter les préparatifs algébriques, je veux transformer le déterminant

$$A_0^{(n+1)} = \begin{vmatrix} a_0 & a_1 & \dots & a_n \\ a_1 & a_2 & \dots & a_{n+1} \\ \vdots & \vdots & & \vdots \\ a_n & a_{n+1} & \dots & a_{2n} \end{vmatrix}.$$

A cet effet j'envisage la suite des polynomes

$$Q_0^*(z) = 1,$$

$$Q_1^*(z) = t_1^{(1)}z+1,$$

$$Q_2^*(z) = t_2^{(2)}z^2+t_1^{(2)}z+1,$$

$$\dots \quad \dots \quad \dots$$

Je pose $Q_k^*(z)\,f(z) = \square_k a_0 + \square_k a_1 z + \square_k a_2 z^2 + \dots + \square_k a_m z^m + \dots,$

$$Q_k^*(z)\,Q_l^*(z) f(z) = \square_k\square_l a_0 + \square_k\square_l a_1 z + \dots + \square_k\square_l a_m z^m + \dots :$$

on a

$$(4) \qquad \square_k a_m = a_m + t_1^{(k)} a_{m-1} + \dots + t_k^{(k)} a_{m-k},$$

pour $m \geqslant k$,

$$(5) \qquad \square_k\square_l a_m = \square_l(\square_k a_m) = \square_k a_m + t_1^{(l)}\square_k a_{m-1} + \dots + t_l^{(l)}\square_k a_{m-l},$$

pour $m \geqslant l$. Si $m \geqslant k+l$ les $\square_k a_{m-\nu}$ entrant dans la formule (5) se calculent d'après la formule (4).

Multiplions la première, la seconde, ... l'avant-dernière colonne du déterminant $A_0^{(n+1)}$ respectivement par $t_n^{(n)}, t_{n-1}^{(n)}, \dots, t_1^{(n)}$ et ajoutons à la dernière. On obtient

$$A_0^{(n+1)} = \begin{vmatrix} a_0 & a_1 & \dots & a_{n-1} & \square_n a_n \\ a_1 & a_2 & \dots & a_n & \square_n a_{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_n & a_{n+1} & \dots & a_{2n-1} & \square_n a_{2n} \end{vmatrix}.$$

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l'aide de (2) : lorsque l'on a

$$A_m^{(q+1)} = A_{m+1}^{(q+1)} = A_{m+2}^{(q+1)} = A_{m+3}^{(q+1)} = \dots = 0,$$

les déterminants $A_{m+1}^{(q)}, A_{m+2}^{(q)}, A_{m+3}^{(q)}, \dots$ sont ou bien tous différents de zéro ou bien tous égaux à zéro.

En continuant à ajouter à chaque colonne une combinaison linéaire convenablement choisie des précédentes, on arrive à la formule

$$A_0^{(n+1)} = \begin{vmatrix} a_0 & \Box_1 a_1 & \Box_2 a_2 & \dots & \Box_n a_n \\ a_1 & \Box_1 a_2 & \Box_2 a_3 & \dots & \Box_n a_{n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ a_n & \Box_1 a_{n+1} & \Box_2 a_{n+2} & \dots & \Box_n a_{2n} \end{vmatrix},$$

tous les éléments étant calculés à l'aide de la formule (4), qui est ici applicable, la condition $m \geqslant k$ étant remplie. En opérant sur les lignes de la même manière que sur les colonnes on obtient*

$$A_0^{(n+1)} = \begin{vmatrix} a_0 & \Box_1 a_1 & \dots & \Box_n a_n \\ \Box_1 a_1 & \Box_1 \Box_1 a_2 & \dots & \Box_1 \Box_n a_{n+1} \\ \vdots & \vdots & & \\ \Box_n a_n & \Box_n \Box_1 a_{n+1} & \dots & \Box_n \Box_n a_{2n} \end{vmatrix} = |\Box_k \Box_l a_{k+l}| \quad (k, l = 0, 1, \dots, n),$$

tous les éléments étant calculés d'après la formule (5).

On a

$$\Box_k \Box_l a_{k+l} = \frac{1}{2\pi i} \int \frac{f(z)\, Q^*_k(z)\, Q^*_l(z)\, dz}{z^{k+l+1}}, \tag{6}$$

l'intégrale étant étendue le long de la frontière d'un certain domaine ne contenant à son intérieur que des points réguliers de $f(z)$ et contenant le point $z = 0$. Je pose

$$Q^*_n(z)\, z^{-n} = z^{-n} + t_1^{(n)} z^{-n+1} + \dots + t_n^{(n)} = Q_n(z^{-1}): \tag{7}$$

$Q_n(z^{-1})$ est un polynome de degré n en z^{-1}, le coefficient de z^{-n} étant 1. On a en définitive cette expression du déterminant

$$A_0^{(n+1)} = \left| \frac{1}{2\pi i} \int f(z)\, Q_k(z^{-1})\, Q_l(z^{-1})\, z^{-1} dz \right| \quad (k, l = 0, 1, 2, \dots, n). \tag{8}$$

3. Comment entrera la notion du nombre entier dans la démonstration ? L'ensemble des nombres entiers rationnels a les deux propriétés suivantes :

(A) z' et z'' étant éléments de l'ensemble $z' + z''$, $z' - z''$, $z'z''$ le sont aussi.

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* Le cas particulier où $Q^*_m(z) = (1-z)^m$ est dû à Hankel. Voir p.e. G. Kowalewski, "Einführung in die Determinantentheorie" (Leipzig, 1909), pp. 112-114.

(B) Le point $z=0$ est un point mais il n'est pas un point limite de l'ensemble (envisagé comme ensemble de points).

La démonstration n'utilise que ces deux propriétés (A) et (B) des nombres entiers rationnels. Les nombres entiers d'un corps quadratique imaginaire[*] ont les deux mêmes propriétés, donc la démonstration leur sera aussi applicable. D'après la propriété (A) les déterminants $A_0^{(n+1)}$ sont des entiers si les coefficients $a_0, a_1, a_2, \ldots$ de la série (1) le sont. D'après la propriété (B) on peut conclure de la relation

$$\lim_{n=\infty} A_0^{(n+1)} = 0, \tag{9}$$

que la suite $A_0^{(1)}, A_0^{(2)}, A_0^{(3)}, \ldots$ ne contient qu'un nombre fini des termes différents de zéro. Tout revient donc à démontrer (9).

On pourrait croire qu'en n'utilisant que les propriétés (A) et (B) on utilise trop défectueusement la notion du nombre entier. Il n'en est rien. On peut démontrer facilement que, lorsqu'un ensemble E a les deux propriétés (A) et (B), il ne contient que des nombres entiers pris dans un certain corps quadratique imaginaire éventuellement seulement des entiers rationnels.[†]

4. On apprend du travail de M. Hausdorff cité ci-dessus que l'étude de la représentation conforme n'est pas sans rapport au problème des

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[*] Nombres de la forme $m+n\theta$, où $m, n = 0, \pm 1, \pm 2, \ldots$ et θ est un nombre imaginaire, racine d'une certaine équation $\theta^2+a\theta+b=0$, où a, b sont des nombres entiers rationnels.

[†] En ne s'appuyant que sur une partie de l'hypothèse (A), celle qui concerne $z'+z''$ et $z'-z''$, et laissant de côté ce qui concerne $z'z''$, on démontre dans la théorie des fonctions elliptiques qu'il y a deux cas à distinguer : (1) tous les nombres de l'ensemble E sont compris dans la formule nA, où $n = 0, \pm 1, \pm 2, \pm 3, \ldots$, ou bien : (2) ils sont compris dans la formule $mA+nB$, où $m, n = 0, \pm 1, \pm 2, \pm 3, \ldots$, et le quotient B/A est imaginaire. Utilisons maintenant ce qui a été dit de $z'z''$. Observons dans le premier cas que A^2 doit être de la forme nA ; donc $A=n$, un nombre entier rationnel. Observons dans le second cas que par hypothèse

$$AB = lA+l'B, \quad B^2 = mA+m'B, \quad AB^2 = nA+n'B,$$

l, l', m, m', n, n' étant des nombres entiers rationnels. En éliminant B on obtient

$$mA^2+(lm'-l'm-n)A = ln'-l'n.$$

$ln'-l'n$ appartient donc à E ; si $ln'-l'n=0$, $lm'-l'm-n$ appartient à E ; si ce nombre était aussi $=0$, donc $m=0$, $m'=B\neq 0$ appartient à l'ensemble E. E contient donc nécessairement des nombres entiers rationnels $\neq 0$; le plus petit de ces nombres-là soit

$$R = rA+r'B.$$

Alors le plus grand commun diviseur de r et r' doit être l'unité. Soient s, s' des entiers rationnels tels que $rs'-r's=1$. $S=sA+s'B$ appartient aussi à l'ensemble E et chaque nombre appartenant à E doit être de la forme $nR+n'S$ ($n, n' = 0, \pm 1, \pm 2, \ldots$). En particulier $S^2 = nR+n'S$, c.q.f.d.

séries entières aux coefficients entiers. C'est par la représentation conforme qu'on peut trouver l'ordre de grandeur des polynomes qui s'écartent le moins possible de zéro dans un domaine donné. On doit à M. Faber cette découverte intéressante.* C'est là la véritable raison pour laquelle le rayon de l'image du domaine où la fonction est uniforme joue un rôle décisif dans ces questions. Je me borne aux détails nécessaires à ma démonstration.

Le domaine D situé dans le plan des z dont il était question ci-dessus (voir § 1) correspond point par point à son image conforme, l'aire circulaire $|w| < \rho$, moyennant la fonction

$$\phi(z) = w.$$

Puisque $\phi(0) = 0$, $\phi'(0) = 1$ on a

$$\frac{1}{\phi(z)} = \frac{1}{z} + \lambda_0 + \lambda_1 z + \lambda_2 z^2 + \ldots \tag{10}$$

dans un certain voisinage du point $z = 0$. Je pose

$$\begin{aligned}\left(\frac{1}{\phi(z)}\right)^m &= \frac{1}{z^m} + \frac{l_{m,1}}{z^{m-1}} + \frac{l_{m,2}}{z^{m-2}} + \ldots + \frac{l_{m,m-1}}{z} + l_{m,m} + P^*_m(z) \\ &= P_m(z^{-1}) + P^*_m(z),\end{aligned} \tag{11}$$

où $P_m(z^{-1})$, la partie principale du développement, est un polynome en z^{-1} de degré m et $P^*_m(z)$ une série entière, $P^*_m(0) = 0$. J'affirme que *dans le domaine D on a*

$$\lim_{m=\infty} [\phi(z)]^m P_m(z^{-1}) = 1 \tag{12}$$

la convergence étant uniforme en chaque partie fermée de D.

Remarquons que $z/\phi(z)$ est régulier dans tout le domaine D parce qu'au point $w = 0$ ne correspond que le point $z = 0$. On a

$$P_m(z^{-1}) = \frac{1}{2\pi i}\int_\Gamma \left[\left(\frac{u}{z}\right)^m + \left(\frac{u}{z}\right)^{m-1} + \ldots + \frac{u}{z} + 1\right]\left(\frac{1}{\phi(u)}\right)^m \frac{du}{u}. \tag{13}$$

Le contour Γ peut être une courbe fermée quelconque sans points multiples situés dans D et entourant le point $z = 0$. Soit Γ l'image de la circonférence $|w| = R$, où

$$|\phi(z)| < R < \rho\,; \tag{14}$$

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* G. Faber, "Ueber Tschebyscheffsche Polynome," *Journal f. d. r. u. ang. Math.* (*Crelle*), Vol. 150 (1920), pp. 70–106.

ce qui revient à dire que, lorsque le point u est situé sur Γ, on a $|\phi(u)| = R$. J'écris la formule (13) comme il suit :

$$P_m(z^{-1}) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z^m}\left(\frac{u}{\phi(u)}\right)^m \frac{du}{u-z} - \frac{1}{2\pi i}\int_\Gamma \frac{z}{u-z}\left(\frac{1}{\phi(u)}\right)^m \frac{du}{u}. \tag{15}$$

Le point z est situé à l'intérieur de Γ comme le point $w = \phi(z)$ est à l'intérieur du cercle $|w| = R$, voir (14). Donc on a en calculant le résidu relatif au point $u = z$,

$$P_m(z^{-1}) - \left(\frac{1}{\phi(z)}\right)^m = -\frac{1}{2\pi i}\int_\Gamma \frac{z}{u-z}\left(\frac{1}{\phi(u)}\right)^m \frac{du}{u}.$$

On en tire

$$|P_m(z)\,\phi(z)^m - 1| \leqslant G\left(\frac{|\phi(z)|}{R}\right)^m, \tag{16}$$

où
$$G = \frac{1}{2\pi}\int_\Gamma \left|\frac{z}{u-z}\right| \frac{|du|}{|u|}.$$

G est donc indépendant de m. Vu (14) le théorème énoncé découle de (16).

5. Nous n'avons qu'à enchaîner ces différentes remarques pour obtenir la démonstration de la première partie du théorème énoncé au § 1, où l'on suppose que ρ, le rayon de l'image de D, est plus grand que l'unité.

Je choisis R de manière que

$$1 < R < \rho, \tag{17}$$

et que la courbe $|\phi(z)| = R$ ne passe par aucun point singulier de la fonction $f(z)$. Je désignerai la courbe $|\phi(z)| = R$, image biunivoque et analytique du cercle $|w| = R$, par Γ. Par hypothèse les points singuliers de $f(z)$ n'ont pas de point limite sur Γ ni à l'intérieur de Γ. A l'intérieur de Γ il n'y a donc qu'un nombre fini s de points singuliers de $f(z)$. Désignons ces points par

$$c_1,\ c_2,\ c_3,\ \ldots,\ c_s.$$

(Je me place dans le cas général où $s \geqslant 1$; dans le cas où $s = 0$ la démonstration se simplifie.)

Je considère le polynome en z^{-1},

$$S(z^{-1}) = (z^{-1} - c_1^{-1})(z^{-1} - c_2^{-1}) \ldots (z^{-1} - c_s^{-1}).$$

$P_m(z^{-1})$ étant défini par (11) et α désignant une fraction positive dont la valeur sera fixée plus tard, je pose*

$$P_{n-s[\alpha n]}(z^{-1})\{S(z^{-1})\}^{[\alpha n]} = Q_n(z^{-1}). \tag{18}$$

Cette notation n'est pas en désaccord avec celle du § 2, parce que $Q_n(z^{-1})$ n'était assujetti qu'à la condition d'être un polynome de degré n en z^{-1} dans lequel le coefficient de z^{-n} est $= 1$, voir (7), condition qui est bien remplie par (18).

Fixons maintenant la valeur de α. A cet effet je choisis deux grandeurs positives h et r assujetties respectivement aux conditions

$$1/R < h < 1, \tag{19}$$

$$r > 0, \quad r < |c_\sigma| \quad (\sigma = 1, 2, 3, \ldots, s). \tag{20}$$

Les polynomes de degré $s-1$ en z^{-1},

$$(z^{-1}-c_\sigma^{-1})^{-1} S(z^{-1}) \quad (\sigma = 1, 2, 3, \ldots, s),$$

sont réguliers dans le domaine $|z| \geqslant r$. Il existe donc une grandeur positive N telle que, pour $|z| \geqslant r$,

$$|S(z^{-1})| < N, \quad |S(z^{-1})| < |z^{-1}-c_\sigma^{-1}| N \quad (\sigma = 1, 2, \ldots, s). \tag{21}$$

Je choisis α de telle sorte que les deux conditions suivantes

$$0 < \alpha < 1/s, \tag{22}$$

$$R^{-1+s\alpha} N^\alpha < h, \tag{23}$$

soient satisfaites. Ce choix est possible. Il uffit d'observer que la limite de la quantité dans le membre gauche de (23) pour $\alpha = 0$ est $< h$ en vertu de (19).

Il me reste à fixer le chemin d'intégration des intégrales qui figurent dans la formule (8). L'image du cercle $|z| = r$ dans le plan des w est une courbe entourant le point $w = 0$. Soit ω le point de cette courbe qui est le plus rapproché de $w = 0$. Donc on a

$$|\phi(z)| \geqslant |\omega| \quad (|z| \geqslant r). \tag{24}$$

Je considère les cercles

$$|z^{-1}-c_\sigma^{-1}| = \epsilon \quad (\sigma = 1, 2, \ldots, s). \tag{25}$$

Je choisis le nombre positif ϵ assez petit pour que ces cercles (25) soient extérieurs les uns aux autres et entièrement contenus dans la couronne comprise entre le cercle $|z| = r$ et la courbe Γ, voir (20), et pour qu'on ait en outre

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* $[x]$ est le nombre entier satisfaisant aux inégalités $x-1 < [x] \leqslant x$.

$$|\omega|^{-1+s\alpha} N^{\alpha} \epsilon^{\alpha} < h. \tag{26}$$

Ce choix est évidemment possible, α étant > 0, voir (22). Je désigne le cercle (25) par $\Gamma_\sigma (\sigma = 1, 2, \ldots, s)$. On a d'accord avec (6), (7), (8)

$$\square_k \square_l a_{k+l} = \frac{1}{2\pi i} \left(\int_\Gamma - \int_{\Gamma_1} - \int_{\Gamma_2} - \ldots - \int_{\Gamma_s} \right) f(z)\, Q_k(z^{-1})\, Q_l(z^{-1})\, z^{-1} dz. \tag{27}$$

Toutes les courbes $\Gamma, \Gamma_1, \Gamma_2, \ldots, \Gamma_s$ sont décrites dans le sens positif.

Le long des courbes $\Gamma, \Gamma_1, \Gamma_2, \ldots, \Gamma_s$ la fonction $f(z)$ est régulière. Il existe donc une grandeur K telle que

$$|f(z)| < K \tag{28}$$

lorsque z est situé sur l'une quelconque des courbes $\Gamma, \Gamma_1, \ldots, \Gamma_s$. A l'intérieur et sur la courbe Γ la relation limite (12) a lieu uniformément. Il existe donc une grandeur M telle que

$$|P_m(z^{-1})| < M |\phi(z)|^{-m} \quad (m = 1, 2, 3, \ldots). \tag{29}$$

Cette inégalité est valable sur l'ensemble des courbes $\Gamma, \Gamma_1, \ldots, \Gamma_s$. Remarquons qu'il en va de même des inégalités (21), (24), (28).

Posons

$$L = \int_\Gamma |z|^{-1} |dz|, \qquad L_\sigma = \int_{\Gamma_\sigma} |z|^{-1} |dz| \quad (\sigma = 1, 2, \ldots, s). \tag{30}$$

On a

$$\begin{aligned} &\left| \int_\Gamma f(z)\, Q_k(z^{-1})\, Q_l(z^{-1})\, z^{-1} dz \right| \\ &\quad = \left| \int_\Gamma f(z) P_{k-s[\alpha k]}(z^{-1}) P_{l-s[\alpha l]}(z^{-1}) \left\{ S(z^{-1} \right\}^{[\alpha k]+[\alpha l]} z^{-1} dz \right| \\ &\quad \leqslant K \int_\Gamma |z|^{-1} M^2 |\phi(z)|^{-k-l+s[\alpha k]+s[\alpha l]} N^{[\alpha k]+[\alpha l]} |dz| \\ &\quad = KLM^2 N^{[\alpha k]+[\alpha l]} R^{-k-l+s[\alpha k]+s[\alpha l]}, \end{aligned} \tag{31}$$

en observant d'abord (18), puis en tenant compte des inégalités (28), (29), (21), de l'égalité (30), et enfin de ce que $|\phi(z)| = R$ lorsque z se meut sur Γ.

On a

$$(32)\quad \int_{\Gamma_\sigma} f(z)\, Q_k(z^{-1})\, Q_l(z^{-1})\, z^{-1} dz$$

$$\leqslant K \int_{\Gamma_\sigma} |z|^{-1} M^2 |\phi(z)|^{-k-l+s[ak]+s[al]} (|z^{-1}-c_\sigma^{-1}|N)^{[ak]+[al]} |dz|$$

$$\leqslant K L_\sigma M^2 |\omega|^{-k-l+s[ak]+s[al]} (\epsilon N)^{[ak]+[al]},$$

en vertu tout d'abord des inégalités (28), (29), (21) (second cas), puis de l'inégalité (24), et enfin de la définition de Γ_σ, voir (25). Les derniers termes des inégalités (31) et (32) sont, abstraction faite d'un facteur qui reste borné, égaux respectivement à

$$(N^a R^{-1+sa})^{k+l}, \qquad (|\omega|^{-1+sa} \epsilon^a N^a)^{k+l}.$$

Vu les inégalités (23), (26), nous arrivons à cette conclusion qu'il existe une quantité positive G telle que

$$|\Box_k \Box_l a_{k+l}| < G h^{k+l} \quad (k,\ l = 0, 1, 2, 3, \ldots),$$

voir (27). Donc le déterminant

$$A_0^{(n+1)} = |\Box_k \Box_l a_{k+l}| \quad (k,\ l = 0, 1, 2, \ldots, n),$$

est plus petit que

$$(n+1)!\, G^{n+1} h^{2(1+2+3+\ldots+n)} < (nGh^n)^{n+1}.$$

Cette dernière expression tend évidemment vers 0 avec $1/n$. C.q.f.d.

6. J'admets maintenant que le rayon ρ de l'image de D ne surpasse pas l'unité. La fonction $\phi(z)$ qui transforme le domaine D en cercle de rayon ρ satisfait à l'inégalité

$$|\phi(z)| < \rho \leqslant 1.$$

Par conséquent, si la suite $c_0, c_1, c_2, c_3, \ldots$ est bornée, la série

$$(33)\qquad c_0 + c_1\phi(z) + c_2\phi(z)^2 + \ldots + c_n\phi(z)^n + \ldots = F(z)$$

est convergente dans D, uniformément convergente dans chaque partie fermée de D, et $F(z)$ y est analytique. Posons d'accord avec (10),

$$(34)\qquad \phi(z) = z + \mu_2 z^2 + \mu_3 z^3 + \ldots.$$

Dans un certain cercle de centre $z = 0$ on a

$$F(z) = \sum_{\nu=0}^{\infty} c_\nu (z+\mu_2 z^2+\mu_3 z^3+\dots)^\nu \tag{35}$$

$$= a_0+a_1 z+a_2 z^2+\dots.$$

Cette dernière série s'obtient en ordonnant les termes de la première suivant les puissances croissantes de z, d'après un théorème connu de Weierstrass. J'affirme que *les coefficients* $a_0, a_1, a_2, \dots$ *peuvent être des entiers de la forme* $m+ni$, *où* m *et* n *sont des entiers rationnels, sans que la fonction* $F(z)$ *soit rationnelle.* On pourrait d'ailleurs choisir les coefficients $a_0, a_1, \dots$ dans un corps quadratique imaginaire autre que celui de Gauss. On verra que $a_0, a_1, a_2, \dots$ peuvent être rationnels si les coefficients $\mu_2, \mu_3, \dots$ de la série (34) sont réels.

Soit donné une suite infinie

$$\eta_0, \eta_1, \eta_2, \dots, \eta_n, \dots, \tag{36}$$

la valeur de η_n étant *ou bien* 0 *ou bien* 1 ($n = 0, 1, 2, 3, \dots$). Je pose $c_0 = \eta_0$, ce qui donne $c_0 = a_0 = 0$ ou 1 suivant le cas. Je pose de même $c_1 = \eta_1$, ce qui donne $c_1 = a_1 = 0$ ou 1 suivant le cas. Admettons que $c_0, c_1, c_2, \dots, c_{n-1}$ soient déjà déterminés. On a, en vertu de (35),

$$a_n = c_n + C_{n-1},$$

C_{n-1} étant une certaine expression linéaire en $c_0, c_1, \dots, c_{n-1}$, le coefficient de z^n dans le développement de la fonction

$$\sum_{\nu=0}^{n-1} c_\nu (z+\mu_2 z^2+\mu_3 z^3+\dots)^\nu.$$

Soit
$$C_{n-1} = A_{n-1}+iB_{n-1},$$

A_{n-1} et B_{n-1} étant réels. Je pose

$$c_n = [A_{n-1}]+\eta_n-A_{n-1}+i([B_{n-1}]-B_{n-1}),$$

$$a_n = [A_{n-1}]+i[B_{n-1}]+\eta_n,$$

ce qui donne évidemment

$$|c_n| < \sqrt{2}. \tag{37}$$

Il suit de (37) qu'à chaque suite déterminée (36) correspond une série convergente (35), les coefficients $a_0, a_1, a_2, \dots$ étant des entiers de Gauss. A deux suites différentes $\eta_0, \eta_1, \eta_2, \dots$ correspondent deux suites différentes $c_0, c_1, c_2, \dots$ (observons les premiers termes distincts), donc deux fonctions

$F(z)$ différentes. Mais la puissance de l'ensemble des suites de la forme (36) est celle du continu, tandis que l'ensemble des fonctions rationnelles à coefficients appartenant au corps $\sqrt{-1}$ est évidemment dénombrable : ce qui établit le fait en question.

7. Dans ce paragraphe je réserverai le mot "domaine" pour désigner un ensemble situé dans le plan des z, ne contenant que des points intérieurs, simplement connexe et contenant le point $z = 0$. On connaît le théorème suivant : si le domaine D est une partie du domaine D^* (partie proprement dite) le rayon de l'image de D est plus petit que le rayon de l'image de D^* (égalité exclue).

Considérons un domaine D dont l'image a le rayon un. Admettons que la frontière de D soit une courbe continue sans points multiples. Soit $f(z)$ la fonction dont il est question dans le § 1 ; $f(z)$ est donc développable en série entière à coefficients entiers dans un cercle de centre $z = 0$, elle est uniforme dans D et n'y a que des points singuliers isolés. Si la frontière de D n'est pas coupure pour la fonction $f(z)$, celle-ci possédera les mêmes propriétés dans un domaine plus étendu D^*. Mais le rayon de l'image de D^* est nécessairement plus grand que 1, d'après ce que je viens de dire, puisque celui de l'image de D est $= 1$. $f(z)$ est donc une fonction rationnelle, c.q.f.d. Ce qui précède s'applique en particulier au cas le plus simple et le plus intéressant quand le domaine D est le cercle $|z| < 1$ identique à son image.

On pourrait traiter de la même manière d'autres cas où la frontière de D est suffisamment "dégagée" bien qu'elle ne soit pas une courbe continue sans points multiples. Je ne m'arrête pas à épuiser cette question secondaire. Je me borne à donner un exemple montrant que, sans quelque hypothèse supplémentaire sur la nature de la frontière, l'alternative piquante, fonction rationnelle ou à domaine d'existence limité, n'a pas lieu.

Je désigne par B le domaine qui contient tous les points du plan des z sauf ceux situés sur l'axe réel positif entre $\frac{1}{4}$ et ∞, limite comprise. La relation

$$z = \frac{w}{(1+w)^2} = w - 2w^2 + 3w^3 - \dots, \tag{38}$$

fait correspondre B au cercle $|w| < 1$. Le rayon de l'image de B est donc 1. Mais la fonction de z qu'on obtient en résolvant (38) par rapport à w,

$$w = \frac{1}{2z} - 1 - \frac{\sqrt{(1-4z)}}{2z} = \sum_{n=1}^{\infty} \binom{2n}{n+1} \frac{z^n}{n},$$

a des coefficients entiers et elle est régulière en B sans être ni une fonction rationnelle ni à domaine d'existence limité. Remarquons comme explication qu'aucun point sur la frontière de B n'est " dégagé."

En faisant usage de la représentation conforme (38) on démontre facilement le théorème suivant : supposons que la série entière

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots,$$

à coefficients entiers a_0, a_1, a_2, ... soit prolongeable indéfiniment le long de chaque demi-droite issue de l'origine excepté de l'axe réel positif. Si la fonction $f(z)$ n'est pas rationnelle, son point singulier le plus rapproché de l'origine se trouve sur le segment $0 < z \leqslant \frac{1}{4}$. Si le point singulier le plus rapproché est $\frac{1}{4}$, il n'y a que trois cas possibles : ou bien (1) la partie de l'axe réel positif comprise entre $\frac{1}{4}$ et l'infini est une coupure pour la fonction $f(z)$, ou bien (2) $f(z)$ est une fonction algébrique à deux valeurs et à deux points d'embranchement, $\frac{1}{4}$ et ∞, ou bien (3) la fonction $f(z)$ est rationnelle.

8. Je ne veux pas discuter point par point ce que la démonstration exposée dans les §§ 2 à 5 ajoute de nouveau aux raisonnements développés dans les travaux successifs de M. Borel, de moi-même et de M. Carlson. Je me contente d'indiquer le point de contact entre le raisonnement actuel et celui de M. Carlson.

Soit α une grandeur positive arbitrairement petite. M. Carlson construit un polynome $P(z)$ tel que $P(0) = 1$ et $|P(z)| < 1$ quand $|z| = 1$ et $\alpha \leqslant \arg z \leqslant 2\pi - \alpha$, c. à d. le long d'un arc qui constitue *presque* un cercle complet de centre $z = 0$. Le rayon de ce cercle est sans importance. C'est ce qu'on voit en considérant le polynome $P(r^{-1}z)$ sur la circonférence $|z| = r$. L'existence d'un tel polynome peut être montrée de manières très diverses, chaque mathématicien exercé dans la théorie des fonctions en trouvera une autre. Voici la méthode qui découle des considérations du § 4. Considérons le domaine D limité par l'arc de cercle en question ($|z| = 1$, $\alpha \leqslant \arg z \leqslant 2\pi - \alpha$), et par une courbe continue qui relie les deux extrémités de cet arc, située dans le domaine $|z| > 1$. Le rayon ρ de l'image de ce domaine D *surpasse l'unité* (voir § 7). On a donc $|\phi(z)|^{-1} < 1$ sur l'arc de cercle $|z| = r < 1$, $\alpha \leqslant \arg z \leqslant 2\pi - \alpha$ dès que r s'approche suffisamment de l'unité. Sur le même arc on a aussi $|P_n(z^{-1})| < 1$, n étant choisi assez grand, en vertu de (12), et à fortiori $|z^n P_n(z^{-1})| < 1$. Mais $z^n P_n(z^{-1})$ est un polynome de degré n se réduisant, en vertu de la formule (11), à 1 pour $z = 0$. C.q.f.d.

9. L'examen des séries particulières à coefficients entiers peut provo-

quer des remarques intéressantes ou tout au moins faire trouver des exercices sortant un peu du cadre habituel. Je veux en donner un exemple auquel on pourrait ajouter beaucoup d'autres.

M. Hansen, dans sa Thèse,* a désigné comme probable un théorème qui légèrement généralisé peut être énoncé ainsi :

Je suppose que les nombres entiers

(39) $$a_1, a_2, a_3, \ldots, a_n, \ldots,$$

satisfont à l'inégalité double

(40) $$0 \leqq a_n < b,$$

b étant un entier donné. Si la suite (39) *contient une infinité de termes différents de zéro, le domaine d'existence des fonctions*

$$L(z) = \sum_{n=1}^{\infty} \frac{a_n z^n}{1-z^n}, \qquad H(z) = \sum_{n=1}^{\infty} \frac{a_n z^n}{1+z^n},$$

est le cercle $|z| < 1$.

(M. Hansen ne parle que du cas où $b = 2$.) Les fonctions $L(z)$ et $H(z)$ peuvent être développées en série entière à coefficients entiers convergente dans le cercle $|z| < 1$. Donc pour démontrer la supposition de M. Hansen on n'a qu'à établir que les fonctions $L(z)$ et $H(z)$ ne sont pas rationnelles. Mais ce fait n'est pas évident bien que sa démonstration n'est qu'un exercice. Remplaçons l'inégalité (40) par $-1 \leqq a_n \leqq +1$ et $L(z)$ peut être rationnelle. Exemple :

$$\sum_{n=1}^{\infty} \frac{\mu(n) z^n}{1-z^n} = z,$$

$\mu(n)$ désigne le symbole connu de Möbius qui entre dans le développement de $\zeta(s)^{-1}$. Ne conservons qu'une moitié de l'inégalité double (40) en la remplaçant par $a_n \geqq 0$ et $L(z)$ peut être rationnelle. Exemple :

$$\sum_{n=1}^{\infty} \frac{\phi(n) z^n}{1-z^n} = \frac{z}{(1-z)^2},$$

$\phi(n)$ désignant comme d'habitude le nombre des entiers inférieurs et premiers à n. Remplaçons $L(z)$ par un type de séries légèrement différent : $\sum_{n=1}^{\infty} \frac{a_n z^n}{1-z^{2n}}$ et on arrivera tout en conservant (40) à des fonctions rationnelles. Exemple :

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* D. Hansen, "Om en gruppe hele transcendente Funktioner," Thèse (Kopenhague, 1904). Voir p. 65, Thèse 5.

$$\frac{z}{1-z^2}+\frac{z^2}{1-z^4}+\frac{z^4}{1-z^8}+\dots=\frac{z}{1-z}.$$

Mais voilà comment on peut démontrer en admettant (40) que $L(z)$ n'est pas une fonction rationnelle. On constate d'abord que

$$\lim_{z=1}(1-z)^2L(z)=0, \tag{41}$$

z se rapprochant du point 1 le long de l'axe réel positif. C'est sur ce rayon du cercle $|z|<1$ que $L(z)$ prend ses plus grandes valeurs. Si $L(z)$ était rationnelle ses points singuliers sur la circonférence $|z|=1$ seraient des pôles ; ces pôles ne sauraient pas être doubles ou d'un ordre supérieur d'après (41) ; ils sont donc nécessairement simples. Lorsqu'une série entière converge dans le cercle de rayon 1, et n'a sur ce cercle d'autres points singuliers que des pôles simples, la suite de ses coefficients est *bornée.* Mais voici la contradiction. Soient

$$a_{\nu_1},\ a_{\nu_2},\ a_{\nu_3},\ \dots,\ a_{\nu_k},\ \dots,$$

les nombres entiers figurant dans $L(z)$ qui sont différents de zéro ; leur nombre est infini par hypothèse. Dans la série de Maclaurin de la fonction $L(z)$ le coefficient de $z^{\nu_1\nu_2\nu_3\dots\nu_k}$ n'est pas inférieur à

$$a_{\nu_1}+a_{\nu_2}+a_{\nu_3}+\dots+a_{\nu_k}\geqslant k;$$

donc la suite des coefficients de cette série *n'est pas bornée.* Admettre que $L(z)$ est rationnelle est donc contradictoire.

Je parle maintenant de $H(z)$. Je distribue les nombres

$$1,\ 2,\ 3,\ 4,\ 5,\ \dots,\ n,\ \dots,$$

en une infinité de classes différentes que je désignerai par

$$Z,\ P_0,\ P_1,\ P_2,\ \dots,\ P_l,\ \dots.$$

n appartient à la classe Z si $a_n=0$.

n appartient à la classe P_l si $a_n>0$, si n est divisible par 2^l et n'est pas divisible par 2^{l+1}.

Je distingue deux cas.

(1) Il existe une classe P_l qui contient une infinité d'entiers. Admettons donc que $P_0, P_1, P_2, \dots, P_{l-1}$ ne contiennent qu'un nombre fini d'entiers tandis que P_l contienne les entiers $2^lj_1, 2^lj_2, 2^lj_3, \dots, j_1, j_2, j_3, \dots$ étant impaires. En considérant le coefficient de $z^{2^lj_1j_2j_3\dots j_k}$ et en supposant

$H(z)$ rationnelle, on arrive à une contradiction du même genre que celle qu'on a rencontrée en supposant $L(z)$ rationnelle.

(2) Chaque classe $P_0, P_1, P_2, \ldots, P_l, \ldots,$

contient un nombre fini d'entiers, respectivement

$$p_0, p_1, p_2, \ldots, p_l, \ldots,$$

entiers. Désignons par $N(x)$ le nombre des entiers *non*-contenus dans Z et ne surpassant pas x. On a l'inégalité

$$N(x) \leqslant p_0+p_1+\ldots+p_{l-1}+\left[\frac{x}{2^l}\right],$$

d'où l'on conclut $\overline{\lim\limits_{x=\infty}}\, x^{-1}N(x) \leqslant 2^{-l},$

pour $l=1, 2, 3, \ldots,$ donc

$$\lim_{x=\infty} x^{-1}N(x) = 0. \tag{42}$$

Il s'en suit de (42) que $\lim\limits_{z=1} (1-z)H(z) = 0,$

z s'approchant de l'unité par valeurs positives. Mais on a évidemment

$$\lim_{z=1} H(z) = \infty.$$

Donc le point $z=1$ est un point singulier, mais n'est pas un pôle de $H(z)$, et ainsi $H(z)$ n'est pas rationnelle. C.q.f.d.

THE APPROXIMATE FUNCTIONAL EQUATION IN THE THEORY OF THE ZETA-FUNCTION, WITH APPLICATIONS TO THE DIVISOR-PROBLEMS OF DIRICHLET AND PILTZ

By G. H. HARDY *and* J. E. LITTLEWOOD.

[Read March 10th, 1921.]

1. *Introduction.*

1.1. Our first object in this paper is to prove the following theorem.

THEOREM A.—*Suppose that H and K are positive constants, and*

(1.11) $$s=\sigma+it, \quad -H \leqslant \sigma \leqslant H,$$

(1.12) $$x>K, \quad y>K, \quad 2\pi xy=|t|.$$

Then (i)

(1.13) $$\zeta(s)=\sum_{n<x} n^{-s}+\chi \sum_{n<y} n^{s-1}+R,$$

where

(1.14) $$\chi=2(2\pi)^{s-1}\sin\tfrac{1}{2}s\pi\,\Gamma(1-s),$$

(1.15) $$R=O(x^{-\sigma})+O(y^{\sigma-1}|t|^{\frac{1}{2}-\sigma}),$$

and $O[\phi(x, y, \sigma, t)]$ denotes a function which is, for all values of the variables which satisfy (1.11) *and* (1.12), *numerically less than $A\phi(x, y, \sigma, t)$, $A=A(H, K)$ being a number which depends on H and K alone ;*

(ii) *if, in addition to* (1.11) *and* (1.12), $x \geqslant y$, *then*

(1.16) $$R=O\left(\frac{x^{1-\sigma}}{|t|^{\frac{1}{2}}+x\vartheta(y)}\right),$$

where $\vartheta(y)$ denotes the absolute value of the difference between y and the integer nearest y ;

(iii) *if, in addition to* (1.11) *and* (1.12), $x \leqslant y$, *then*

$$R = O\left(\frac{|t|^{\frac{1}{2}-\sigma} y^{\sigma}}{|t|^{\frac{1}{2}} + y\,\vartheta(x)}\right). \tag{1.17}$$

It is to be understood that, in (1.13) and all similar formulæ, the lower limit of summation is 1.

It is part (i) of this theorem that, in a memoir published recently in the *Mathematische Zeitschrift*,* we called the "approximate functional equation". The purpose of the theorem is to facilitate the study of $\zeta(s)$ in the "critical strip" $0 \leqslant \sigma \leqslant 1$. The function is represented for $\sigma > 1$ by the formula

$$\zeta(s) = \Sigma n^{-s},$$

and for $\sigma < 0$ by the formula

$$\zeta(s) = \chi\zeta(1-s) = \chi\Sigma n^{s-1}.$$

The formula (1.13) is, so to say, a compromise between the two, and it seems to have many important applications.

The theorem which we proved in our memoir just quoted was less complete than part (i) of Theorem A in two respects. In the first place it involved error terms

$$O(x^{-\sigma}\log|t|), \quad O(y^{\sigma-1}|t|^{\frac{1}{2}-\sigma}\log|t|),$$

containing a superfluous factor $\log|t|$. Secondly, the result was proved only for a fixed σ lying between 0 and 1.†

Our former theorem was sufficient for the end which we had then in view, and it would be sufficient also for the new applications which we give here. But it seems very desirable that Theorem A should be proved in its most perfect form.‡ The defects are not easily removed if we adhere to the method which we used before, and our argument here proceeds on entirely different lines.

1.2. We give the proof of Theorem A in § 2. In § 3 we apply the theorem to the proof that

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* G. H. Hardy and J. E. Littlewood, "The zeros of Riemann's Zeta-function on the critical line", *Math. Zeitschrift*, Vol. 10 (1921), pp. 283-317.

† It would have been easy to modify our argument so as to secure uniformity for the strip $0 < \delta \leqslant \sigma \leqslant 1-\delta < 1$.

‡ The unnecessary logarithm in the error terms is at any rate a serious æsthetic blemish. The extension to values of σ outside the critical strip is no doubt of secondary importance. But it is important to have a result which holds uniformly for the whole strip, and the attempt to obtain one leads naturally to the more complete form of the theorem.

$$\int_{-T}^{T} |\zeta(\tfrac{1}{2}+it)|^4\,dt = O[T(\log T)^4], \tag{1.21}$$

an equation which marks at any rate a further stage in the direction of "Lindelöf's hypothesis",* viz. that

$$\zeta(\tfrac{1}{2}+it) = O(|t|^{\epsilon}) \tag{1.22}$$

for every positive ϵ.

In § 4 we apply the result of § 3 to the divisor problem of Piltz, and obtain a new result as to the order of the error term in the classical formula.

2. *Proof of Theorem* A.

2.11. A few preliminary explanations are necessary.

We use A (or A', A_1, ...) throughout to denote a positive number $A(H, K)$ depending upon H and K alone. If particular values were assigned to H and K, all the A's would become absolute constants; and, as the number of different A's is finite, it would then be possible to assign absolute constants α and β such that

$$0 < \alpha < A < \beta$$

for every A. The meaning of O corresponds to that of A, and is stated explicitly in the enunciation of the theorem.

We shall sometimes find it convenient to use "generalised integrals", of the type

$$\int_0 f(z)\,z^{-s}\,dz, \tag{2.111}$$

which are not convergent in the ordinary sense. Such integrals have

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[*] Lindelöf's hypothesis would be proved if we could prove that

$$\int_{-T}^{T} |\zeta(\tfrac{1}{2}+it)|^{2p}\,dt = O(T^{1+\epsilon}),$$

for every p. The most known before in this direction was that

$$\int_{-T}^{T} |\zeta(\tfrac{1}{2}+it)|^{2}\,dt \sim 2T\log T.$$

This result was proved in our memoir "Contributions . . . .", *Acta Mathematica*, Vol. 41 (1917), pp. 119–196 (pp. 151–156).

been used for various purposes by other writers, notably Cauchy and Hadamard.*

Suppose that $f(z)$ is regular for $z=0$, that

$$f(z) = a_0+a_1z+a_2z^2+\dots,$$

near $z=0$, and that $\qquad p \leqslant \Re(s) < p+1,$

where p is an integer. If $p \leqslant 0$, the integral (2 . 111) is convergent in the ordinary sense, along any regular path with an extremity at the origin. If $p > 0$, we write

$$f_p(z) = a_0+a_1z+\dots+a_{p-1}z^{p-1},$$

$$\phi_p(\epsilon) = \int_\epsilon f_p(z)z^{-s}\,dz = \sum_{k=0}^{p-1} a_k \frac{\epsilon^{k+1-s}}{s-k-1},$$

it being understood that, when $s=p$, the last term in $\phi_p(\epsilon)$ is to be replaced by

$$-a_{p-1}\log\epsilon,$$

and that ϵ^{-s} and $\log\epsilon$ have their principal values. This being so, the "generalised" integral (2 . 111) is defined by

$$\int_0 f(z)z^{-s}\,dz = \lim_{\epsilon\to 0}\left(\int_\epsilon f(z)z^{-s}\,dz - \phi_p(\epsilon)\right), \tag{2.112}$$

ϵ tending to zero along C.

These generalised integrals possess (as regards addition, transformation, partial integration, and so forth) many of the properties of ordinary integrals. It is hardly necessary to set out these properties in a succession of formal theorems, for the integrals occur only in rather trivial parts of our argument, and the reader will easily verify any property which we assume. But their use enables us to extend the domain of validity of a number of standard formulæ, and so to avoid various irrelevant and tiresome complications. In particular it will easily be verified that (as was shown by Cauchy) Euler's formula

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}\,dx \tag{2.113}$$

is valid for all values of s except

$$s = 0,\ -1,\ -2,\ \dots,$$

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* A. L. Cauchy, *Exercices de mathématiques*, Vol. 1, pp. 57–65, Vol. 2, pp. 92, 93; J. Hadamard, "Théorie des équations aux dérivées partielles et du problème de Cauchy", *Acta Mathematica*, Vol. 31 (1908), pp. 333–380 (p. 339).

and that a similar generalisation may be made of the standard formulæ involving e^{-ax}, $\cos ax$, $\sin ax$, ..., commonly deduced from (2.113).

2.2. Our main aim is to establish part (ii) of Theorem A; the rest of the theorem follows readily from this. We observe first that, on grounds of conjugacy, it is enough to consider the case when t is positive (so that $t > A$). We show next that, in proving (ii), we may suppose that

$$x \equiv \tfrac{1}{2} \pmod{1}.$$

For, assuming the result for this case, let x be unrestricted,

$$x_1 = [x] + \tfrac{3}{2},$$

and y_1 and R_1 the y and R corresponding to x_1. Then

$$x_1 > x \geqslant y > y_1,$$

and
$$0 \leqslant y - y_1 < A_1 t x^{-2}.$$

Hence

(2.21)
$$\vartheta(y) \leqslant \vartheta(y_1) + A_1 t x^{-2}.$$

There are two cases.

(*a*) *No integer is contained between* y *and* y_1 *(inclusive).*

Here we have

(2.22)
$$R = R_1 + O(x_1^{-\sigma}) = O(x_1^{-\sigma}) + O\left(\frac{x_1^{1-\sigma}}{\sqrt{t} + x_1 \vartheta(y_1)}\right)$$
$$= O(x^{-\sigma}) + O\left(\frac{x^{1-\sigma}}{\sqrt{t} + x\,\vartheta(y_1)}\right).$$

Hence, provided $tx^{-2} < 1/4A_1^2 = A_2$,

we have
$$\frac{1}{\sqrt{t} + x\vartheta(y_1)} \leqslant \frac{1}{\sqrt{t} - A_1 t x^{-1} + x\vartheta(y)} < \frac{1}{\frac{1}{2}\sqrt{t} + x\vartheta(y)},$$
$$R = O(x^{-\sigma}) + O\left(\frac{x^{1-\sigma}}{\sqrt{t} + x\vartheta(y)}\right).$$

If, on the other hand, $tx^{-2} \geqslant A_2$,

we have
$$A\sqrt{t} < x < A\sqrt{t}, \quad x\vartheta(y) < A\sqrt{t},$$

and so, from (2.22),

$$R = O(x^{-\sigma}) + O\left(\frac{x^{1-\sigma}}{\sqrt{t}}\right) = O(x^{-\sigma}) + O\left(\frac{x^{1-\sigma}}{\sqrt{t} + x\vartheta(y)}\right),$$

the result desired.

(*b*) *There are integers between y and y_1.*

Since $$y - y_1 < A_1 t x^{-2}, \quad x > A\sqrt{t},$$

there are fewer than A_3 such integers. Further,

$$\vartheta(y) \leqslant y - y_1 \leqslant A_1 t x^{-2}.$$

Thus

$$\frac{1}{\sqrt{t} + x\vartheta(y)} \geqslant \frac{1}{\sqrt{t} + A_1 t x^{-1}} \geqslant \frac{A_4}{\sqrt{t}}. \tag{2.23}$$

We have therefore

$$|R| \leqslant |R_1| + A x^{-\sigma} + A_3 |\chi| A_5 y_1^{\sigma-1}$$

$$< \frac{A x^{1-\sigma}}{\sqrt{t} + x\vartheta(y_1)} + A x^{-\sigma} + A t^{\frac{1}{2}-\sigma} t^{\sigma-1} x_1^{1-\sigma},$$

since $$\chi = O(t^{\frac{1}{2}-\sigma});$$

and this is less than

$$\frac{A x^{1-\sigma}}{\sqrt{t}} + A x^{-\sigma} = O\left(\frac{x^{1-\sigma}}{\sqrt{t}}\right) = O\left(\frac{x^{1-\sigma}}{\sqrt{t} + x\vartheta(y)}\right),$$

by (2.23), which completes the proof.

2.3. We suppose then in what follows that

$$2\pi x y = t, \quad x \geqslant y \geqslant K, \quad x \equiv \tfrac{1}{2} \pmod{1}; \tag{2.31}$$

and we now write

$$y_1 = [y] + 1. \tag{2.32}$$

Suppose in the first instance that $\sigma > 1$, and denote by L_1 and L_2 the lines from x to infinity which make angles $\frac{1}{4}\pi$ and $-\frac{1}{4}\pi$ with the positive real axis. Then

$$(2.33) \qquad \zeta(s) - \sum_{n<x} n^{-s}$$

$$= \sum_{n>x} n^{-s} = \left(\frac{1}{2i}\int_{L_2} - \frac{1}{2i}\int_{L_1}\right) z^{-s} \cot \pi z \, dz$$

$$= \frac{1}{2i}\int_{L_2} (\cot \pi z - i) z^{-s} dz - \frac{1}{2i}\int_{L_1} (\cot \pi z + i) z^{-s} dz - \frac{x^{1-s}}{1-s}$$

$$= I_2 + I_1 - \frac{x^{1-s}}{1-s},$$

say. These integrals are convergent for all values of s, and we have plainly

$$(2.34) \qquad \zeta(s) - \sum_{n<x} n^{-s} = I_1 + I_2 + O\left(\frac{x^{1-\sigma}}{t}\right) = I_1 + I_2 + O(x^{-\sigma}),$$

throughout the domain

$$-H \leqslant \sigma \leqslant H, \quad t > A.$$

2.41. In I_1 we write

$$-\cot \pi z - i = 2i \sum_{\nu \leqslant y_1} e^{2\nu z \pi i} + 2i \frac{e^{2(y_1+1) z\pi i}}{1 - e^{2z\pi i}},$$

so that

$$(2.411) \qquad I_1 = \sum_{\nu \leqslant y_1} \int_{L_1} e^{2\nu z \pi i} z^{-s} dz + \int_{L_1} \frac{e^{2(y_1+1) z\pi i}}{1 - e^{2z\pi i}} z^{-s} dz$$

$$= \sum_{\nu \leqslant y_1} j_{\nu,1} + J_1,$$

say. In J_1 we have $\qquad z = x + re^{\frac{1}{4}\pi i} \quad (r \geqslant 0),$

$$z^{-s} = O(|z^{-s}|) = O[\exp\{-\mathfrak{R}(\sigma+it)\log(x+re^{\frac{1}{4}\pi i})\}]$$

$$= O\left[(x^2 + xr\sqrt{2} + r^2)^{-\frac{1}{2}\sigma} \exp\left(t \operatorname{arc\,tan} \frac{r}{x\sqrt{2}+r}\right)\right]$$

$$= O(x^{-\sigma})\left(1 + \frac{r^H}{x^H}\right) \exp\left(\frac{tr}{x\sqrt{2}}\right),$$

since $\qquad \operatorname{arc\,tan} \dfrac{\xi}{1+\xi} \leqslant \xi \quad (\xi \geqslant 0).$

Also $\qquad \dfrac{1}{1-e^{2z\pi i}} = O(1),$

$$e^{2(y_1+1)z\pi i} = O(e^{-\pi(y_1+1)r\sqrt{2}}) = O(e^{-\pi(y+1)r\sqrt{2}}).$$

Hence

$$(2.412)\quad J_1 = O(x^{-\sigma})\int_0^\infty \left(1+\frac{r^H}{x^H}\right)\exp\left(\frac{tr}{x\sqrt{2}}-\pi y r\sqrt{2}-\pi r\sqrt{2}\right)dr$$

$$= O(x^{-\sigma})\int_0^\infty \left(1+\frac{r^H}{x^H}\right)\exp(-\pi r\sqrt{2})\,dr = O(x^{-\sigma}).$$

2.42. In I_2 we write

$$\cot \pi z - i = 2i \sum_{\nu \leqslant y_1} e^{-2\nu z\pi i} + 2i\,\frac{e^{-2(y_1+1)z\pi i}}{1-e^{-2z\pi i}},$$

$$(2.421)\quad I_2 = \sum_{\nu \leqslant y_1}\int_{L_2} e^{-2\nu z\pi i} z^{-s}\,dz + \int_{L_2}\frac{e^{-2(y_1+1)z\pi i}}{1-e^{-2z\pi i}}\,dz$$

$$= \sum_{\nu \leqslant y_1} j_{\nu,2} + J_2,$$

say. The discussion of J_2 is simpler than that of J_1. We have, in fact,

$$z = x + r e^{-\frac{1}{4}\pi i} \quad (r \geqslant 0),$$

$$z^{-s} = O\left[x^{-\sigma}\left(1+\frac{r^H}{x^H}\right)\exp\left\{-t \operatorname{arc\,tan}\frac{r}{x\sqrt{2}+r}\right\}\right] = O(x^{-\sigma})\left(1+\frac{r^H}{x^H}\right),$$

$$\frac{1}{1-e^{-2z\pi i}} = O(1),$$

$$e^{-2(y_1+1)z\pi i} = O(e^{-\pi r\sqrt{2}}),$$

$$(2.422)\quad J_2 = O\left[x^{-\sigma}\int_0^\infty \left(1+\frac{r^H}{x^H}\right)e^{-\pi r\sqrt{2}}\,dr\right] = O(x^{-\sigma}).$$

Collecting our results from (2.34), (2.411), (2.412), (2.421), and (2.422), we have

$$(2.423)\quad \zeta(s) - \sum_{n<x} n^{-s} - \sum_{\nu \leqslant y_1}(j_{\nu,1}+j_{\nu,2}) = O(x^{-\sigma}).$$

2.5. If now C_1 and C_2 denote the contours formed by adding to L_1 and L_2 respectively the stretch $(0, x)$ of the real axis, we have

$$\int_{C_1} e^{2\nu z\pi i} z^{-s}\,dz = (2\nu\pi)^{s-1} e^{\frac{1}{2}\pi i(1-s)}\,\Gamma(1-s),$$

$$\int_{C_2} e^{-2\nu z\pi i} z^{-s}\,dz = (2\nu\pi)^{s-1} e^{-\frac{1}{2}\pi i(1-s)}\,\Gamma(1-s).$$

Hence

$$j_{\nu,1}+j_{\nu,2} = 2(2\nu\pi)^{s-1}\sin\tfrac{1}{2}s\pi\,\Gamma(1-s)-k_{\nu,1}-k_{\nu,2}, \tag{2.51}$$

where

$$k_{\nu,1} = \int_0^x e^{2\nu z\pi i} z^{-s}\,dz, \qquad k_{\nu,2} = \int_0^x e^{-2\nu z\pi i} z^{-s}\,dz. \tag{2.52}$$

These integrals are generalised integrals if $\sigma \geqslant 1$.

Let now

$$y_2 = [y]-1. \tag{2.53}$$

Then we have, from (2.423) and (2.51),

(2.54)

$$\zeta(s)-\sum_{n<x} n^{-s}-\chi\sum_{n<y_2} n^{s-1} = \sum_{\nu=y_2}^{y_1}(j_{\nu,1}+j_{\nu,2})-\sum_{\nu<y_2}(k_{\nu,1}+k_{\nu,2})+O(x^{-\sigma}),$$

where terms for which ν or n are negative or zero are simply to be omitted.

2.61. We now deform the integrals (2.52) into integrals taken along (1) the segment $(0, -ix)$ of the imaginary axis, and (2) the straight line from $(0, -ix)$ to $(x, 0)$. We call these lines L_3 and L_4 respectively. The integrals along L_3 are generalised integrals if $\sigma \geqslant 1$. And we write

$$\sum_{\nu<y_2}(k_{\nu,1}+k_{\nu,2}) = S = J_{3,1}+J_{4,1}+J_{3,2}+J_{4,2}, \tag{2.611}$$

where

$$J_{3,1} = \int_{L_3}\Big(\sum_{\nu<y_2} e^{2\nu z\pi i}\Big) z^{-s}\,dz, \qquad J_{4,1} = \int_{L_4}\Big(\sum_{\nu<y_2} e^{2\nu z\pi i}\Big) z^{-s}\,dz, \tag{2.612}$$

$$J_{3,2} = \int_{L_3}\Big(\sum_{\nu<y_2} e^{-2\nu z\pi i}\Big) z^{-s}\,dz, \qquad J_{4,2} = \int_{L_4}\Big(\sum_{\nu<y_2} e^{-2\nu z\pi i}\Big) z^{-s}\,dz. \tag{2.613}$$

2.62. We consider first the integral $J_{3,1}$. On L_3 we have

$$z = -ir \quad (0 \leqslant r \leqslant x),$$

$$z^{-s} = e^{-(\sigma+it)(\log r-\frac{1}{2}\pi i)} = r^{-s}e^{\frac{1}{2}\pi i\sigma}e^{-\frac{1}{2}\pi t},$$

$$e^{2\nu z\pi i} = e^{2\nu r\pi}.$$

We have therefore

$$ie^{-\frac{1}{2}\pi i\sigma}J_{3,1} = e^{-\frac{1}{2}\pi t}\sum_{\nu<y_2}\int_0^x e^{2\nu r\pi}r^{-s}\,dr = e^{-\frac{1}{2}\pi t}\sum_{\nu<y_2} u_\nu, \tag{2.621}$$

say.

If
$$p \leqslant \sigma < p+1,$$
we have
$$u_\nu = \int_0^x e^{2\nu r\pi} r^{-s}\,dr$$
$$= \lim_{\epsilon \to 0} \left\{ \int_\epsilon^x e^{2\nu r\pi} r^{-s}\,dr - \sum_{k=0}^{p-1} \frac{(2\nu\pi)^k}{k!} \frac{\epsilon^{k+1-s}}{s-k-1} \right\}$$
$$= \lim_{\epsilon \to 0} \left\{ \int_\epsilon^x e^{2\nu r\pi} r^{-s}\,dr - \sum_{k=0}^{p} \frac{(2\nu\pi)^k}{k!} \frac{\epsilon^{k+1-s}}{s-k-1} \right\}^*$$
$$= \lim_{\epsilon \to 0} \left\{ \int_\epsilon^x \left(e^{2\nu r\pi} - \sum_{k=0}^{p} \frac{(2\nu r\pi)^k}{k!} \right) r^{-s}\,dr - \sum_{k=0}^{p} \frac{(2\nu\pi)^k}{k!} \frac{x^{k+1-s}}{s-k-1} \right\}$$
$$= \int_0^x \left(e^{2\nu r\pi} - \sum_{k=0}^{p} \frac{(2\nu r\pi)^k}{k!} \right) r^{-s}\,dr - \sum_{k=0}^{p} \frac{(2\nu\pi)^k}{k!} \frac{x^{k+1-s}}{s-k-1}.$$

The second term here is plainly
$$O(\nu^A x^A) = O(t^A).$$

Also

(2 . 622)
$$e^{2\nu r\pi} - \sum_{k=0}^{p} \frac{(2\nu r\pi)^k}{k!} < \frac{(2\nu r\pi)^{p+1}}{(p+1)!} e^{2\nu r\pi},$$

so that the first term is
$$O\left(\nu^A e^{2xy_2\pi} \int_0^x r^{p+1-\sigma}\,dr \right) = O(t^A e^t),$$

by (2 . 31) and (2 . 53). Hence (2 . 621) gives

(6 . 623)
$$J_{3,1} = O(e^{-\frac{1}{2}\pi t} t^A e^t) = O(e^{-At}).$$

2 . 63. The integral $J_{3,2}$ may be disposed of in the same way; the only difference is that $-2\nu r\pi$ occurs instead of $2\nu r\pi$, and that we use instead of (2 . 622) the inequality
$$\left| e^{-2\nu r\pi} - \sum_{k=0}^{p} \frac{(-2\nu r\pi)^k}{k!} \right| < \frac{(2\nu r\pi)^{p+1}}{(p+1)!}.$$

Thus

(2 . 631)
$$J_{3,2} = O(e^{-At});$$

and so, by (2 . 611), (2 . 623), and (2 . 631),

(2 . 632)
$$\sum_{\nu < y_2} (k_{\nu,1} + k_{\nu,2}) = S = J_{4,1} + J_{4,2} + O(e^{-At}).$$

---

* It is to be understood that the sums with respect to k disappear if $p-1 < 0$ or $p < 0$. The term which we have added plainly tends to zero with ϵ.

2.64. On L_4 we have

$$z = x - re^{\frac{1}{4}\pi i} \quad (0 \leqslant r \leqslant x\sqrt{2}),$$

$$\begin{aligned} z^{-s} = O(|z^{-s}|) &= O\left[\exp\{-\mathfrak{R}(\sigma+it)\log(x-re^{\frac{1}{4}\pi i})\}\right] \\ &= O\left[(x^2 - xr\sqrt{2}+r^2)^{-\frac{1}{2}\sigma}\exp\left\{-t\arctan\frac{r}{x\sqrt{2}-r}\right\}\right] \\ &= O\left(x^{-\sigma}\exp\left\{-\frac{tr}{x\sqrt{2}}\right\}\right) = O\left(x^{-\sigma}\exp\{-\pi yr\sqrt{2}\}\right), \end{aligned}$$

since
$$\arctan\frac{\xi}{1-\xi} \geqslant \xi \quad (0 \leqslant \xi \leqslant 1).$$

Also
$$\begin{aligned} \sum_{\nu<y_2} e^{2\nu z\pi i} &= \frac{e^{2z\pi i}}{1-e^{2z\pi i}}(1-e^{2(y_2-1)z\pi i}) = O(|1-e^{2(y_2-1)z\pi i}|) \\ &= O(1)+O(e^{\pi(y_2-1)r\sqrt{2}}) = O(1)+O(e^{\pi(y-1)r\sqrt{2}}). \end{aligned}$$

Hence

$$J_{4,1} = O(x^{-\sigma})\int_0^{x\sqrt{2}}(e^{-\pi yr\sqrt{2}}+e^{-\pi r\sqrt{2}})\,dr = O(x^{-\sigma}). \tag{2.641}$$

2.65. Finally, $J_{4,2}$ differs from $J_{4,1}$ in that $e^{2z\pi i}$ is everywhere replaced by $e^{-2z\pi i}$. We have

$$\frac{1}{1-e^{-2z\pi i}} = O(1),$$

$$e^{-2z\pi i} - e^{-2y_2 z\pi i} = O(1),$$

and so

$$\begin{aligned} J_{4,2} &= O\left(x^{-\sigma}\int_0^{x\sqrt{2}}\exp\left\{-t\arctan\frac{r}{x\sqrt{2}-r}\right\}dr\right) \\ &= O\left(x^{1-\sigma}\int_0^1\exp\left\{-t\arctan\frac{\xi}{1-\xi}\right\}d\xi\right) \\ &= O\left(x^{1-\sigma}\int_0^1 e^{-At\xi}\,d\xi\right) = O\left(\frac{x^{1-\sigma}}{t}\right) = O(x^{-\sigma}). \end{aligned} \tag{2.651}$$

2.71. From (2.54), (2.611), (2.632), (2.641), and (2.651), we deduce

$$\zeta(s) - \sum_{n<x} n^{-s} - \chi\sum_{n<y_2} n^{s-1} = \sum_{\nu=y_2}^{y_1}(j_{\nu,1}+j_{\nu,2}) + O(x^{-\sigma}), \tag{2.711}$$

and it remains to discuss the first term on the right-hand side. There are at most $y_1-y_2+1 \leqslant 3$ values of ν involved, and these satisfy $|\nu-y| \leqslant 3$. Consider first the case

$$\nu = y-a \quad (a > 0).$$

We have, by (2.51),

$$j_{\nu,1}+j_{\nu,2} = \chi\nu^{s-1}-(k_{\nu,1}+k_{\nu,2}), \tag{2.712}$$

and we apply the argument of § 2.61, replacing the integral from 0 to x of $k_{\nu,1}$ and $k_{\nu,2}$ by a sum of integrals along L_3 and L_4.

The integrals along L_3 contribute $O(e^{-At})$ as before. Hence

$$k_{\nu,1}+k_{\nu,2} = \int_{L_4} (e^{2\nu z\pi i}+e^{-2\nu z\pi i})\, z^{-s}\,dz+O(e^{-At}). \tag{2.713}$$

Writing $z = x-re^{\frac{1}{4}\pi i}$, where $0 \leqslant r \leqslant x\sqrt{2}$, we have, as before,

$$z^{-s} = O(x^{-\sigma}) \exp\left(-t \operatorname{arc\,tan} \frac{r}{x\sqrt{2}-r}\right). \tag{2.714}$$

Also $\quad e^{2\nu z\pi i}+e^{-2\nu z\pi i} = O(|e^{2\nu z\pi i}|) = O(e^{\pi\nu r\sqrt{2}})$

$$= O(e^{\pi y r\sqrt{2}-\pi a r\sqrt{2}}) = O\left(\exp\left\{\frac{tr}{x\sqrt{2}}-\pi a r\sqrt{2}\right\}\right),$$

(2.715) $k_{\nu,1}+k_{\nu,2}$

$$= O(x^{-\sigma})\int_0^{x\sqrt{2}} \exp\left(-t \operatorname{arc\,tan} \frac{r}{x\sqrt{2}-r}+\frac{tr}{x\sqrt{2}}-\pi a r\sqrt{2}\right) dr+O(e^{-At}).$$

Writing $$\xi = \frac{r}{x\sqrt{2}},$$

and observing that $\quad -\operatorname{arc\,tan}\dfrac{\xi}{1-\xi}+\xi \leqslant -A\xi^2 \quad (0 \leqslant \xi \leqslant 1),$

we obtain $\quad k_{\nu,1}+k_{\nu,2} = O(x^{-\sigma})\displaystyle\int_0^{x\sqrt{2}} \exp\left(-A\frac{r^2t}{x^2}-aAr\right) dr.$

The integral is less than either of

$$\int_0^\infty \exp\left(-A\frac{r^2t}{x^2}\right) dr, \quad \int_0^\infty \exp(-aAr)\,dr,$$

and is therefore of both the forms $O(xt^{-\frac{1}{2}})$ and $O(a^{-1})$. Hence

$$k_{\nu,1}+k_{\nu,2} = O\left(\frac{x^{-\sigma}}{x^{-1}\sqrt{t}+a}\right) = O\left(\frac{x^{1-\sigma}}{\sqrt{t}+ax}\right). \tag{2.716}$$

2.72. We consider now the case

$$\nu = y + \beta \quad (\beta \geqslant 0).$$

Here we have

$$j_{\nu,1} + j_{\nu,2} = \int_{L_1} e^{2\nu z\pi i} z^{-s}\,dz + \int_{L_2} e^{-2\nu z\pi i} z^{-s}\,dz$$

$$= O\left(\left|\int_{L_1} e^{2\nu z\pi i} z^{-\sigma-it}\,dz\right|\right) + O\left(\left|\int_{L_1} e^{2\nu z\pi i} z^{-\sigma+it}\,dz\right|\right)^{*}.$$

Now on L_1 we have

$$e^{2\nu z\pi i} z^{-\sigma\pm it} = O(x^{-\sigma})\left(1+\frac{r^H}{x^H}\right)\exp\left(-\pi\nu r\sqrt{2} + t\arctan\frac{r}{x\sqrt{2}+r}\right)$$

$$= O(x^{-\sigma})\left(1+\frac{r^H}{x^H}\right)\exp\left(-\frac{tr}{x\sqrt{2}} + t\arctan\frac{r}{x\sqrt{2}+r} - \pi\beta r\sqrt{2}\right).$$

Writing
$$\xi = \frac{r}{x\sqrt{2}},$$

and observing that
$$-\xi + \arctan\frac{\xi}{1+\xi} \leqslant -\frac{A\xi^2}{1+\xi} \quad (\xi \geqslant 0),$$

we obtain
$$j_{\nu,1} + j_{\nu,2} = O(x^{-\sigma})\,U;$$

where U is of the form

$$O\left[\int_0^\infty \left(1+\frac{r^H}{x^H}\right)\exp\left\{-At\frac{r^2}{2x^2+rx\sqrt{2}} - A\beta r\right\}dr\right]$$

$$= O\left(\int_0^\infty \exp\left\{-A(t-A)\frac{r^2}{2x^2+rx\sqrt{2}} - A\beta r\right\}dr\right).$$

We may replace $t-A$ here by t, when U is seen to be of both the forms

$$O(x)\int_0^\infty \exp\left(-At\frac{\rho^2}{2+\rho\sqrt{2}}\right)d\rho = O\left(\frac{x}{\sqrt{t}}\right),$$

and
$$O\left[\int_0^\infty \exp(-A\beta r)\,dr\right] = O\left(\frac{1}{\beta}\right),$$

unless $\beta = 0$, when of course it is of the first form only. Hence

(2.721) $$j_{\nu,1} + j_{\nu,2} = O\left(\frac{x^{-\sigma}}{x^{-1}\sqrt{t}+\beta}\right) = O\left(\frac{x^{1-\sigma}}{\sqrt{t}+\beta x}\right).$$

---

* Replacing every element of the integral along L_2 by its conjugate.

2.73. The least of the numbers α and β is $\vartheta(y)$. Thus we obtain, from (2.711), (2.712), (2.716), and (2.721),

$$\zeta(s) = \sum_{n<x} n^{-s} + \chi \sum_{n<y} n^{s-1} + R,$$

where
$$R = O(x^{-\sigma}) + O\left(\frac{x^{1-\sigma}}{\sqrt{t} + x\vartheta(y)}\right).$$

If $x \geqslant y$, then $x > A\sqrt{t}$, and so

$$x^{-\sigma} = O\left(\frac{x^{1-\sigma}}{\sqrt{t} + x\vartheta(y)}\right),$$

$$R = O\left(\frac{x^{1-\sigma}}{\sqrt{t} + x\vartheta(y)}\right), \tag{2.731}$$

This is (1.16), and we have thus established part (ii) of Theorem A.

2.74. If we simply omit the term $x\vartheta(y)$ in the denominator of (2.731), we obtain the result of part (i), when $x \geqslant y$. In order to remove the restriction, we take this result, with $x \geqslant y$, and change s into $1-s$.* We thus obtain

$$\zeta(1-s) = \sum_{n<x} n^{s-1} + \chi_1 \sum_{n<y} n^{-s} + O(x^{\sigma-1}) + O(y^{-\sigma}|t|^{\sigma-\frac{1}{2}}), \tag{2.741}$$

where
$$\chi_1 = 2(2\pi)^{-s} \cos \tfrac{1}{2}s\pi\, \Gamma(s).$$

But, by the classical functional equation,

$$\zeta(1-s) = 2(2\pi)^{-s} \cos \tfrac{1}{2}s\pi\, \Gamma(s)\, \zeta(s) = \chi_1 \zeta(s),$$

and
$$\chi\chi_1 = 4(2\pi)^{-1} \cos \tfrac{1}{2}s\pi \sin \tfrac{1}{2}s\pi\, \Gamma(s)\, \Gamma(1-s) = 1.$$

Hence, when we divide (2.741) by χ_1, we obtain

$$\zeta(s) = \sum_{n<y} n^{-s} + \chi \sum_{n<x} n^{s-1} + O\left(\frac{x^{\sigma-1}}{|\chi_1|}\right) + O\left(\frac{y^{-\sigma}|t|^{\sigma-\frac{1}{2}}}{|\chi_1|}\right) \tag{2.742}$$

$$= \sum_{n<y} n^{-s} + \chi \sum_{n<x} n^{s-1} + O(y^{-\sigma}) + O(x^{\sigma-1}|t|^{\frac{1}{2}-\sigma}),$$

since
$$|\chi_1| \sim A\,|t|^{\sigma-\frac{1}{2}}.$$

This is simply part (i) of Theorem A, with x and y interchanged. Hence the restriction that $x \geqslant y$ is irrelevant in (i). It is evident that the same argument serves to deduce part (iii) from part (ii). This completes the proof of Theorem A.

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* This involves a change in the constant H of the hypothesis of the theorem, but we may suppose the original H to be increased by 1 without prejudice to the results so far obtained.

2.8. It is worth while to note explicitly the effect of the term $\vartheta(y)$ in (1.16). If x and y are of the same order (more precisely, if

$$A < x/y < A',$$

or if

$$A\sqrt{t} < x < A'\sqrt{t}),$$

the error terms (1.15) and (1.16) in (i) and (ii) are equivalent, and may be written in the form

$$O(t^{-\frac{1}{2}\sigma}).$$

If x is large compared with y, and $\vartheta(y)$ is not too small—in particular, if $\vartheta(y) > A$—then (ii) gives more precise information than (i). In particular, if

$$(2.81) \qquad y < A, \quad A < \vartheta(y) < 1 - A',$$

then

$$\zeta(s) - \sum_{n<x} n^{-s} = \chi \sum_{n<y} n^{s-1} + O(|t|^{-\sigma}).$$

In the still more particular case $y < A < 1$, the term in χ disappears and the inequalities (2.81) are satisfied; and we obtain the theorem :—

If

$$C|t| < x < C'|t|,$$

where

$$C > 1/2\pi,$$

then

$$\zeta(s) = \sum_{n<x} n^{-s} + O(|t|^{-\sigma}),$$

the constant implied in the O depending only on H, C, and C'.

This result appears in Lemma 2 of our memoir referred to in § 1. The proof there given does not reveal the intimate relations of the result with the approximate functional equation.

2.91. The error term (1.16) represents the sum of various parts, of which one is a term $O(x^{-\sigma})$; and on this there seems little prospect of improving. The remainder of the error term, however, can be expressed in a more explicit form. The result has a certain intrinsic interest, and may possibly find applications. It should in any case be included for the sake of completeness.

Suppose that $t > 0$, that y is not an integer, and that $x \equiv \frac{1}{2} \pmod{1}$; and let us return to the discussion, in § 2.7, of the sum on the right-hand side of (2.711), concentrating our attention on contributions to the error term (as opposed to contributions to the term in χ), and to such contributions as are not certainly of the form $O(x^{-\sigma})$. It will be found that such contributions arise only from $-k_{\nu_1,1}$ and $j_{\nu_2,1}$, where

$$\nu_1 = y-\eta, \quad \nu_2 = y+1-\eta, \quad \eta = y-[y].$$

Further it appears that

$$k_{\nu_1,1} = \int_{L_4} e^{2\nu_1 z\pi i} z^{-s} dz + O(x^{-\sigma}).$$

Hence

$$(2.911) \quad \zeta(s) - \sum_{n<x} n^{-s} - \chi \sum_{n<y} n^{s-1} = O(x^{-\sigma}) - \int_{L_4} e^{2\nu_1 z\pi i} z^{-s} dz + \int_{L_1} e^{2\nu_2 z\pi i} z^{-s} dz,$$

if $x \geqslant y$. We proceed to study the last two integrals in greater detail.

In the first place we shall show that the restriction $x \equiv \frac{1}{2} \pmod 1$ on (2.911) may be replaced by a restriction of another kind. We have

$$\frac{t}{2\pi}\left(\frac{1}{[x]-\frac{3}{2}} - \frac{1}{[x]+\frac{3}{2}}\right) < A\frac{t}{x^2} < \tfrac{1}{2},$$

provided only that $x > A_1 y$.

Let $A_2 = \text{Max}(2, A_1)$,

and suppose now that $x > A_2 y$,

but that x is no longer subject to any congruence condition. Let

$$x' = [x]-\tfrac{3}{2}, \quad x'' = [x]+\tfrac{3}{2};$$

and let y', η', ν_1', ν_2', L_1', L_4' correspond to x', and similarly for x''. We have clearly

$$y' > y > y'';$$

and there is not more than one integer between y' and y''. Hence $[y]$ is equal to $[y']$ or to $[y'']$: let us suppose that

$$[y] = [y'].$$

Then $\nu_1' = \nu_1, \quad \nu_2' = \nu_2,$

and we have, as a particular case of (2.911),

$$\zeta(s) - \sum_{n<x} n^{-s} - \chi \sum_{n<y} n^{s-1} = O(x'^{-\sigma}) - \int_{L_4'} e^{2\nu_1 z\pi i} z^{-s} dz + \int_{L_1'} e^{2\nu_2 z\pi i} z^{-s} dz.$$

Now $x'^{-\sigma} = O(x^{-\sigma})$,

and $$\int_{L_1'} - \int_{L_1} = \int_{x'}^{x} e^{2\nu_2 z\pi i} z^{-s}\, dz = O\left(\int_{x'}^{x} |z^{-s} dz|\right) = O(x^{-\sigma});$$

and similarly for $\int_{L_4}$. Thus (2.911) is true under the new conditions.

2.92. On L_1, $$z = x + re^{\frac{1}{4}\pi i},$$

and $$z^{-s} e^{2\nu_2 z\pi i} = O(|z^{-s} e^{2yz\pi i}|)$$

$$= O(|z|^{-\sigma}) \exp\left(t \operatorname{arc\,tan} \frac{r}{x\sqrt{2}+r} - \pi y r\sqrt{2}\right)$$

$$= O(|z|^{-\sigma}) \exp\left\{t\left(\operatorname{arc\,tan} \frac{r}{x\sqrt{2}+r} - \frac{r}{x\sqrt{2}}\right)\right\}.$$

Let $$\rho = xt^{-\frac{3}{8}}.$$

Then it is easily verified that

$$\operatorname{arc\,tan} \frac{r}{x\sqrt{2}+r} - \frac{r}{x\sqrt{2}} \leqslant -\frac{Ar^2}{2x^2 + rx\sqrt{2}} \leqslant -A\frac{\rho^2}{x^2} = -At^{-\frac{3}{4}},$$

for $r \geqslant \rho$; and so that

(2.921)

$$\int_{L_1} e^{2\nu_2 z\pi i} z^{-s}\, dz = e^{\frac{1}{4}\pi i} \int_0^{\rho} (x + re^{\frac{1}{4}\pi i})^{-s} \exp\{2\nu_2 (x + re^{\frac{1}{4}\pi i})\pi i\}\, dr + O(e^{-At^{\frac{1}{4}}}).$$

A similar argument shows that

(2.922)

$$-\int_{L_4} e^{2\nu_1 z\pi i} z^{-s}\, dz = -e^{\frac{1}{4}\pi i} \int_0^{\rho} (x - re^{\frac{1}{4}\pi i})^{-s} \exp\{2\nu_1 (x - re^{\frac{1}{4}\pi i})\pi i\}\, dr + O(e^{-At^{\frac{1}{4}}}).$$

Now

(2.923) $$(x + re^{\frac{1}{4}\pi i})^{-s} \exp\{2\nu_2 (x + re^{\frac{1}{4}\pi i})\pi i\} = x^{-s} e^{2\nu_2 x\pi i + \Phi},$$

where

(2.924) $$\Phi = -(\sigma + it) \log\left(1 + \frac{r}{x} e^{\frac{1}{4}\pi i}\right) - 2\nu_2 \pi r e^{-\frac{1}{4}\pi i}.$$

If $0 \leqslant r \leqslant \rho$, we have*

(2 . 9241)
$$\Phi = -\frac{\sigma r}{x} e^{\frac{1}{4}\pi i} - it\left(\frac{r}{x} e^{\frac{1}{4}\pi i} - \frac{ir^2}{2x^2} + \frac{r^3}{3x^3} e^{\frac{3}{4}\pi i}\right)$$
$$-2y\pi r e^{-\frac{1}{4}\pi i} - 2(1-\eta)\pi r e^{-\frac{1}{4}\pi i} + O(t^{-\frac{1}{2}}).$$

The terms involving tr/x and yr cancel; and so

(2 . 925)
$$e^{\Phi} = e^{\Phi_0}\left\{1 + O\left(\frac{r}{x}\right) + O\left(\frac{tr^3}{x^3}\right) + O(t^{-\frac{1}{2}})\right\}$$
$$= e^{\Phi_0}\left\{1 + O\left(\frac{r}{x}\right) + \frac{r^3}{\rho^3} O(t^{-\frac{1}{8}}) + O(t^{-\frac{1}{2}})\right\},$$

where

(2 . 9251)
$$\Phi_0 = -2(1-\eta)\pi r e^{-\frac{1}{4}\pi i} - \frac{tr^2}{2x^2}.$$

Now

(2 . 926)
$$\int_0^{\rho} e^{\Phi} dr = \int_0^{\rho} e^{\Phi_0} dr + R = \int_0^{\infty} e^{\Phi_0} dr + O(e^{-At^{\frac{1}{4}}}) + R,$$

where

(2 . 9261
$$R = O(x^{-1}) \int_0^{\rho} e^{-tr^2/2x^2} r\, dr + O\left(t^{-\frac{1}{8}} \rho^{-3} \int_0^{\rho} e^{-tr^2/2x^2} r^3 dr\right) + O\left(t^{-\frac{1}{2}} \int_0^{\rho} e^{-tr^2/2x^2} dr\right)$$
$$= O\{x^{-1}(xt^{-\frac{1}{2}})^2\} + O\{t^{-\frac{1}{8}}(xt^{-\frac{3}{8}})^{-3}(xt^{-\frac{1}{2}})^4\} + O\{t^{-\frac{1}{2}}(xt^{-\frac{1}{2}})\}$$
$$= O\left(\frac{x}{t}\right) = O(1).$$

Collecting our results from (2 . 921) and (2 . 923)–(2 . 9261), we obtain

(2 . 927)
$$\int_{L_1} e^{2\nu_2 z\pi i} z^{-s} dz = O(x^{-\sigma}) + e^{\frac{1}{4}\pi i} x^{-s} e^{2\nu_2 x\pi i} \int_0^{\infty} \exp\left\{-2(1-\eta)\pi r e^{-\frac{1}{4}\pi i} - \frac{tr^2}{2x^2}\right\} dr.$$

Similarly

(2 . 928)
$$-\int_{L_4} e^{2\nu_1 z\pi i} z^{-s} dz = O(x^{-\sigma}) - e^{\frac{1}{4}\pi i} x^{-s} e^{2\nu_1 x\pi i} \int_0^{\infty} \exp\left\{-2\eta\pi r e^{-\frac{1}{4}\pi i} - \frac{tr^2}{2x^2}\right\} dr.$$

---

* Since
$$\frac{r^2}{x^2} = O\left(\frac{\rho^2}{x^2}\right) = O(t^{-\frac{3}{4}}),$$
and
$$\frac{tr^4}{x^4} = O\left(\frac{t\rho^4}{x^4}\right) = O(t^{-\frac{1}{2}}).$$

2.93. Hence

$$\zeta(s) - \sum_{n<x} n^{-s} - \chi \sum_{n<y} n^{s-1} = O(x^{-\sigma}) + P + Q, \tag{2.931}$$

where P, Q are the main terms on the right-hand side of (2.927) and (2.928) respectively. This result is proved on the assumption $x \geqslant A_2 y$. But it is now easy to see that this restriction may be replaced by $x \geqslant y$. For the integrals in P and Q are each of the form

$$O\left(\int_0^\infty e^{-tr^2/2x^2}\,dr\right) = O\left(\frac{x}{\sqrt{t}}\right);$$

so that, when $y \leqslant x \leqslant A_2 y$, and therefore $x < At^{\frac12}$, P and Q are of the form $O(x^{-\sigma})$. But Theorem A (ii) shows that, when these inequalities are satisfied, the left-hand side is also of this form, so that P and Q may be retained or omitted at our pleasure.

If we write $r = xu$ and substitute for ν_1 and ν_2 in terms of y and η, we obtain

$$P = e^{\frac14\pi i} x^{1-s} e^{it+2(1-\eta)x\pi i} J\Big((1-\eta)x,\ t\Big), \tag{2.932}$$

$$Q = -e^{\frac14\pi i} x^{1-s} e^{it-2\eta x\pi i} J(\eta x,\ t), \tag{2.933}$$

where

$$J(\lambda,\ t) = \int_0^\infty \exp\left(-2\pi e^{-\frac14\pi i}\lambda u - \tfrac12 t u^2\right) du. \tag{2.934}$$

Finally, we observe that, if $1-\eta \geqslant \frac12$,*

$$P = O\left(x^{1-\sigma}\int_0^\infty \exp(-Axu)\,du\right) = O(x^{-\sigma}),$$

and P may be omitted. Similarly Q may be omitted if $\eta \geqslant \frac12$.

Summing up, we have

THEOREM B.—*If* $-H \leqslant \sigma \leqslant H$, $x > K$, $y > K$,

$$2\pi xy = t, \quad x \geqslant y,$$

y is not an integer, and $y = [y] + \eta$,

then

$$\zeta(s) = \sum_{n<x} n^{-s} + \chi \sum_{n<y} n^{s-1} + e^{\frac14\pi i - 2\eta x\pi i + it} x^{1-s}\left[e^{2x\pi i} J\Big((1-\eta)x,\ t\Big) - J(\eta x,\ t)\right] + O(x^{-\sigma}),$$

---

* Or indeed $1-\eta \geqslant A$.

where J is defined by (2.934). *The first term involving J may be omitted when $\eta \leqslant \frac{1}{2}$, and the second when $\eta \geqslant \frac{1}{2}$.*

There are of course corresponding results when $t < 0$, and again when $x \leqslant y$.

We have supposed y non-integral. If y tends from below to an integral value m, the second member of the right-hand side tends to its value for $y = m$, and we obtain

$$\zeta(s) = \sum_{n<x} n^{-s} + \chi \sum_{n<y} n^{s-1} + e^{\frac{1}{4}\pi i + it} x^{1-s} J(0, t) + O(x^{-\sigma})$$

$$= \sum_{n<x} n^{-s} + \chi \Big(\sum_{n<m} n^{s-1} + \tfrac{1}{2} m^{s-1} \Big) + O(x^{-\sigma}).$$

When y increases beyond m the second member gains a new term, and the third changes its sign. It may easily be verified that these two discontinuities are, as they should be, equal and opposite, to order $O(x^{-\sigma})$.

2.94. We return to Theorem A, and suppose in particular that $0 \leqslant \sigma \leqslant H$. By Stirling's theorem,

$$\chi = \left(\frac{|t|}{2\pi e}\right)^{-it} \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\sigma} e^{\frac{1}{4}\pi i \operatorname{sgn} t} \left\{ 1 + O\left(\frac{1}{|t|}\right) \right\},$$

and $$O\Big(t^{-\frac{1}{2}-\sigma} \sum_{n<y} n^{\sigma-1} \Big) = O(t^{-\frac{1}{2}-\sigma} y^{\sigma} \log y) = O(x^{-\sigma}) \quad (t > 0).$$

THEOREM C.—*We have*

(2.941) $$\zeta(s) = \sum_{n<x} n^{-s} + \chi' \sum_{n<y} n^{s-1} + O(x^{-\sigma}) + O(|t|^{\frac{1}{2}-\sigma} y^{\sigma-1}),$$

where $$\chi' = \left(\frac{|t|}{2\pi e}\right)^{-it} \left(\frac{|t|}{2\pi}\right)^{\frac{1}{2}-\sigma} e^{\frac{1}{4}\pi i \operatorname{sgn} t},$$

uniformly for $$0 \leqslant \sigma \leqslant H.$$

In particular, when $\sigma = \frac{1}{2}$, and

(2.942) $$x = y = \sqrt{\left(\frac{|t|}{2\pi}\right)},$$

we have

(2.943) $$\zeta(\tfrac{1}{2}+it) = \sum_{n<x} n^{-\frac{1}{2}-it} + \left(\frac{|t|}{2\pi e}\right)^{-it} e^{\frac{1}{4}\pi i \operatorname{sgn} t} \sum_{n<x} n^{-\frac{1}{2}+it} + O(|t|^{-\frac{1}{4}}).$$

It is (2.941) that we shall use in the sequel. We may note in passing that we have, as an obvious corollary,

$$\zeta(\tfrac{1}{2}+it) = O(t^{\frac{1}{4}}).$$

3. *The mean value of* $|\zeta(\frac{1}{2}+it)|^4$.

3.1. Our main object in this section is to prove

THEOREM D.—*We have* (i)

$$\frac{1}{2T}\int_{-T}^{T}|\zeta(\sigma+it)|^4\,dt \sim \sum_{1}^{\infty}\frac{[d(n)]^2}{n^{2\sigma}} = \frac{[\zeta(2\sigma)]^4}{\zeta(4\sigma)}, \tag{3.11}$$

if $\frac{1}{2} < \sigma < 1$ *and* $T \to \infty$; (ii)

$$\int_{-T}^{T}|\zeta(\tfrac{1}{2}+it)|^4\,dt = O[T(\log T)^4]; \tag{3.12}$$

and (iii)

$$\int_{2}^{T}|\zeta(\sigma+it)|^4\,dt = O[T(\log T)^4], \tag{3.13}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2$, $T > 2$.

But before proceeding to the proof of this theorem we insert a proof of one already known, viz.:

THEOREM E.—*When* $T \to \infty$,

$$\int_{-T}^{T}|\zeta(\tfrac{1}{2}+it)|^2\,dt \sim 2T\log T. \tag{3.14}$$

This theorem is proved in § 2.4 of our memoir in the *Acta Mathematica*,* but the proof there given is difficult and indirect, and the simpler proof which we give here affords an excellent illustration of the use of the approximate functional equation.

Take, in (2.941), $\sigma = \frac{1}{2}$, $t > 2$, and

$$x = \frac{t}{2\pi\sqrt{(\log t)}}, \qquad y = \sqrt{(\log t)}.$$

Since $\chi' = O(1)$ when $\sigma = \frac{1}{2}$, we obtain

$$\begin{aligned}\zeta(s) &= \sum_{n<x} n^{-s} + \chi'\sum_{n<y} n^{s-1} + O[(\log t)^{-\frac{1}{4}}]\\ &= \sum_{n<x} n^{-s} + O[(\log t)^{\frac{1}{4}}] = Z + O[(\log t)^{\frac{1}{4}}],\end{aligned}$$

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* G. H. Hardy and J. E. Littlewood, "Contributions to the theory of the Riemann Zeta-function and the theory of the distribution of primes", *Acta Mathematica*, Vol. 41 (1917), pp. 119–196 (pp. 151–156).

say. Suppose now that we have proved that

$$\int_0^T |Z|^2\,dt \sim T\log T. \tag{3.15}$$

Then

$$\int_0^T |\zeta|^2\,dt = \int_0^T |Z|^2\,dt + O\left(\int_0^T |\log t|^{\frac{1}{4}} |Z|\,dt\right) + O\left(T(\log T)^{\frac{1}{2}}\right)$$

$$= \int_0^T |Z|^2\,dt + O\left\{(\log T)^{\frac{1}{4}}\, T^{\frac{1}{2}} \left(\int_0^T |Z|^2\,dt\right)^{\frac{1}{2}}\right\} + O\left(T(\log T)^{\frac{1}{2}}\right)$$

$$= \int_0^T |Z|^2\,dt + O\left(T(\log T)^{\frac{3}{4}}\right) \sim T\log T;$$

and a similar argument may be applied to the integral from $-T$ to 0. Thus it is sufficient for our purpose to establish (3.15).

Now $$\int_0^T |Z|^2\,dt = \int_0^T \left(\sum_{m<x} m^{-\frac{1}{2}-it}\right)\left(\sum_{n<x} n^{-\frac{1}{2}+it}\right) dt.$$

Here we invert the order of integration and summation, and this requires a little care, since x is a function of t. The term in (m, n) occurs if

$$x = \frac{t}{2\pi\sqrt{(\log t)}}$$

is larger than the larger of m and n. Hence, if

$$X = \frac{T}{2\pi\sqrt{(\log T)}},$$

and if $T_1 = T_1(m, n)$ is given by

$$\frac{T_1}{2\pi\sqrt{(\log T_1)}} = \operatorname{Max}(m, n),$$

we obtain

$$\int_0^T |Z|^2\,dt = \sum_{m,\,n<X} \int_{T_1}^T m^{-\frac{1}{2}-it}\, n^{-\frac{1}{2}+it}\,dt$$

$$= \sum_{n<X} \frac{T-T_1(n, n)}{n} + \sum_{m,\,n<X}{}' \frac{1}{\sqrt{(mn)}} \int_{T_1}^T \left(\frac{n}{m}\right)^{it} dt$$

$$= T\sum_{n<X} \frac{1}{n} + O\left(\sum_{n<X} \frac{T_1(n, n)}{n}\right) + O\left(\sum_{m,\,n<X}{}' \frac{1}{\sqrt{(mn)}\left|\log \frac{n}{m}\right|}\right),$$

the dash denoting as usual that terms for which $m = n$ are omitted. The first term here is

$$T \log X + O(T) = T \log T + o(T \log T);$$

the second is

$$O\left(\sum_{n<X}\left(\frac{1}{n} n\sqrt{(\log n)}\right)\right) = O\left(X\sqrt{(\log X)}\right) = O(T) = o(T \log T),$$

and the third is*

$$O(X \log X) = O\left(T\sqrt{(\log T)}\right) = o(T \log T).$$

This proves (3 . 15), and so Theorem E.

3 . 2. We require some preliminary lemmas for the proof of Theorem D.

LEMMA α: *If* $$D(n, \sigma) = \sum_{1}^{n} \frac{d(\nu)}{\nu^{\sigma}},$$

then

$$D(n, \sigma) = O(1)\ (\sigma > 1), = O\left((\log n)^2\right)(\sigma = 1), = O(n^{1-\sigma} \log n)\ (\tfrac{1}{2} \leqslant \sigma < 1),$$

if σ *is fixed and* $n \to \infty$; *and*

$$D(n, \sigma) = O\left(n^{\mu}(\log n)^2\right),$$

where $\mu = \text{Max}\,(1-\sigma, 0),$

uniformly for $\tfrac{1}{2} \leqslant \sigma \leqslant 2.$

These equations follow at once by partial summation from the familiar result that

$$d(1)+d(2)+\ldots+d(n) \sim n \log n.$$

LEMMA β: *If* $\sigma > 1$, *then*

$$\sum_{1}^{\infty} \frac{\left(d(n)\right)^2}{n^s} = \frac{\left(\zeta(s)\right)^4}{\zeta(2s)}.$$

This formula is due to Ramanujan.†

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* By Lemma 5 of our memoir in the *Mathematische Zeitschrift.*

† S. Ramanujan, "Some formulæ in the analytic theory of numbers", *Messenger of Mathematics,* Vol. 45 (1916), pp. 81–84. Ramanujan gives no proofs; but proofs of all his results have been worked out by Mr. B. M. Wilson, and will be published shortly in these *Proceedings.*

LEMMA γ: $\sum_1^n \left(d(\nu)\right)^2 = O\left(n(\log n)^3\right).$

In fact Ramanujan showed* that

$$\sum_1^n \left(d(\nu)\right)^2 \sim \frac{1}{\pi^2} n(\log n)^3.$$

LEMMA δ: *If* $\mathfrak{D}(n, \sigma) = \sum_1^n \frac{\left(d(\nu)\right)^2}{\nu^\sigma},$

then $\mathfrak{D}(n, \sigma) = O(1) \quad (\sigma > 1), \qquad = O\left((\log n)^4\right) \quad (\sigma = 1),$

$$= O\left(n^{1-\sigma}(\log n)^3\right) \quad (\tfrac{1}{2} \leqslant \sigma < 1),$$

if σ *is fixed and* $n \rightarrow \infty$; *and*

$$\mathfrak{D}(n, \sigma) = O\left(n^\mu (\log n)^4\right),$$

where $\mu = \text{Max}(1-\sigma, 0),$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2.$

These results follow by partial summation from Lemma γ.

LEMMA ϵ: *If* $K > 0$, *then*

$$\sum_1^n d(\nu)\, d(\nu \pm r) = O\left(n(\log n)^3\right),$$

uniformly for $0 \leqslant r \leqslant Kn.$

It is understood here that $d(n) = 0$ if $n \leqslant 0$. To prove the lemma we have only to observe that

$$\sum_1^n d(\nu)\, d(\nu \pm r) \leqslant \sqrt{\left\{\sum_1^n \left(d(\nu)\right)^2 \sum_1^n \left(d(\nu \pm r)\right)^2\right\}}$$

$$\leqslant \sqrt{\left\{\sum_1^n \left(d(\nu)\right)^2 \sum_1^{(K+1)n} \left(d(\nu)\right)^2\right\}} = O\left\{n(\log n)^3\right\},$$

by Lemma δ.

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* Ramanujan, *l.c.* (where he gives a much more precise result).

LEMMA ζ:
$$\sum_{1}^{n} \frac{d(\nu)\, d(\nu \pm r)}{\nu^{\sigma}} = O\left(n^{1-\sigma} (\log n)^4\right),$$

uniformly for
$$0 \leqslant r \leqslant Kn, \quad 0 \leqslant \sigma \leqslant 1.$$

This follows from Lemma ϵ by partial summation.

3.31. We proceed now to the proof of (3.13), of which (3.12) is an obvious corollary. In (2.941) we suppose $\sigma \geqslant \frac{1}{2}$, $t > 0$, and take

$$x = y = \sqrt{\left(\frac{t}{2\pi}\right)} = \tau.$$

Observing that the two error terms are both of the form

$$O(t^{-\frac{1}{2}\sigma}) = O(t^{-\frac{1}{4}}),$$

we obtain
$$\zeta(\sigma+it) = \sum_{n<\tau} n^{-s} + \chi' \sum_{n<\tau} n^{s-1} + O(t^{-\frac{1}{4}}),$$

$$|\zeta|^4 \leqslant A \left| \sum_{n<\tau} n^{-s} \right|^4 + A \left| \chi' \sum_{n<\tau} n^{s-1} \right|^4 + O\left(\frac{1}{t}\right).$$

The contribution of the last term to our integral is plainly negligible, and it is sufficient to prove that

(3.311)
$$J_1 = \int_2^T \left| \sum_{n<\tau} n^{-s} \right|^4 dt = O\left(T (\log T)^4\right).$$

(3.312)
$$J_2 = \int_2^T \left| \chi' \sum_{n<\tau} n^{s-1} \right|^4 dt = O\left(T (\log T)^4\right).$$

3.32. Write

(3.321)
$$\mathfrak{T} = \sqrt{\left(\frac{T}{2\pi}\right)}, \quad T_1 = 2\pi \operatorname{Max}(l^2, m^2, n^2, p^2).$$

Then

(3.322)
$$J_1 = \int_2^T \sum_{l,\, m,\, n,\, p < \tau} (lmnp)^{-\sigma} \left(\frac{np}{lm}\right)^{it} dt$$

$$= \sum_{l,\, m,\, n,\, p < \mathfrak{T}} (lmnp)^{-\sigma} \int_{T_1}^{T} \left(\frac{np}{lm}\right)^{it} dt$$

$$= O\left(T \sum_{lm=np} (lmnp)^{-\sigma}\right) + O\left(\sum_{lm \neq np} \frac{(lmnp)^{-\sigma}}{|\log (np/lm)|}\right)$$

$$= O(T\Sigma_1) + O(\Sigma_2),$$

say. It is understood that in these sums l, m, n, p run from 1 to $\mathfrak{T}$.

Now the number of positive integral solutions of the equations

$$lm = np = \nu$$

can plainly not exceed $\bigl(d(\nu)\bigr)^2$. Hence

$$\Sigma_1 \leqslant \sum_{\nu<\mathfrak{T}^2} \frac{\bigl(d(\nu)\bigr)^2}{\nu^{2\sigma}} < \sum_{\nu<AT} \frac{\bigl(d(\nu)\bigr)^2}{\nu} = O(\log T)^4, \tag{3.323}$$

by Lemma δ.

Similarly

$$\Sigma_2 \leqslant \sum_{\mu,\,\nu<\mathfrak{T}^2}{}' \frac{d(\mu)\,d(\nu)}{(\mu\nu)^\sigma\,|\log(\mu/\nu)|} = \Sigma_{21}+\Sigma_{22}, \tag{3.324}$$

say, where Σ_{21} includes the terms for which

$$\left|\frac{\mu}{\nu}-1\right| \leqslant \tfrac{1}{2},$$

and Σ_{22} the remainder. In Σ_{22}, $|\log(\mu/\nu)| > A$, and so

$$\Sigma_{22} = O\left(\sum_{\mu,\,\nu<\mathfrak{T}^2} \frac{d(\mu)\,d(\nu)}{(\mu\nu)^\sigma}\right) = O\left(\sum_{\mu<AT} \frac{d(\mu)}{\mu^\sigma}\right)^2 = O\bigl(T(\log T)^4\bigr), \tag{3.325}$$

by Lemma α. On the other hand

$$\begin{aligned} \Sigma_{21} &\leqslant \sum_{\mu<\mathfrak{T}^2}\ \sum_{0<|r|<\frac{1}{2}\mu} \frac{d(\mu)\,d(\mu+r)}{\{\mu(\mu+r)\}^\sigma\left|\log\left(1+\dfrac{r}{\mu}\right)\right|} \\ &= O\left(\sum_{\mu<AT}\ \sum_{0<|r|<\frac{1}{2}\mu} \frac{d(\mu)\,d(\mu+r)}{\mu^{2\sigma}}\,\frac{\mu}{|r|}\right) \\ &= O\left(\sum_{0<|r|<AT} \frac{1}{|r|} \sum_{\mu<AT} d(\mu)\,d(\mu+r)\right) \\ &= O\bigl(\log T\,.\,T(\log T)^3\bigr) = O\bigl(T(\log T)^4\bigr), \end{aligned} \tag{3.326}$$

by Lemma ϵ. From (3.322)–(3.326) we deduce (3.311).

3.33. It is plain that, in discussing J_2, we may write $O(t^{\frac{1}{2}-\sigma})$ for χ'; we have, in fact,

$$J_2 = O\left(\int_2^T t^{2-4\sigma}\,|\Sigma n^{s-1}|^4\,dt\right) \tag{3.331}$$

We consider first

(3 . 332) $$j(T) = \int_2^T |\Sigma n^{s-1}|^4 \, dt;$$

and it is clear, after the preceding discussion, that we have

$$\begin{aligned} j(T) &= O\Bigl(T \sum_{lm=np} (lmnp)^{\sigma-1}\Bigr) + O\Bigl(\sum_{lm\neq np} \frac{(lmnp)^{\sigma-1}}{|\log(np/lm)|}\Bigr) \\ &= O(T\Sigma_1') + O(\Sigma_2'), \end{aligned}$$

say, l, m, n, and p again running from 1 to $\mathfrak{T}$.

Arguing as in § 3 . 32, we find

$$\Sigma_1' = O\Bigl(\sum_{\nu<AT} \nu^{2\sigma-2} \bigl(d(\nu)\bigr)^2\Bigr) = O\bigl(T^{2\sigma-1}(\log T)^4\bigr),$$

$$\Sigma_2' = \Sigma_{21}' + \Sigma_{22}',$$

$$\Sigma_{22}' = O\Bigl(\sum_{\mu<AT} \mu^{\sigma-1} d(\mu)\Bigr)^2 = O\bigl(T^{2\sigma}(\log T)^4\bigr),$$

$$\begin{aligned} \Sigma_{21}' &= O\Bigl(\sum_{\mu<AT} \sum_{0<|r|<\frac{1}{2}\mu} \mu^{2\sigma-2} d(\mu)\, d(\mu+r) \frac{\mu}{|r|}\Bigr) \\ &= O\bigl(T^{2\sigma-1}.T(\log T)^4\bigr) = O\bigl(T^{2\sigma}(\log T)^4\bigr), \end{aligned}$$

and so

(3 . 333) $$j(T) = O\bigl(T^{2\sigma}(\log T)^4\bigr).$$

From (3 . 331), (3 . 332), and (3 . 333), we obtain

(3 . 334)

$$\begin{aligned} J_2 &= O\Bigl(\int_2^T t^{2-4\sigma} \frac{dj(t)}{dt}\, dt\Bigr) = O\bigl(T^{2-2\sigma}(\log T)^4\bigr) + O\Bigl(\int_2^T t^{1-2\sigma}(\log t)^4\, dt\Bigr) \\ &= O\bigl(T^{2-2\sigma}(\log T)^4\bigr) + O\bigl(T^{2-2\sigma}(\log T)^4\bigr) + O(T^{\epsilon}) = O\bigl(T(\log T)^4\bigr), \end{aligned}$$

which is (3 . 312); and this completes the proof of (3 . 12).

3 . 34. It remains to prove (3 . 11). If we suppose now that σ is fixed and $\frac{1}{2} < \sigma < 1$, and write

(3 . 341) $$\zeta(s) = \sum_{n<\tau} n^{-s} + \chi' \sum_{n<\tau} n^{s-1} + O(t^{-\frac{1}{4}}) = \sum_{n<\tau} n^{-s} + Z_2 = Z_1 + Z_2,$$

it follows from §§ 3 . 32 and 3 . 33 that

$$\int_{-T}^{T} |Z_2|^4\, dt = O(T^{1-\delta}), \tag{3 . 342}$$

where $\delta = \delta(\sigma) > 0$. Now

$$\int_{-T}^{T} |\zeta|^4\, dt = \int_{-T}^{T} |Z_1|^4\, dt + O\left(\int_{-T}^{T} |Z_1|^3\, |Z_2|\, dt\right) + O\left(\int_{-T}^{T} |Z_2|^4\, dt\right).$$

The last term here is $O(T^{1-\delta})$, by (3 . 342), and the other error term is*

$$O\left\{\left(\int_{-T}^{T} |Z_1|^4\, dt\right)^{\frac{3}{4}} \left(\int_{-T}^{T} |Z_2|^4\, dt\right)^{\frac{1}{4}}\right\}$$

$$= O\left\{\left(T(\log T)^4\right)^{\frac{3}{4}} (T^{1-\delta})^{\frac{1}{4}}\right\} = O\left(T^{1-\frac{1}{4}\delta} (\log T)^3\right),$$

by (3 .311) and (3 . 342). Thus it is enough to prove that

$$J = \int_{-T}^{T} |Z_1|^4\, dt \sim \frac{\left(\zeta(2\sigma)\right)^4}{\zeta(4\sigma)} T = AT. \tag{3 . 343}$$

3 . 35. We consider the half of J corresponding to the interval $(0, T)$. and apply to it the transformation of § 3 . 32. We obtain

$$\tfrac{1}{2}J = \sum_{lm=np} (T-T_1)(lmnp)^{-\sigma} + O\left(\sum_{lm \neq np} \frac{(lmnp)^{-\sigma}}{|\log(np/lm)|}\right) = \Sigma_1 + \Sigma_2, \tag{3 .351}$$

say. Arguing with Σ_2 as in § 3 . 32, but now taking account of the fact that $\sigma > \frac{1}{2}$, and using Lemma δ, we find that

$$\Sigma_2 = O(T^{1-\eta}), \tag{3 . 352}$$

where $$\eta = \eta(\sigma) > 0.$$

The discussion is thus reduced to that of Σ_1.

The number of solutions of the equations $lm = np = \nu$ is exactly

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* Using the generalised inequality of Schwarz, viz.

$$\left|\int ab\, dx\right| \leqslant \left(\int |a|^p\, dx\right)^{1/p} \left(\int |b|^{p/(p-1)}\, dx\right)^{(p-1)/p},$$

with $p = \frac{4}{3}$.

$\big(d(\nu)\big)^2$, if $\nu < \mathfrak{T}$. Hence

$$(3\,.\,353)\quad T \sum_{lm=np} (lmnp)^{-\sigma} = T \sum_{\nu<\mathfrak{T}} \frac{\big(d(\nu)\big)^2}{\nu^{2\sigma}} + O\left(T \sum_{\mathfrak{T}\leqslant\nu<AT} \frac{\big(d(\nu)\big)^2}{\nu^{2\sigma}}\right)$$

$$\sim T \sum_{1}^{\infty} \frac{\big(d(\nu)\big)^2}{\nu^{2\sigma}} = AT,$$

by Lemma β.

Finally we have

$$\sum_{lm=np} T_1 (lmnp)^{-\sigma} < \sum_{lm=np} 2\pi(l^2+m^2+n^2+p^2)(lmnp)^{-\sigma},$$

and the right-hand side, by considerations of symmetry, is

$$8\pi \sum_{lm=np} l^2 (lmnp)^{-\sigma} \leqslant 8\pi \Sigma l^2 (lm)^{-2\sigma}\, d(lm)$$

$$= O(T^{\epsilon})\, \Sigma\, l^{2-2\sigma} m^{-2\sigma} = O(T^{\epsilon})\, \Sigma\, l^{2-2\sigma}\, \Sigma\, m^{-2\sigma}$$

$$= O(T^{\epsilon})\, \{O(T^{\frac{1}{2}(3-2\sigma)} \log T) + O(\log T)\}\, O(1)$$

$$= O(T^{\frac{3}{2}-\sigma+\epsilon}) + O(T^{\epsilon}).$$

Hence

$$(3\,.\,354)\qquad \sum_{lm=np} T_1 (lmnp)^{-\sigma} = O(T^{1-\eta'}),$$

where $\eta' = \eta'(\sigma) > 0$.

From (3.351)–(3.352) we deduce (3.343), and the proof of Theorem D is thus completed.*

4. *Applications to the divisor problems.*†

4.1. Suppose that $d_k(n)$ is the number of decompositions of n into

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* It is the equations (3.12) and (3.13) which we shall require in the sequel, and we have therefore omitted some of the details of the proof of (3.11).

† For references to the literature of these problems up to 1915, see G. H. Hardy: (1) "On Dirichlet's divisor problem," *Proc. London Math. Soc.*, Ser. 2, Vol. 15 (1915), pp. 1–25; (2) "On the average order of the arithmetical functions $P(n)$ and $\Delta(n)$," *ibid.*, pp. 192–213; (3) "Additional note on two problems in the analytic theory of numbers", *ibid.*, Vol. 18 (1919), pp. 201–204. The following memoirs have appeared still more recently:—

E. Landau: (1) "Über die Gitterpunkte in einem Kreise", *Math. Zeitschrift*, Vol. 5 (1919), pp. 319–320; (2) "Über Dirichlets Teilerproblem", *Göttinger Nachrichten*, 1920, pp. 13–32; (3) "Über die Gitterpunkte in einem Kreise (III)", *ibid.*, pp. 109–134:

E. Landau and J. G. van der Corput: "Über Gitterpunkte in ebenen Bereichen", *Göttinger Nachrichten*, 1920, pp. 135–171:

J. G. van der Corput: (1) "Over roosterpunkten in het platte vlak (De beteekenis van de

k factors, so that $d_2(n) = d(n)$ is the number of divisors of n.* It was proved by Piltz that

$$(4\,.\,11)\qquad D_k(n) = \sum_1^n d_k(\nu) = x\,\{a_{k,1}(\log x)^{k-1} + \ldots + a_{k,k}\} + \Delta_k(x),$$

where the a's are constants and

$$(4\,.\,12)\qquad \Delta_k(x) = O\,\{x^{(k-1)/k}(\log x)^{k-2}\}.$$

The last equation was improved by Landau to

$$(4\,.\,13)\qquad \Delta_k(x) = O\,\{x^{(k-1)/(k+1)}(\log x)^{k-1}\};$$

and this is still the best result that is known, though it is easily proved that

$$(4\,.\,14)\qquad \Delta_k(x) = O(x^{\frac{1}{2}+\epsilon}),$$

for every k and every positive ϵ, if Lindelöf's hypothesis (1 . 22) is true (and in particular if the Riemann hypothesis is true).

Let us denote by α_k the least number such that

$$(4\,.\,15)\qquad \Delta_k(x) = O(x^{\alpha_k+\epsilon}),$$

for every positive ϵ. It follows from (4 . 13) that

$$(4\,.\,16)\qquad \alpha_k \leqslant \frac{k-1}{k+1},$$

and from (4 . 14) that (on Lindelöf's hypothesis)

$$(4\,.\,17)\qquad \alpha_k \leqslant \tfrac{1}{2}.$$

On the other hand it has been shown by Hardy† that

$$(4\,.\,18)\qquad \Delta_k(x) \neq o(x^{(k-1)/2k}),$$

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methoden van Voronoï en Pfeiffer)", *Inaugural dissertation*, Groningen, 1919; (2) "Über Gitterpunkte in der Ebene", *Math. Annalen*, Vol. 81 (1920), pp. 1–20; (3) "Zahlentheoretische Abschätzungen mit der Piltzschen Methode", *Math. Zeitschrift*, Vol. 10 (1921), pp. 105–120; (4) "Zahlentheoretische Abschätzungen", *Math. Annalen*, Vol. 84 (1921), pp. 53–79:

S. Wigert: "Über das Problem der Gitterpunkte in einem Kreise", *Math. Zeitschrift*, Vol. 5 (1919), pp. 310–318.

* Thus $d_4(4) = 10$, since 4 may be expressed in 4 forms of the type 4 . 1 . 1 . 1, and 6 of the type 2 . 2 . 1 . 1.

† *L.c.* (1), p. 14. The most important of the inequalities (4 . 19), viz. $\alpha_2 \geqslant \frac{1}{4}$, was found independently by Hardy and Landau: see Hardy, *l.c.* (1), p. 23, and Landau, "Über die Gitterpunkte in einem Kreise (II)", *Göttinger Nachrichten*, 1915, pp. 161–171.

so that

$$a_k \geqslant \frac{k-1}{2k}. \tag{4.19}$$

4.2. We show here how it is possible, by using Theorem D, in the less exact form

$$\int_{-T}^{T} |\zeta(\tfrac{1}{2}+it)|^4\, dt = O(T^{1+\epsilon}), \tag{4.21}$$

to improve on the inequality (4.16) for every value of k greater than 3. We prove in fact that

$$a_k \leqslant \frac{k-2}{k} \quad (k > 3). \tag{4.22}$$

It is possible to go further, and prove that

$$a_k \leqslant \frac{k-1}{k+2} \quad (k > 3); \tag{4.23}$$

but this requires the use of the equation

$$\zeta(\tfrac{1}{2}+it) = O(t^{\frac{1}{6}+\epsilon}), \tag{4.24}$$

which lies very deep, and of which no proof has yet been published.

The relations of these numbers may be seen from the table :—

$k =$	2,	3,	4,	5,	6,	7,	...,
$\frac{k-1}{2k} =$	$\frac{1}{4}$,	$\frac{1}{3}$,	$\frac{3}{8}$,	$\frac{2}{5}$,	$\frac{5}{12}$,	$\frac{3}{7}$,	...,
$\frac{k-1}{k+2} =$	*[*],	*,	$\frac{1}{2}$,	$\frac{4}{7}$,	$\frac{5}{8}$,	$\frac{2}{3}$,	...,
$\frac{k-2}{k} =$	*,	*,	$\frac{1}{2}$,	$\frac{3}{5}$,	$\frac{2}{3}$,	$\frac{5}{7}$,	...,
$\frac{k-1}{k+1} =$	$\frac{1}{3}$,	$\frac{1}{2}$,	$\frac{3}{5}$,	$\frac{2}{3}$,	$\frac{5}{7}$,	$\frac{3}{4}$,	....

4.3. We shall make use of the following lemmas.

LEMMA η: *We have*

$$\int_{2}^{T} |\zeta(\tfrac{1}{2}+it)|^4 \frac{dt}{t} = O(T^{\epsilon}), \tag{4.31}$$

for every positive ϵ.

---

[*] We omit the values in the cases to which our result does not apply.

This follows from (4 . 21) by partial integration.

Lemma θ: *Suppose that* δ *and* ϵ *are positive and less than unity. Then there is a number* $A = A(\epsilon)$ *which possesses the following property: given an arbitrary* ω', *greater than unity, there is an* ω *between* ω' *and* $2\omega'$ *for which*

$$\int_{\frac{1}{2}}^{1+\delta} |\zeta(\sigma+i\omega)|^4\, d\sigma < A\omega^{\epsilon}. \tag{4.32}$$

This follows at once from the fact that

$$\int_{\omega'}^{2\omega'} dt \int_{\frac{1}{2}}^{1+\delta} |\zeta(\sigma+it)|^4\, d\sigma = \int_{\frac{1}{2}}^{1+\delta} d\sigma \int_{\omega'}^{2\omega'} |\zeta(\sigma+it)|^4\, dt = O(\omega'^{(1+\epsilon)}),$$

by (3 . 13).

Lemma ι: *Suppose that* δ *is positive and fixed, that*

$$a_n = O(n^{\epsilon})$$

for every positive ϵ, *that*

$$f(s) = \sum_{1}^{\infty} \frac{a_n}{n^s} \quad (\sigma > 1),$$

that x *and* ω *are large, and that* x *is of the form* $\sqrt{\{m(m+1)\}}$, *where* m *is an integer. Then*

$$A(x) = \sum_{n<x} a_n = \frac{1}{2\pi i}\int_{1+\delta-i\omega}^{1+\delta+i\omega} f(s)\frac{x^s}{s}\, ds + O\left(\frac{x^{1+\delta}}{\omega}\right). \tag{4.33}$$

This is a corollary of formula (1), p. 826, of Landau's *Handbuch*, and *Hilfssatz* 3, p. 841.

4 . 4. We now prove

Theorem F.—*If* $k \geqslant 4$, *then*

$$\Delta_k(x) = O(x^{\frac{k-2}{k}+\epsilon}), \tag{4.41}$$

for every positive ϵ; *i.e.*

$$\alpha_k \leqslant \frac{k-2}{k} \quad (k \geqslant 4). \tag{4.42}$$

We take, in (4 . 33),

$$a_n = d_k(n), \quad f(s) = \left(\zeta(s)\right)^k.$$

We may plainly suppose x of the form $\sqrt{\{m(m+1)\}}$. Then

$$D_k(x) = \sum_{n<x} d_k(n) = \frac{1}{2\pi i}\int_{1+\delta-i\omega}^{1+\delta+i\omega} \zeta^k \frac{x^s}{s}\,ds + O\left(\frac{x^{1+\delta}}{\omega}\right). \tag{4.43}$$

We write $x=\omega'^{\alpha}$, where α is a positive constant which will be determined later, and we choose $\epsilon = \delta$, and ω, between ω' and $2\omega'$, so as to satisfy (4.32). Thus

$$A\omega^{\alpha} < x < B\omega^{\alpha}.$$

Now
$$\Delta_k(x) = D_k(x) - \Phi_k(x),$$
where
$$\Phi_k(x) = x\,\{a_{k,1}(\log x)^{k-1}+\ldots+a_{k,k}\}$$
is the residue of the integrand at the pole $s=1$. Hence

$$\Delta_k(x) = \frac{1}{2\pi i}\int_C \zeta^k \frac{x^s}{s}\,ds + O(\omega^{\alpha-1+\alpha\delta}), \tag{4.44}$$

where C is a path passing from $1+\delta-i\omega$ to $1+\delta+i\omega$ between 0 and 1. We take C to be formed by the straight lines joining the points $1+\delta-i\omega$, $\frac{1}{2}-i\omega$, $\frac{1}{2}+i\omega$, $1+\delta+i\omega$, and write

$$\frac{1}{2\pi i}\int_C = \frac{1}{2\pi i}\left(\int_{1+\delta-i\omega}^{\frac{1}{2}-i\omega} + \int_{\frac{1}{2}-i\omega}^{\frac{1}{2}+i\omega} + \int_{\frac{1}{2}+i\omega}^{1+\delta+i\omega}\right) = I_1+I_2+I_3, \tag{4.45}$$

say.

4.5. We consider first the integral I_3. We have

$$\begin{aligned} I_3 &= O\left(\int_{\frac{1}{2}}^{1+\delta} |\zeta(\sigma+i\omega)|^k \frac{x^{\sigma}}{|\sigma+i\omega|}\,d\sigma\right) \\ &= O\left(\frac{M}{\omega}\int_{\frac{1}{2}}^{1+\delta} |\zeta(\sigma+i\omega)|^4\,d\sigma\right) \\ &= O(M\omega^{\delta-1}), \end{aligned} \tag{4.51}$$

by (4.32), M being defined by

$$M = \underset{\frac{1}{2}\leqslant\sigma\leqslant 1+\delta}{\mathrm{Max}} \left(x^{\sigma}\,|\zeta(\sigma+i\omega)|^{k-4}\right).$$

Now
$$\zeta(\sigma+i\omega) = O(\omega^{\mu}),$$
where μ is the greater of $\frac{1}{2}-\frac{1}{2}\sigma+\delta$ and δ; and so

$$M = O\left(\mathrm{Max}\,(x^{\sigma}\omega^{(k-4)\mu})\right) = O\left(\mathrm{Max}\,\omega^{\beta}\right), \tag{4.52}$$

where

$$\beta = a\sigma+(k-4)(\tfrac{1}{2}-\tfrac{1}{2}\sigma+\delta) \quad (\tfrac{1}{2} \leqslant \sigma \leqslant 1), \tag{4.531}$$

$$\beta = a\sigma+(k-4)\,\delta \quad (1 \leqslant \sigma \leqslant 1+\delta). \tag{4.532}$$

As β is a linear function of σ for $\sigma \leqslant 1$, and increasing for $\sigma \geqslant 1$, it must assume its maximum for $\sigma = \frac{1}{2}$ or $\sigma = 1+\delta$, when its values are

$$\tfrac{1}{2}a+\tfrac{1}{4}k-1+(k-4)\delta, \quad a(1+\delta)+(k-4)\delta.$$

Substituting these values in (4 . 52) and (4 . 51), and observing that I_1 and I_3 are conjugate, we obtain

$$I_1+I_3 = O(\omega^{\gamma_1})+O(\omega^{\gamma_2}), \tag{4.54}$$

where

$$\gamma_1 = \tfrac{1}{2}a+\tfrac{1}{4}k-2+(k-3)\delta, \quad \gamma_2 = a-1+(k-3+a)\delta. \tag{4.55}$$

4 . 6. Finally, it is plain that

$$\begin{aligned} I_2 &= O(\sqrt{x})+O\left(\sqrt{x}\int_2^\omega |\zeta(\tfrac{1}{2}+it)|^k \frac{dt}{t}\right) \\ &= O(\omega^{\frac{1}{2}a})+O\left(\omega^{\frac{1}{2}a}\mathfrak{M}\int_2^\omega |\zeta(\tfrac{1}{2}+it)|^4 \frac{dt}{t}\right) \\ &= O(\omega^{\frac{1}{2}a})+O(\omega^{\frac{1}{2}a+\delta}\mathfrak{M}), \end{aligned} \tag{4.61}$$

by Lemma η; $\mathfrak{M}$ being defined by

$$\mathfrak{M} = \max_{2 \leqslant t \leqslant \omega} |\zeta(\tfrac{1}{2}+it)|^{k-4}. \tag{4.62}$$

This is $O(\omega^{(k-4)(\frac{1}{4}+\delta)})$; and so

$$I_1 = O(\omega^{\frac{1}{2}a})+O(\omega^{\gamma_3}), \tag{4.63}$$

where

$$\gamma_3 = \tfrac{1}{2}a+\tfrac{1}{4}k-1+(k-3)\,\delta. \tag{4.64}$$

From (4 . 45), (4 . 54), (4 . 55), (4 . 63), and (4 . 64) we deduce

$$\frac{1}{2\pi i}\int_C = O(\omega^{\frac{1}{2}a})+O(\omega^{\gamma_2})+O(\omega^{\gamma_3}), \tag{4.65}$$

since $\gamma_1 < \gamma_3$. And from (4 . 44) and (4 . 65) we obtain

$$\Delta_k(x) = O(\omega^{\gamma}), \tag{4.66}$$

where γ is the greatest of the four numbers

$$(4\,.\,67)\qquad a-1+a\delta,\ \tfrac{1}{2}a,\ a-1+(k-3+a)\delta,\ \tfrac{1}{2}a+\tfrac{1}{4}k-1+(k-3)\delta.$$

4 . 7. The parts of the numbers (4 . 67) which do not involve a factor δ are $a-1$, $\frac{1}{2}a$, $a-1$, and $\frac{1}{2}a+\frac{1}{4}k-1$. It is plain that, if we choose

$$a=\tfrac{1}{2}k,$$

the first, third, and fourth of these are equal to $\frac{1}{2}k-1$, and the second is not greater. Hence

$$\Delta_k(x)=O(\omega^{\frac{1}{2}k-1+\epsilon}),$$

for every positive ϵ; and this is equivalent to (4 . 41), since

$$\omega=O(x^{2/k}).$$

Throughout the preceding argument we assumed, as regards the order of $\zeta(\frac{1}{2}+it)$, nothing more than the well known theorem that

$$\zeta(\tfrac{1}{2}+it)=O(t^{\frac{1}{4}+\epsilon}).$$

If we had assumed instead the much deeper result expressed by (4 . 24), we could have replaced $\frac{1}{2}-\frac{1}{2}\sigma+\delta$, in (4 . 531) by $\frac{1}{3}-\frac{1}{3}\sigma+\delta$, and we should have found

$$\gamma_1=\tfrac{1}{2}a+\tfrac{1}{6}k-\tfrac{5}{3}+(k-3)\delta.$$

Similarly, instead of (4 . 64), we should have found

$$\gamma_3=\tfrac{1}{2}a+\tfrac{1}{6}k-\tfrac{2}{3}+(k-3)\delta.$$

We should have chosen a so that

$$a-1=\tfrac{1}{2}a+\tfrac{1}{6}k-\tfrac{2}{3},$$

which gives $\qquad a=\frac{1}{3}(k+2),\quad a-1=\frac{1}{3}(k-1)$;

and our final result would have been

$$\Delta_k(x)=O(\omega^{\frac{1}{3}(k-1)+\epsilon})=O(x^{(k-1)/(k+2)+\epsilon}),$$

which is equivalent to (4 . 23).

4 . 8. The most that can be asserted about the numbers α_k, if we take (4 . 24) as proved, is that

$$\tfrac{1}{4}\leqslant\alpha_2\leqslant\tfrac{1}{3},\quad \tfrac{1}{3}\leqslant\alpha_3\leqslant\tfrac{1}{2},\quad \tfrac{3}{8}\leqslant\alpha_4\leqslant\tfrac{1}{2},$$

and $\qquad \dfrac{k-1}{2k}\leqslant\alpha_k\leqslant\dfrac{k-1}{k+2}\quad(k>4)$;

and the exact determination of a_k (even on the assumption of the truth of the Riemann hypothesis) no doubt demands analysis deeper than any we have as yet at our command.

A somewhat easier problem is that of the *average order* of $\Delta_k(x)$. We may say that

$$\Delta_k(x) = O_2(x^{\beta_k+\epsilon}),$$

or $\Delta_k(x)$ is *of average order* $x^{\beta_k+\epsilon}$, if β_k is the least number such that

$$\int_1^x \left(\Delta_k(y)\right)^2 dy = O(x^{1+2\beta_k+\epsilon}),$$

for every positive ϵ. It is plain that

$$\beta_k \leqslant a_k.$$

The results of § 3 enable us to prove fairly simply that

$$\beta_2 = \tfrac{1}{4},\text{*}$$

and this is a certain presumption in favour of the hypothesis that $a_2 = \frac{1}{4}$. It can also be proved that, if *Lindelöf's hypothesis is true* (and in particular if the Riemann hypothesis is true) then

$$\beta_k = \frac{k-1}{2k}$$

for every k. The proof of this is reasonably straightforward, but it does not depend upon the ideas of § 1; and, in view of the length to which this paper has extended, we must postpone it. We will only observe that the three hypotheses

$$(1) \quad \beta_k \leqslant \tfrac{1}{2} \textit{ for every } k,$$

$$(2) \quad \beta_k = \frac{k-1}{2k} \textit{ for every } k,$$

and

$$(3) \quad \zeta(\tfrac{1}{2}+it) = O(t^{\epsilon}) \textit{ for every } \epsilon,$$

are exactly equivalent to one another. This is rather curious, for (2) is on the face of it a highly specialised form of (1).

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* This was proved by Hardy in his paper (2) referred to on p. 67, but by much more intricate analysis which depended essentially on the difficult "explicit formula" of Voronoï.

ON A NEW SET OF CONDITIONS FOR A FORMULA FOR AN AREA

By W. H. YOUNG.

[Read December 9th, 1920.]

1. In a previous communication[*] to the Society, I have shown how, by actually summing the areas of triangles which together make up the directed area of a polygon inscribed in a curve, and proceeding to the limit with the sum, the well-known formula for an area,

$$A = \int_a^c \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv,$$

may be proved to hold under extremely general conditions, involving, relatively speaking, little more than the existence of the latter integral. The analytical artifices there employed appear not to render more extended conclusions possible.

Recently, while engaged on the problem of volume, I had occasion to consult Brouwer's[†] discussion of Stokes' theorem in the general form given to it by Poincaré,[‡] applicable for any number of dimensions, when the manifold itself has everywhere continuously varying tangent planes, and the functions integrated over the manifold, as well as their differential coefficients, are continuous. It was in extending the scope of these results that I was led to employ a new method which I perceived would render equally valuable service in dealing with the formula for an area.

Perhaps the most interesting result of my earlier paper is that the formula holds when, beside the simpler requirements of absolute continuity of x and y with respect to each variable u, v, separately, all that is needed is that the derivates $|\partial x/\partial u|$ and $|\partial y/\partial u|$ should each be less than a summable function $\mu(u)$ of u alone, while the analogous condition, $|\partial x/\partial v| < \lambda(v)$, $|\partial y/\partial v| < \lambda(v)$, holds for the other derivates, conditions which, in a very simple and obvious manner, secure the existence of the double integral of the Jacobian.

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[*] W. H. Young, "On the Triangulation Method of Defining the Area of a Surface," *Proc. London Math. Soc.*, Ser. 2, Vol. 19 (1919), pp. 117–152.

[†] L. E. J. Brouwer: (1) "Polydimensional vectors distributions", *Proc. Amsterdam Acad.*, Vol. 9 (1906), p. 66; (2) "Remark on multiple integrals", *ibid.*, Vol. 22 (1920), p. 150.

[‡] M. Poincaré, *Les méthodes nouvelles de la mécanique céleste*, Vol. 3, p. 10.

By employing the processes explained below, I have been enabled to establish the truth of the formula under other, in some respects more, general, and not less symmetrical, conditions, namely when *the pair of functions* $\partial x/\partial u$ *and* $\partial y/\partial v$ *are such that either* (*a*) *one is bounded and the other summable with respect to* (u, v), *or else* (*b*) *they have what I call "associated summabilities" with respect to* (u, v), *while the same is true for* $\partial y/\partial u$ *and* $\partial x/\partial v$. For the definition of this term I must refer my readers to the body of my paper and to an earlier communication to the Royal Society,* where illustrations of the utility of the concept in the Theory of Fourier series will be found.

The associated summabilities need not of course be the same in the two cases or in the whole (u, v)-rectangle. Practically the Formula for an Area is now seen to be true in all the cases in which the double integral of the Jacobian is known to exist, though this does not preclude the possibility of other cases being discovered requiring further treatment.

2. We require the following Lemmas :—

Lemma 1.—*If* $f(x, y)$ *is summable with respect to* (x, y) *in the rectangle* $(a, b\,;\ c, d)$ *the integral*

$$J(f, h, k) \equiv \int_\alpha^\gamma \int_\beta^\delta |f(x+h, y+k) - f(x, y)|\, dx\, dy \qquad (1)$$

has the unique double limit zero, as $(h, k) \rightarrow (0, 0)$, *the convergence being uniform, where* $(\alpha, \beta\,;\ \gamma, \delta)$ *is any rectangle in the interior of* $(a, b\,;\ c, d)$.

Case 1.—When $f(x, y)$ is a continuous function of (x, y), the result is obvious, since, given ϵ, we can find e, so that, for all values of $|h|$ and $|k|$ less than e, such that $(x+h, y+k)$ lies in $(a, b\,;\ c, d)$, the integrand is less than $\epsilon/(c-a)(d-b)$, and consequently the integral is less than ϵ. Thus the convergence is uniform and the limit zero, as stated.

Case 2.—Next let $f(x, y)$ be any positive bounded l (or u) function, and

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* W. H. Young, "On Classes of Summable Functions and their Fourier Series," *Proc. Roy. Soc.*, (A), Vol. 87, p. 226. A very special case is when one of the factors has its $(1+p)$-th power summable and the other its $(1+1/p)$-th. In a still more special case, $p = 1$, the squares are summable.

$f_1(x, y), f_2(x, y), \ldots, f_n(x, y), \ldots$ any monotone ascending (or descending) bounded sequence of continuous functions with $f(x, y)$ as limit. Then, since the modulus of a sum does not exceed the sum of the moduli,

$$\begin{aligned} J(f, h, k) &\leqslant J(f_n, h, k) + J(f - f_n, h, k) \\ &\leqslant J(f_n, h, k) + 2\int_a^c \int_b^d |f(x, y) - f_n(x, y)|\, dx\, dy. \qquad (2) \end{aligned}$$

Now the monotone descending sequence of positive bounded functions

$$|f(x, y) - f_n(x, y)|$$

has zero for limit. Integrating term-by-term,

$$\int_a^c \int_b^d |f(x, y) - f_n(x, y)|\, dx\, dy \to 0 \qquad (n \to \infty).$$

Therefore, given ϵ, we can find N, so that, for $n \geqslant N$, the integral on the left is less than ϵ. Moreover, by the proof in Case 1, we can then choose e, so that, if $|h| < e$, $|k| < e$,

$$J(f_N, h, k) < \epsilon,$$

and therefore, by (2), $\quad J(f, h, k) < 3\epsilon.$

This proves the Lemma in this case.

CASE 3.—Let $f(x, y)$ be any bounded *lu*- or *ul*-function.

Precisely the same proof holds as in Case 2, *l*- and *u*-functions replacing continuous functions.

CASE 4.—Let $f(x, y)$ be any bounded function.

We can then find a bounded *lu*-function which differs from it only at a set of values of (x, y) of plane content zero.

If $\phi(x, y)$ be this *lu*-function, $f(x, y) - \phi(x, y)$ is zero except at a set of content zero. Hence

$$|J(f, h, k) - J(\phi, h, k)| \leqslant J(f - \phi, h, k) \leqslant 2\int_a^c \int_b^d |f(x, y) - \phi(x, y)|\, dx\, dy = 0.$$

Hence, by Case 3, the Lemma is true in this case.

CASE 5.—Let $f(x, y)$ be any unbounded summable function.

Let $f_n(x, y) = f(x, y)$ when $|f(x, y)| \leqslant n$, and $f_n(x, y) = 0$ elsewhere. Then if E denote the set of points where $f(x, y) > n$,

$$J(f - f_n, h, k) \leqslant 2\int_a^c \int_b^d |f - f_n|\, dx\, dy \leqslant \iint_E f(x, y)\, dx\, dy \leqslant \epsilon,$$

provided n is so large that the content of E is less than a certain small quantity e determined by $f(x, y)$ and ϵ.

Hence, by Case 4, the Lemma is true in this final case.

Lemma 2.—*If $g(x, y)$ is a bounded function of (x, y), and $f(x, y)$ is a summable function of (x, y),*

$$\int_\alpha^\gamma \int_\beta^\delta f(x+h,\ y+k)\, g(x, y)\, dx\, dy \to \int_\alpha^\gamma \int_\beta^\delta f(x, y)\, g(x, y)\, dx\, dy,$$

the convergence being uniform, as $(h, k) \to (0, 0)$ in any manner.

In fact the difference of the two sides of this relation is

$$\leqslant \int_a^c \int_b^d |f(x+h,\ y+k) - f(x, y)|\, |g(x, y)|\, dx\, dy \leqslant MJ(f, h, k),$$

where M is the upper bound of $|g(x, y)|$.

Hence, by Lemma 1, the result at once follows.

Lemma 3.—*If $f(x, y)$ and $g(x, y)$ are a pair of functions satisfying one of the following alternative conditions :—*

(1) *One of the two functions is bounded and the other is summable with respect to (x, y) ; or*

(1*a*) *$f(x, y)$ and $g(x, y)$ have associated summabilities, then*

$$\int_\alpha^\gamma \int_\beta^\delta \{f(x+h,\ y+k) - f(x, y)\}\, g(x, y)\, dx\, dy \to 0,$$

the convergence to the double limit being uniform, as $(h, k) \to (0, 0)$.

We have seen that the lemma is true when $g(x, y)$ is bounded and $f(x, y)$ merely summable (Lemma 2). It is therefore true when we replace $g(x, y)$ by $g_m(x, y)$, a function equal to $g(x, y)$ wherever $g(x, y) \leqslant m$, and equal to zero at the points of the set E at which $g(x, y) > m$. But, denoting our integral by I, and the integral with ϕ_m in place of ϕ by I_m,

$$I - I_m = \iint_E \{f(x+h,\ y+k) - f(x, y)\}\, g(x, y)\, dx\, dy,$$

and therefore

$$|I - I_m| \leqslant \left| \iint_E f(x+h,\ y+k)\, g(x, y)\, dx\, dy \right| + \left| \int_E f(x, y)\, g(x, y)\, dx\, dy \right|. \quad (3)$$

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* For the definition, see below in the text.

Now, if $f(x, y)$ is bounded, say numerically $\leqslant M$, each of these two integrals is numerically

$$\leqslant M \iint_E |g(x, y)|\, dx\, dy,$$

which, since $g(x, y)$ is summable, tends uniformly to zero as $E \rightarrow 0$, that is $m \rightarrow \infty$.

This proves the required result in case the condition (1) is verified.

We proceed to examine the cases when (1a) is verified.

Our lemma will evidently be proved, if we can show that, whatever values h and k may have, including zero values, we can find m so large that, for this and all greater values, the first integral on the right in (3) is as small as we please, say less than ϵ.

We may treat $f(x, y)$ and $g(x, y)$ as being positive, since, in the contrary case, we merely replace them by the difference of two such functions.

Now, by *the definition of associated summabilities, there are certain functions*

$$\phi\{f(x, y)\}, \quad \psi\{g(x, y)\}$$

which are summable with respect to (x, y), *and satisfy the conditions*

(1) $\phi(z) = \int_{z_0}^{z} U(t)\, dt, \quad \psi(z) = \int_{z_1}^{z} V(t)\, dt$;

(2) $U(t)$ *is a positive monotone increasing function of* t, *possessing a differential coefficient* $U'(t)$ *which is greater than zero*;

(3) $V(t)$ *is the inverse function of* $U(t)$, *so that, if*

$$Z = U(z),$$

then

$$z = V(Z).$$

By a known theorem,* if we write $f_1(x, y)$ for $f(x+h, y+k)$,

$$\begin{aligned} f_1(x, y)\, g(x, y) - z_0 z_1 &\leqslant \int_{z_0}^{f_1(x, y)} U(z)\, ds + \int_{z_1}^{g(x, y)} V(z)\, dz \\ &\leqslant \phi\{f_1(x, y)\} + \psi\{g(x, y)\} \\ &\leqslant \phi\{f(x+h, y+k)\} + \psi\{g(x, y)\}. \end{aligned}$$

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* "On Classes of Summable Functions and their Fourier Series," *loc. cit.*, p. 226.

Hence

$$\iint_E f(x+h,\, y+k)\, g(x,\, y)\, dx\, dy \leqslant z_0 z_1 E + \iint_F \phi\,\{f(x,\, y)\}\, dx\, dy$$
$$+\iint_E \psi\,\{g(x,\, y)\}\, dx\, dy, \qquad (4)$$

where F denotes the set obtained from E by moving it through a displacement h parallel to the axis of x and k parallel to the axis of y.

Now the content of E, and therefore of F, is as small as we please, if we choose m large enough. Therefore, by the characteristic property of summable functions, we can, given ϵ, find m, independent of h or k, such that the right-hand side of (4) is less than ϵ. Hence also the same is true of the left-hand side, whatever values h and k may have.

This proves the Lemma.

3. We shall first apply these lemmas to prove the following general theorem:—

Theorem 1.—*If* $\qquad x = x(u,\, v), \quad y = y(u,\, v)$

be one-valued functions of $(u,\, v)$, *possessing the following properties:—*

(1) $x(u,\, v)$ *and* $y(u,\, v)$ *are absolutely continuous functions of each of the variables separately;*

(2) *their partial derivates* $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, $\partial y/\partial v$ *have absolutely convergent Lebesgue double integrals; and either*

(3*a*) *in each pair of functions* $(\partial x/\partial u,\, \partial y/\partial v)$ *and* $(\partial x/\partial v,\, \partial y/\partial u)$ *one function is bounded; or else*

(3*b*) *the functions in each pair have associated summabilities;** *then the formula for an area holds,*

$$A = \int_C x\, dy = \int_a^c \int_b^d \frac{\partial(x,\, y)}{\partial(u,\, v)}\, du\, dv.$$

Here the double integral of the Jacobian is taken over the rectangle $(a,\, b;\ c,\, d)$ of the $(u,\, v)$-plane, and the single integral round the contour of the curve image C of the perimeter of this rectangle in the $(x,\, y)$-plane.

Denoting by t a parameter connecting the coordinates of a current

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* The summabilities may be differently associated in different parts of the plane.

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point on C with those of a point on this perimeter, we may write

$$\int_C x\,dy = \int_{t_0}^{t_1} x(t)\,dy(t). \tag{5}$$

We shall take the parameter t so that on the sides parallel to the axes of x

$$x = x(t) = x(u, b), \quad \text{or} \quad x(u, d),$$

and on the other sides

$$x = x(t) = x(a, v), \quad \text{or} \quad x(c, v),$$

and similarly for y.

4. In proving this theorem we assume that the result is known when $x(u, v)$ and $y(u, v)$ satisfy certain simple conditions, and, to begin with, these conditions shall be that *the partial derivates of x and y with respect to u and v are continuous functions of (u, v) and are accordingly differential coefficients.* This case is so simple that the proof need not be repeated here.*

We then pass to the general case when $x(u, v)$ and $y(u, v)$ only satisfy the requirements of the enunciation, by modifying the simple conditions first for one, and subsequently for the other, of these functions. This we are enabled to do by employing an artifice which involves securing the simultaneous approach to one of the functions and to its derivates by a suitably selected function and its continuous differential coefficients. This mode of treatment takes its rise in a certain sense in a general method of operation which I have found efficacious in various similar circumstances. The argument is applicable because we have to deal with an equation, both sides of which can be varied at once. It would fail, for example, if we attempted to use it in the problem of estimating the sum of our triangles taken positively, as we then have no contour integral to fall back upon.

5. Let $$Q(u, v) = \int_{u_0}^{u} dU \int_{v_0}^{v} x(U, V)\,dV, \tag{6}$$

which, when (2) holds for x, is the double integral of $x(U, V)$; and

$$R(u, v) = \frac{\partial Q}{\partial u} = \int_{v_0}^{v} x(u, V)\,dV = \int_{v_0}^{v} dV \left\{ \int_{u_0}^{u} \frac{\partial x(U, V)}{\partial U}\,dU + x(u_0, V) \right\}, \tag{7}$$

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* See § 18 of "On a Formula for an Area," *loc. cit.*

in which we use condition (1) of the enunciation, in so far as $x(U, V)$ is absolutely continuous with respect to V and with respect to U.

Then writing

$$\bar{Q}(u, v) = \{Q(u+h, v+k) - Q(u, v+k) - Q(u+h, v) + Q(u, v)\}/hk$$

$$= \frac{1}{hk}\int_u^{u+h} dU \int_v^{v+k} x(U, V)\, dV, \tag{8}$$

we shall show first that

$$\int_{t_0}^{t} \bar{Q}(t)\, dy(t) \to \int_{t_0}^{t} x(t)\, dy(t),$$

when first $h \to 0$ and then $k \to 0$.

Now since

$$\int_v^{v+k} x(u, V)\, dV = \int_v^{v+k} dV \left\{ \int_{u_0}^{u} \frac{\partial x(U, V)}{\partial U}\, dU + x(u_0 V) \right\},$$

this is a double integral and therefore certainly continuous with respect to u. Therefore, by (8),

$$\underset{h\to 0}{\mathrm{Lt}}\; \bar{Q}(u, v) = \frac{1}{k}\int_v^{v+k} x(u, V)\, dV,$$

and the convergence is uniform.

But $x(u, v)$ is a continuous function of v. Therefore

$$\underset{k\to 0}{\mathrm{Lt}} \left\{ \underset{h\to 0}{\mathrm{Lt}}\; \bar{Q}(u, v) \right\} = x(u, v), \tag{9}$$

and the convergence in the case of each variable is uniform. Hence, since along each side of the (u, v)-rectangle the variable t is equal to one of the coordinates (u, v) while the other is constant, we have on the perimeter of the (u, v) rectangle

$$\underset{k\to 0}{\mathrm{Lt}} \left\{ \underset{h\to 0}{\mathrm{Lt}}\; \bar{Q}(t) \right\} = x(t), \tag{10}$$

the convergence being uniform with respect to each of the variables h and k separately.

By the theory of integration with respect to a function of bounded variation, our required result follows

$$\underset{k\to 0}{\mathrm{Lt}} \left\{ \underset{h\to 0}{\mathrm{Lt}} \int_{t_0}^{t} \bar{Q}(t)\, dt \right\} = \int_{t_0}^{t} x(t)\, dy(t). \tag{11}$$

7. So far we have only used conditions (1) and (2) of the enunciation for x. Now

$$\frac{\partial \bar{Q}}{\partial u} = \{R(u+h,\ v+k)-R(u+h,\ v)-R(u,\ v+k)+R(u,\ v)\}/hk$$

$$= \frac{1}{hk}\int_v^{v+k} \{x(u+h,\ V)-x(u,\ V)\}\,dV$$

$$= \frac{1}{hk}\int_v^{v+k} dV \int_u^{u+h} \frac{\partial x(U,\ V)}{\partial U}\,dU. \tag{12}$$

Under our condition (2) therefore $\partial\bar{Q}/\partial u$ is a double integral, and as such continuous with respect to $(u,\ v)$.

Under similar circumstances $\partial\bar{Q}/\partial v$ will be continuous with respect to $(u,\ v)$.

Therefore, since the theorem we are in search of has been supposed proved under the simple conditions which are satisfied by $y\,(u,\ v)$ and $\bar{Q}\,(u,\ v)$ in place of $x\,(u,\ v)$, we have

$$\int_{t_0}^{t} \bar{Q}(t)\,dy(t) = \int_a^c\int_b^d \frac{\partial(\bar{Q},\ y)}{\partial(u,\ v)}\,du\,dv, \tag{13}$$

and therefore, by (11),

$$\operatorname*{Lt}_{k\to 0}\left\{\operatorname*{Lt}_{h\to 0}\int_a^c\int_b^d \frac{\partial(\bar{Q},y)}{\partial(u,\ v)}\,du\,dv\right\} = \int_{t_0}^{t_1} x(t)dy(t). \tag{14}$$

8. We have now to identify the left-hand side of (14) with the double integral of the Jacobian. We shall take each of the two terms in the Jacobian separately. Thus our required result is to show that

$$I \equiv \int_a^c\int_b^d \left\{\frac{\partial\bar{Q}}{\partial u}-\frac{\partial x}{\partial u}\right\}\frac{\partial y}{\partial v}\,du\,dv \to 0, \tag{15}$$

when first $h\to 0$ and then $k\to 0$. We shall show that this is in fact true however $(h,\ k)\to(0,\ 0)$, so that the limit is a unique double limit, and that the approach to the limit is uniform.

Now, by (12),

$$I = \int_a^c du\int_b^d dv\int_0^1 dt_1\int_0^1 dt_2\,\frac{\partial}{\partial u}\,\{x(u+t_1h,\ v+t_2k)-x(u,\ v)\}\,\frac{\partial y}{\partial v}. \tag{16}$$

Now $\partial y/\partial v$ is continuous and therefore bounded with respect to $(u,\ v)$, and by condition (2) the other factor of the integrand is summable with respect to (u, v), the integrand is summable with respect to the ensemble $(u, v, t_1 t_2)$, and we may therefore change the order of integration and write

$$\left.\begin{aligned} I &= \int_0^1 \int_0^1 K(t_1, t_2)\, dt_1 dt_2, \\ \text{where} \quad K(t_1, t_2) &= \int_a^c \int_b^d \frac{\partial}{\partial u} \{x(u+t_1 h,\ v+t_2 k) - x(u, v)\} \frac{\partial y}{\partial v}\, du\, dv. \end{aligned}\right\} \tag{17}$$

But, by Lemma 3, $K(t_1, t_2)$ tends uniformly to zero as $(h, k) \to (0, 0)$ Therefore also I tends uniformly to zero, which is the required result (15).

9. Interchanging u and v, which is allowable when (2) holds, since Q is then a double integral and therefore symmetrical, the same is true when we work with the other term of the Jacobian. Therefore, when $(h, k) \to (0, 0)$,

$$\int_a^c \int_b^d \frac{\partial(\bar{Q}, y)}{\partial(u, v)}\, du\, dv \to \int_a^c \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv. \tag{18}$$

By (18) the right-hand side of (13) has a unique limit given by the double integral of the Jacobian. Therefore the left-hand side also has a unique double limit, which is therefore the same as the repeated limit found in (14). Hence

$$\int_{t_0}^{t} x(t)\, dy(t) = \int_a^c \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv, \tag{19}$$

which proves the formula for an area when $x(u, v)$ *obeys the conditions* (1), (2) *of the enunciation, while the derivates of* $y(u, v)$ *are continuous functions of* (u, v).

10. Since $x(t)$ and $y(t)$ are integrals by the condition (1),

$$\int_{t_0}^{t_1} x(t)\, dy(t) = -\int_{t_0}^{t_1} y(t)\, dx(t),$$

the curve image of the rectangle being a closed curve. We may therefore throughout exchange x and y, *which proves the formula for an area when* $y(u, v)$ *obeys the conditions* (1) *and* (2) *of the enunciation, while the derivates of* $x(u, v)$ *are continuous functions of* (u, v).

11. The argument already used then serves to extend this result to the case when the derivates of $x(u, v)$ are no longer continuous, but

merely summable with respect to (u, v), while those of $y(u, v)$ are bounded. Condition (1) is supposed still to hold. The derivates of $Q(u, v)$ are therefore still continuous with respect to (u, v), so that by § 10 the formula (14) for an area holds, the coordinate $x(u, v)$ being here replaced by $Q(u, v)$. Moreover, in § 8 no further assumption was in point of fact necessary in order to show that the change of order of integration in (16) was permissible; it was the boundedness, and not the continuity of $\partial y/\partial v$ which served to clinch the argument. Again Lemma 3 can be applied under our present hypotheses. Thus the whole argument is valid, and *the formula for an area is accordingly established when conditions* (1), (2), *and* (3*a*) *hold.*

12. Hence again we may extend the formula, so that (3*b*) takes the place of (3*a*) in our conditions. For (3*b*) ensures equally the summability of the integrand in (16) and justifies our use of Lemma 3.

This proves the theorem.

13. In extending the preceding result, we require an n-dimensional theorem which has, as far as I know, only been enunciated for one dimension. Anyone who has read Part III of my paper "On a Formula for an Area" should have no difficulty in adapting the proof given in "Successions of Integrals and Fourier Series."* The theorem in its primitive form was given by Vitali:† in the extended form here required it is as follows:—

THEOREM (Extended Vitali's Theorem).—*Let* $x = (x_1, x_2, \ldots, x_n)$ *denote the ensemble of any number n of variables, and E a set of points in the n-dimensional interval S in which x moves, then if*

(1) $\int_E f_n(x)\,dx$ *has no negative double limit, as* $E \to 0$, $n \to \infty$, *and*

(2) $f_n(x) \to f(x)$, *except at a set of content zero, and*

(3) $\int_S f_n(x)\,dx \to \int_S f(x)\,dx$,

then the succession $g(x) f_n(x)$ *is absolutely integrable term-by-term, and*

$$\int_E g(x) f_n(x)\,dx \to 0, \quad (E \to 0,\ n \to \infty),$$

where $g(x)$ *is any bounded function.*

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 11, p. 60 top.

† *Rend. di Palermo*, Vol. 23, p. 149.

14. This enables us to prove the following Lemma :—

LEMMA 4.—*If* $\xi(u, v)$ *is any positive function, which is summable with respect to* v, *and whose integral with respect to* v *is bounded, and* $\mu(u)$ *is the integral of a positive summable function* $\mu'(u) \geqslant 0$, *then*

$$\underset{E\to 0,\, h\to}{\mathrm{Lt}} \iint_E \xi(u, v)\, \frac{\mu(u+h)-\mu(u)}{h}\, du\, dv = 0. \tag{20}$$

By the above extension of Vitali's theorem this will be true—since the integrand is $\geqslant 0$, and tends, as $h \to 0$, to $\xi(u, v)\,\mu'(u)$ almost everywhere—provided, as $h \to 0$,

$$\int_a^c \int_b^d \xi(u, v)\, \frac{\mu(u+h)-\mu(u)}{h}\, du\, dv \to \int_a^c \int_b^d \xi(u, v)\, \mu'(u)\, du\, dv, \tag{21}$$

that is, if we write, $x(u, d)-x(u, b) = \int_b^a \xi(u, v)\, dv,$

if $$\int_a^c \{x(u, d)-x(u, b)\}\, \frac{\mu(u+h)-\mu(u)}{h}\, du \to \int_a^c \{x(u, d)-x(u, b)\}\, \mu'(u)\, du, \tag{21'}$$

these integrals, as well as those in (21), necessarily existing since $x(u, v)$ is bounded, and the other factor is in each case, summable, while the integrands are both $\geqslant 0$.

The truth of this relation follows again by Vitali's theorem, for, as $h \to 0$,

$$0 \leqslant \frac{\mu(u+h)-\mu(u)}{h} \to \mu'(u),$$

and $$\int_a^c \frac{\mu(u+h)-\mu(u)}{h}\, du = \frac{1}{h}\int_c^{c+h} \mu(u)\, du - \frac{1}{h}\int_a^{a+h} \mu(u)\, du \to \mu(c)-\mu(a)$$

$$= \int_a^c \mu'(u)\, du,$$

and therefore $$\int_E \{x(u, d)-x(u, b)\}\, \frac{\mu(u+h)-\mu(u)}{h}\, du \to 0, \quad (h \to 0).$$

Hence, since the integrand of the last integral converges almost everywhere to $\{x(u, d)-x(u, b)\}\,\mu'(u)$, the succession is absolutely integrable* and we have the required result (21).

15. One of the theorems† of my earlier paper involves conditions,

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* "On a Formula for an Area," § 32, p. 367.

† Theorem VI, p. 365, *ibid.*

ensuring the existence of the double integral of the Jacobian, and securing the truth of the formula for an area, which, in light of the theorem of § 3, seem somewhat one-sided. In that paper I showed that if

(1) *x, y are absolutely continuous with respect to u and also with respect to v;*

(2) $\left|\dfrac{\partial x}{\partial u}\right| \leqslant \mu'(u)$, $\left|\dfrac{\partial y}{\partial u}\right| \leqslant \mu'(u)$, *where $\mu'(u)$ is a summable function of u alone;*

(3) $\iint \left|\dfrac{\partial x}{\partial v}\right| \mu'(u)\,du\,dv$, $\iint \left|\dfrac{\partial y}{\partial v}\right| \mu'(u)\,du\,dv$ *exist;*

then the formula for an area holds.

We are now able to supplement this result by proving the complementary theorem, in which $\partial y/\partial u$ is interchanged with $\partial x/\partial v$. Our method of proof obliges us however to postulate the summability of the partial derivates with respect to (u, v), which was avoided in the earlier paper. Taking account of both these results, the latter of which alone is proved below, the theorem we obtain is the following :—

THEOREM 2.—*If*

(i) *$x(u, v)$ and $y(u, v)$ are absolutely continuous functions of each of the variables u and v separately;*

(ii) *in each pair of functions $(\partial x/\partial u, \partial y/\partial v)$ and $(\partial x/\partial v, \partial y/\partial u)$ one function is numerically less than a summable function $\mu'(t)$* of a single variable, namely of that variable $t = u$ or $t = v$ with respect to which the derivation is performed;*

(iii) *the other derivate, which completes the pair in each of the two cases, possesses an absolutely convergent integral with respect to (u, v), and this remains true when the derivate is multiplied by $\mu'(t)$;*

then the formula for an area holds,

$$A = \int x(t)\,dy(t) = \int_a^c \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv.$$

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* It is unnecessary to take a different function in each case, since the function which is the greater of two summable functions is a summable function.

We give the proof in the case when

$$\left|\frac{\partial x(u,v)}{\partial u}\right| \leqslant \mu'(u), \quad \left|\frac{\partial x(u,v)}{\partial v}\right| \leqslant \mu'(v),$$

while $\iint \left|\frac{\partial y}{\partial v}\right| \mu'(u)\,du\,dv$ and $\iint \left|\frac{\partial y}{\partial u}\right| \mu'(v)\,du\,dv$ as well as $\iint \left|\frac{\partial y}{\partial u}\right| du\,dv$ and $\iint \left|\frac{\partial y}{\partial v}\right| du\,dv$ exist.

The derivates of $x(u, v)$ and $y(u, v)$ are accordingly summable with respect to (u, v), so that the conditions (1) and (2) of Theorem 1 are satisfied. Therefore the whole of the argument of §§ 5–7 still holds good, since in it we did not use the conditions (3*a*) or (3*b*).

Thus, applying Theorem 1, we arrive at once at the equation (14),

$$\underset{k\to 0}{\mathrm{Lt}} \left\{ \underset{h\to 0}{\mathrm{Lt}} \int_a^c \int_b^d \frac{\partial(\bar{Q}, y)}{\partial(u, v)}\,du\,dv = \int_{t_0}^{t_1} x(t)\,dy(t), \qquad (14)$$

in which we may invert the order of the limits since $\bar{Q}(u, v)$ is a double integral, namely,

$$\bar{Q}(u, v) = \int_v^{v+k} \int_u^{u+h} x(u, v)\,du\,dv/hk. \qquad (6)$$

16. We have next to show that the left-hand side of (14) converges to the double integral of the Jacobian. We shall take separately the two terms of the Jacobian, and prove that both

$$\underset{h\to 0}{\mathrm{Lt}} \left\{ \underset{k\to 0}{\mathrm{Lt}} \int_a^c \int_b^d \frac{\partial\bar{Q}}{\partial u}\frac{\partial y}{\partial v}\,du\,dv \right\} = \int_a^c \int_b^d \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}\,du\,dv, \qquad (22)$$

and also $$\underset{h\to 0}{\mathrm{Lt}} \left\{ \underset{k\to 0}{\mathrm{Lt}} \int_a^c \int_b^d \frac{\partial\bar{Q}}{\partial v}\frac{\partial y}{\partial u}\,du\,dv \right\} = \int_a^c \int_b^d \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\,du\,dv. \qquad (23)$$

In proving (22) we shall use the condition (ii) of the enunciation as applying to $\partial x/\partial u$, namely,

$$\left|\frac{\partial x}{\partial u}\right| < \mu'(u),$$

where, $\mu(u)$ denoting the integral of $\mu'(u)$,

$$\mu(u) - \mu(u_0) = \int_{u_0}^{u} \mu'(u)\,du. \qquad (24)$$

In proving (23) we use the same condition as with v in place of u.

For convenience of writing out, we shall in (23) interchange u and v, h and k; it then becomes

$$\underset{k\to 0}{\mathrm{Lt}} \left\{ \underset{h\to 0}{\mathrm{Lt}} \int_a^c \int_b^d \frac{\partial Q}{\partial u} \frac{\partial y}{\partial v}\, du\, dv \right\} = \int_a^c \int_b^d \frac{\partial x}{\partial u} \frac{\partial y}{\partial v}\, du\, dv. \tag{23'}$$

The equations to be proved (22) and (23′) are now the same except that the repeated limit is taken in a different order. Moreover, the condition (24) is now valid in both cases.

17. We shall assume in the first instance that both $\partial x/\partial u$ and $\partial y/\partial v$ are positive or zero.

Then, by (12) and (24),

$$0 \leqslant \frac{\partial \bar{Q}}{\partial u} \leqslant \frac{\mu(u+h)-\mu(u)}{h}. \tag{25}$$

Hence

$$0 \leqslant \iint_E \frac{\partial \bar{Q}}{\partial u} \frac{\partial y}{\partial v}\, du\, dv \leqslant \iint_E \frac{\mu(u+h)-\mu(u)}{h} \frac{\partial y}{\partial v}\, du\, dv.$$

But the last integral tends to zero as $E \to 0$, $k \to 0$, since the integrand is independent of k, and also as $E \to 0$, $h \to 0$, by Lemma 4. The application of this Lemma involves the condition (3) of the enunciation, since it requires that $\iint \frac{\partial y}{\partial v} \mu'(u)\, du\, dv$ should exist. This ensures the existence of the further integrals which appear.

Thus
$$\underset{k\to 0,\, E\to 0}{\mathrm{Lt}} \iint_E \frac{\partial \bar{Q}}{\partial u} \frac{\partial y}{\partial v}\, du\, dv = 0, \tag{26}$$

and also the same with k replaced by h.

Now, by (12), the integrand tends to a unique limit except at a set of content zero, whether we make k or h tend to zero, since

$$\underset{k\to 0}{\mathrm{Lt}} \frac{\partial \bar{Q}}{\partial u} = \frac{1}{h} \int_u^{u+h} \frac{\partial x}{\partial u}\, du \leqslant \frac{\mu(u+h)-\mu(u)}{h}, \tag{27}$$

and
$$\underset{h\to 0}{\mathrm{Lt}} \frac{\partial \bar{Q}}{\partial u} = \frac{1}{k} \int_v^{v+k} \frac{\partial x}{\partial u}\, dv \leqslant \mu'(u). \tag{27'}$$

Therefore by a known theorem* we have

$$\operatorname*{Lt}_{k\to 0}\int_a^c\int_b^d \frac{\partial\bar{Q}}{\partial u}\frac{\partial y}{\partial v}\,du\,dv = \int_a^c\int_b^d \operatorname*{Lt}_{k\to 0}\frac{\partial\bar{Q}}{\partial u}\frac{\partial y}{\partial v}\,du\,dv, \tag{28}$$

and the same equation, with k replaced by h.

18. We now have to make $h\to 0$ in (28) and $k\to 0$ in the alternative equation. In both cases the limit of the integrand on the right is unique, except at a set of content zero, and we have by (27) and (27′),

$$\operatorname*{Lt}_{h\to 0}\operatorname*{Lt}_{k\to 0}\frac{\partial\bar{Q}}{\partial u}\frac{\partial y}{\partial v} = \operatorname*{Lt}_{k\to 0}\operatorname*{Lt}_{h\to 0}\frac{\partial\bar{Q}}{\partial u}\frac{\partial y}{\partial v} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v}. \tag{29}$$

Also, by (27) and (27′), we have

$$\iint_E \operatorname*{Lt}_{k\to 0}\frac{\partial\bar{Q}}{\partial u}\frac{\partial y}{\partial v}\,du\,dv \leqslant \iint_E \frac{\mu(u+h)-\mu(u)}{h}\frac{\partial y}{\partial v}\,du\,dv,$$

and $$\iint_E \operatorname*{Lt}_{h\to 0}\frac{\partial\bar{Q}}{\partial u}\frac{\partial y}{\partial v}\,du\,dv \leqslant \iint_E \mu'(u)\frac{\partial y}{\partial v}\,du\,dv.$$

In the former case, when we make $E\to 0$, $h\to 0$, by Lemma 4, and in the latter case, when $E\to 0$, $k\to 0$, by the fundamental property of a summable function, we get the unique limit zero. We can therefore again apply the theorem by which we obtained (28) and its alternative. Therefore

$$\operatorname*{Lt}_{h\to 0}\left\{\operatorname*{Lt}_{k\to 0}\int_a^c\int_b^d \frac{\partial\bar{Q}}{\partial u}\frac{\partial y}{\partial v}\,du\,dv\right\} = \operatorname*{Lt}_{h\to 0}\int_a^c\int_b^d \operatorname*{Lt}_{k\to 0}\frac{\partial\bar{Q}}{\partial u}\frac{\partial y}{\partial v}\,du\,dv$$

$$= \int_a^c\int_b^d \operatorname*{Lt}_{h\to 0}\operatorname*{Lt}_{k\to 0}\frac{\partial\bar{Q}}{\partial u}\frac{\partial y}{\partial v}\,du\,dv,$$

and a similar equation with the order of the limits reversed. By (29) these equations are the same as the relations (22) and (23′) to be proved.

This proves the theorem when the derivates of $x(u, v)$ and $y(u, v)$ are $\geqslant 0$.

19. In the general case, since $x(u, v)$ and $y(u, v)$ are absolutely con-

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* § 32 of "On a Formula for an Area," p. 367. In the more general formula of § 34, q has been obviously misprinted for g in the relation (3).

tinuous with respect to each variable separately

$$x = x_1 - x_2, \quad y = y_1 - y_2,$$

where x_1, x_2, y_1, y_2 satisfy the condition as to the derivates being $\geqslant 0$, and fulfil all that we have so far demanded of x and y. Thus (28) holds, when for x we put x_1 or x_2, and for y we put y_1 or y_2. Adding the relations so obtained with proper signs, we obtain the relation (28) for x and y.

20. We shall in what follows require the following Lemma, which I have already given,* though ostensibly only for a single variable.

LEMMA 5.—*If* $f(x)$ *and* $g(x)$ *are summable functions of the ensemble of* n *variables* $x \equiv (x_1, x_2, \ldots, x_n)$,

$$f(x+y)\, g(y)$$

is summable with respect to y *for each ensemble* x, *excepting only a set of* x*'s of content* (n*-dimensional content*) *zero, and*

$$\int f(x+y)\, g(y)\, dy$$

is a summable function of x, *and*

$$\int dx \int f(x+y)\, g(y)\, dy = \int dy \int f(x+y)\, g(y)\, dx.$$

We have evidently only to prove the lemma for positive functions and it will be true generally.

Now

$$\int_a^x f(x+y)\, g(y)\, dx$$

$$\equiv \int_{a_1}^{x_1}\int_{a_2}^{x_2} \ldots \int_{a_n}^{x_n} f(x_1+y_1, x_2+y_2, \ldots, x_n+y_n)\, g(y_1, y_2, \ldots, y_n)\, dx_1\, dx_2 \ldots dx_n$$

$$= g(y) \int_{a+y}^{x+y} f(x)\, dx,$$

and is therefore a summable function of y. Denoting its integral by I, and by $f(x, y, M)$ the function equal to $f(x+y)\,g(y)$ wherever it is $\leqslant M$, and equal to M elsewhere, we have, since I is finite,

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* W. H. Young, "Sur la généralisation du Théorème de Parseval," *Comptes rendus*, Vol. 155, p. 30, 1 juillet 1912.

$$I = \int_b^y dy \int_a^x f(x+y)\,g(y)\,dx = \int_b^y dy \operatorname*{Lt}_{M\to\infty} \int_a^x f(x, y, M)\,dx$$

$$= \operatorname*{Lt}_{M\to\infty} \int_b^g dy \int_a^x f(x, y, M)\,dx$$

$$= \operatorname*{Lt}_{M\to\infty} \int_a^x dx \int_b^y f(x, y, M)\,dy = \int_a^x dx \operatorname*{Lt}_{M\to\infty} \int_b^y f(x, y, M)\,dy.$$

Therefore the limit of the inside integral is finite except at a set of values of x of content zero, and therefore $f(x+y)\,g(y)$ which is the limit of $f(x, y, M)$, has an integral except at this set, and we have

$$I = \int_a^x dx \int_b^y f(x+y)\,g(y)\,dy.$$

This proves the lemma.

21. We can now prove the following statement :—

If we write $\quad x = x(u+\alpha, v+\beta) = \xi(u, v)$ say,

$$y = y(u, v),$$

where α *and* β *are parameters, the formula for an area*

$$A = \int_{t_0}^{t_1} x(t)\,dy(t) = \int_a^c \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv,$$

holds except for a set of values of (α, β) *of plane content zero, provided our conditions* (1) *and* (2) *hold, namely*

(1) $x(u, v)$ *and* $y(u, v)$ *are absolutely continuous with respect to each of the variables separately;*

(2) *their partial derivates have absolutely convergent double integrals.*

In fact, as we already saw the formula (14) holds, namely

$$\operatorname*{Lt}_{k\to 0} \left\{ \operatorname*{Lt}_{h\to 0} \int_a^c \int_b^d \frac{\partial(\bar{Q}, y)}{\partial(u, v)}\,du\,dv \right\} = \int_{t_0}^{t_1} x(t)\,dy(t), \tag{14}$$

where, using the formulæ already obtained,

$$hk\,\frac{\partial(\bar{Q},\,y)}{\partial(u,\,v)} = \int_u^{u+h}\int_v^{v+k}\left\{\frac{\partial\xi(U,\,V)}{\partial U}\,\frac{\partial y(u,\,v)}{\partial v} - \frac{\partial\xi(U,\,V)}{\partial V}\,\frac{\partial y(u,\,v)}{\partial u}\right\}dU\,dV$$

$$= \int_0^h\int_0^k\left\{\frac{\partial\xi(u+H,\,v+K)}{\partial u}\,\frac{\partial y(u,\,v)}{\partial v} - \frac{\partial\xi(u+H,\,v+K)}{\partial v}\,\frac{\partial y(u,\,v)}{\partial u}\right\}$$

$$\times\, dH\,dK.$$

Denoting the last integrand by $\phi(u,\,v,\,H,\,K)$, we have therefore

$$A = \underset{k\to 0}{\mathrm{Lt}}\;\underset{h\to 0}{\mathrm{Lt}}\;\frac{1}{hk}\int_0^h\int_0^k\left\{\int_a^c\int_b^d\phi(u,\,v,\,H,\,K)\,du\,dv\right\}dH\,dK,$$

where we have changed the order of integration in accordance with Lemma 5.

In our case we must in this expression write for $\xi(u+H,\,v+k)$ the expression $x(u+\alpha+H,\,v+\beta+K)$, that is, write $\alpha+H$ for H and $\beta+K$ for K and ξ for x. Thus, if ϕ becomes ψ,

$$A = \underset{h\to 0}{\mathrm{Lt}}\;\underset{k\to 0}{\mathrm{Lt}}\;\frac{1}{hk}\int_\alpha^{\alpha+k}\int_\beta^{\beta+k}\left\{\int_a^c\int_b^d\psi(u,\,v,\,H,\,K)\,du\,dv\right\}dH\,dK.$$

ϕ differs from ψ only in that the letter ξ in ϕ replaces x in ψ. A is therefore the mixed differential coefficient at $(\alpha,\,\beta)$ of the outside double integral, which, by a known theorem,* is equal to the inside double integral, except at a set of values of $(\alpha,\,\beta)$ of plane content zero, that is

$$A = \int_a^c\int_b^d\psi(u,\,v,\,\alpha,\,\beta)\,du\,dv$$

$$= \int_a^c\int_b^d\left\{\frac{\partial x(u+\alpha,\,v+\beta)}{\partial u}\,\frac{\partial y(u,\,v)}{\partial v} - \frac{\partial x(u+\alpha,\,v+\beta)}{\partial v}\,\frac{\partial y(u,\,v)}{\partial u}\right\}du\,dv, \quad \text{(Z)}$$

except for a set of values of α, β of plane content zero.

This proves the required result.

22. One more remark. We can easily see that A is a continuous

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* W. H. Young, "Sur la dérivation des fonctions à variation bornée," *Comptes rendus*, Vol. 164, p. 622, 23 avril 1917.

function of (α, β). This follows, in fact, from its expression as a contour integral. If, then, we could prove that by a suitable mode of approach of (α, β) to $(0, 0)$, which omits a certain set of content zero, the right-hand side of Z had as limit

$$\int_a^c \int_b^d \left\{ \frac{\partial x(u, v)}{\partial u} \frac{\partial y(u, v)}{\partial v} - \frac{\partial y(u, v)}{\partial v} \frac{\partial x(u, v)}{\partial u} \right\} du\, dv,$$

then our conditions (1) and (2) of the preceding paragraph would suffice for the validity of the formula for an area.

AN EXAMPLE OF AN ORTHOGONAL DEVELOPMENT WHOSE SUM IS EVERYWHERE DIFFERENT FROM THE DEVELOPED FUNCTION

By STEFAN BANACH.

(*Communicated by* G. H. HARDY.)

[Received June 24th, 1921.—Read November 17th, 1921.]

THE purpose of this note is to give an example of a Fourier-like development

$$f \sim c_1\psi_1+c_2\psi_2+\dots$$

of a summable function $f(t)$ defined in (a, b), $\{\psi_n(t)\}$ being a complete set of functions normalised and orthogonal in (a, b), such that

(i) the series $$\sum_{n=1}^{\infty} c_n\psi_n(t)$$

converges throughout (a, b), but

(ii) the sum of the series differs from $f(t)$ in every point t of (a, b).

The c_n are to be understood here as the Fourier constants of f, *i.e.*

$$c_n = \int_a^b f(t)\,\psi_n(t)\,dt \quad (n = 1, 2, \dots), \tag{1}$$

and we shall choose our functions as to assure the existence of the integrals (1).

THEOREM.—*Suppose that* (i) $f(t)$ *is defined throughout* (a, b), (ii) $f(t)$ *is positive, so that*

$$f(t) > 0 \quad (a \leqslant t \leqslant b), \tag{2}$$

(iii) $f(t)$ *is summable in* (a, b), *and* (iv) $f^2(t)$ *is not summable. Then we can determine a complete, orthogonal, and normalised set of functions* $\{\psi_n(t)\}$, *defined and summable in* (a, b), *such that*

$$\int_a^b f(t)\,\psi_n(t)\,dt = 0 \quad (n = 1, 2, \dots). \tag{3}$$

It is evident that our theorem implies the existence of the required example, because the equations (3) and the definitions (1) give immediately

$$\sum_{n=1}^{\infty} c_n \psi_n(t) \equiv 0 \quad (a \leqslant t \leqslant b), \tag{4}$$

and from (2) and (4) it follows that

$$f(t) > \sum_{n=1}^{\infty} c_n \psi_n(t) \quad (a \leqslant t \leqslant b). \tag{5}$$

Proof.—Let $\{\phi_n(t)\}$ be the ordinary complete and normalised *trigonometrical* set corresponding to (a, b)*, and put

$$a_n = -\frac{\int_a^b f(t)\, \phi_n(t)\, dt}{\int_a^b f(t)\, dt} \quad (n = 1, 2, \ldots).\dagger \tag{6}$$

Then

$$\int_a^b [a_n + \phi_n(t)] f(t)\, dt = 0 \quad (n = 1, 2, \ldots). \tag{7}$$

The set $\{a_n + \phi_n(t)\}$ *is complete;* in fact, let $\gamma(t)$ be a function integrable together with its square in (a, b), and let us suppose

$$\int_a^b [a_n + \phi_n(t)]\, \gamma(t)\, dt = 0 \quad (n = 1, 2, \ldots), \tag{8}$$

$$\int_a^b \gamma^2(t)\, dt > 0. \tag{8'}$$

The so-called "Parseval-relation" holds for the trigonometrical set $\{\phi_n(t)\}$ and gives

$$\sum_{n=1}^{\infty} \left(\int_a^b \gamma(t)\, \phi_n(t)\, dt \right)^2 = \int_a^b \gamma^2(t)\, dt, \tag{9}$$

which, compared with (8) and (8′), implies

$$\sum_{n=1}^{\infty} \left(a_n \int_a^b \gamma(t)\, dt \right)^2 = \int_a^b \gamma^2(t)\, dt > 0. \tag{10}$$

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* $\phi_1 = \frac{1}{\sqrt{(b-a)}}, \quad \phi_2 = \sqrt{\left(\frac{2}{b-a}\right)} \sin 2\pi \frac{t-a}{b-a}, \quad \phi_3 = \sqrt{\left(\frac{2}{b-a}\right)} \cos 2\pi \frac{t-a}{b-a}, \quad \ldots.$

† The denominator is positive, by (2).

It follows that $\int_a^b \gamma(t)\,dt \neq 0$, and so that the series

$$\sum_{n=1}^{\infty} a_n^2 \tag{11}$$

is convergent. But this implies, by (6), the convergence of

$$\sum_{n=1}^{\infty} \left(\int_a^b f(t)\,\phi_n(t)\,dt\right)^2, \tag{12}$$

and so the existence of $\int_a^b f^2(t)\,dt$, which must be equal to (12). This is contradictory to our hypotheses; and this contradiction shows that the assumptions (8) and (8′) are incompatible, and so that the set $\{a_n+\phi_n(t)\}$ is complete.

Put

$$a_n+\phi_n(t) = \chi_n(t) \quad (n = 1, 2, \ldots). \tag{13}$$

Then $\{\chi_n(t)\}$ is a *complete set of continuous functions*, and we have, by (7) and (13),

$$\int_a^b f(t)\,\chi_n(t)\,dt = 0 \quad (n = 1, 2, \ldots). \tag{14}$$

We have now only to derive from $\{\chi_n(t)\}$ a new set $\{\psi_n(t)\}$ by the "orthogonalisation method" of Mr. E. Schmidt to get a complete, orthogonal, and normalised set possessing the property (3). The ψ_n are linear finite forms in χ_n, and thus our set is composed of trigonometrical polynomials.

CHORDS OF TWISTED CUBICS

By E. K. Wakeford.

The present paper deals primarily with the twisted cubics which can be drawn with six given lines as chords, and proceeds to a discussion of Cayley's problem of the condition that seven lines should lie on a quartic surface. Possible relations between two twisted cubics come under discussion to a large extent, and in particular the case in which one cubic is said to be "inscribed" in the other (§ 2).

In § 1 a useful transformation with respect to four straight lines in space is studied. § 2 contains two representations of one twisted cubic by means of another, the former having reference to the number of twisted cubics that there are with six given lines as chords, the latter to the possible relations of the chords of one twisted cubic and the lines that lie in two osculating planes of another. An important fact about two cubics both "inscribed" in one is also proved. In § 3 yet another representation of twisted cubics by means of certain others is described, and important relations are established between cubics with six given chords, and cubics containing each of these lines in two osculating planes. The configuration is completed in § 4, where twenty lines associated with the six given lines are found. § 5 brings quartic surfaces on the scene, and thereby obtains some of the previous results more simply than before. Cayley's problem of the seven lines is examined in § 6, where it is proved that in all cases, if seven lines lie on a quartic surface, so do their reciprocals with respect to a conicoid; in other words, the lines possess the reciprocal property of lying on a class quartic surface. § 7 contains a few properties of two cubics, one of which is "inscribed" in the other, together with a proof of a theorem of Mr. Grace's concerning six lines meeting a single line, which is derived from these related cubics; the proof, it is believed, adds a new property to the configuration of lines that arises.

It is feared that several proofs may be condemned as sketchy and vague; for instance, a (1, 1)-correspondence may be found not fully established. In order to understand this, it should be realized that the principal aim in writing out the proofs has been to give a clear idea of the geometrical argument, with notes to show how to deal rigorously with the important analytical problems that arise, and that the question, as to what to leave out and what to put in, is largely a matter of individual taste. It is hoped that no really important point has been neglected, but no attempt has been made to give full logical proofs throughout.

1. Take four lines l_1, l_2, l_3, l_4 in a space, S, of three dimensions. In the general case, which is the case under consideration, no two of the lines intersect and the four lines have just two distinct common intersectors. Consider now cubic surfaces through the four lines. The coefficients of the equation of all such surfaces satisfy sixteen independent linear conditions; hence, as there are only twenty coefficients in the equation of a cubic surface, all these cubic surfaces can be written in the form

$$\lambda_1 C_1 + \lambda_2 C_2 + \lambda_3 C_3 + \lambda_4 C_4 = 0,$$

where C_1, C_2, C_3, C_4 are four linearly independent cubic surfaces of the set, and λ_1, λ_2, λ_3, λ_4 are independent parameters. But λ_1, λ_2, λ_3, λ_4, varying independently, may be regarded as the coordinates of a plane in a three-dimensional space S', and by this means a projective correspondence between cubic surfaces in S containing $l_1 l_2 l_3 l_4$, and planes in S', is obtained. (In examining the correspondence between the spaces S and S', detailed proofs will not be given at every stage.) A set of the cubic surfaces in S of the form $C_1+\lambda C_2$ corresponds to a set of planes in S' of the form $\pi_1+\lambda\pi_2$, that is to say with the planes of S' containing a certain line. Now the cubic surfaces $C_1+\lambda C_2$ have a common curve of intersection, and this curve uniquely determines the system. There is accordingly a projective correspondence between such curves in S, and lines in S'. In order to find what these curves are, it is sufficient to notice that two cubic surfaces containing $l_1 l_2 l_3 l_4$ contain also the two common intersectors of these lines. The remainder of their curve of intersection is accordingly of the third degree. Now when a line forms part of the curve of intersection of two cubic surfaces it cuts the rest of their curve of intersection in four points. Hence each of $l_1 l_2 l_3 l_4$ cuts this curve of the third degree in two points, the other two of the four points being given by their two common intersectors. This curve of the third degree is a twisted cubic and $l_1 l_2 l_3 l_4$ are chords of it. If we start with any twisted cubic having $l_1 l_2 l_3 l_4$ as chords, we obtain a family of cubic surfaces of the form $C+\lambda C'$ passing through it, and thus the corresponding line in S' is found.

Now any cubic surface containing $l_1 l_2 l_3 l_4$, and not containing the common twisted cubic of C and C', cuts this cubic in only one point not lying on $l_1 l_2 l_3 l_4$, so that a family $C+\lambda C'+\lambda' C''$ determines, and is clearly determined by, one point in the space S. But such a family of cubic surfaces leads to a family of planes $\pi+\lambda\pi'+\lambda'\pi''$ in S', that is to say, all the planes passing through one point. The points of S are thus (in general) projectively correspondent with the points of S'. This is not, of course, true for all points, but that does not affect the argument. The correspondence between cubic surfaces and twisted cubics in S, and planes and lines in S', may now be regarded entirely from the point of view of a correspondence between the points of the corresponding surfaces and curves. The natural question now is "What curve in S' corresponds to a line in S?" To answer it, observe that a line cuts a cubic surface in three points, and hence the curve in S' must be such as to cut a plane in three points. It is, in fact, a twisted cubic. In a similar way it may be seen that a plane in S becomes a cubic surface in S'.

We now proceed to complete the correspondence by finding four lines $l_1' l_2' l_3' l_4'$ in S', which are chords of all the twisted cubics representing lines of S, and lie on all cubic surfaces representing planes of S. In short, consider the family of cubic surfaces consisting of the conicoid through l_2, l_3, l_4, and a variable plane through l_1. This family is of the form $C+\lambda C'$, and is accordingly represented by a line in S'. Call this line l_1'. Similarly l_2', l_3', l_4' may be found. Then any cubic surface through l_1, l_2, l_3, l_4, which does not consist in part of the conicoid through l_2, l_3, l_4, will cut this conicoid in one generator (meeting l_2, l_3, l_4), besides l_2, l_3, l_4 and the two common intersectors of l_1, l_2, l_3, l_4. This generator lies on an ∞^2 of cubic surfaces through l_1, l_2, l_3, l_4, and so corresponds to a point on l_1'. There is thus a correspondence between the lines that cut l_2, l_3, l_4 and points on l_1'. Every point in fact on any one of these lines corresponds to the same point of l_1'. Now any line in S cuts the conicoid through l_2, l_3, l_4 in two points, that is to say, it meets two of the generators which are represented by points on l_1'. The twisted cubic in S' representing the line in S accordingly meets l_1' in two points. Similarly it meets l_2', l_3', l_4' in two points each. It is now easy to see that a plane in S corresponds to a cubic surface in S' containing l_1', l_2, l_3', l_4'. The converses of these correspondences are not difficult to prove.

There is then a symmetrical correspondence between the spaces S and S'. Points correspond with points, lines of one space with twisted cubics, with four given chords, in the other, and planes of one space with cubic surfaces through four fixed lines in the other. The degree of a curve or surface is in general multiplied by three by this birational transformation. If a surface contains l_1, l_2, l_3, l_4, eight must be deducted from this result in order to allow for the four conicoids each containing three of the lines. In the same way allowance must be made for each intersection of a curve with any of the lines l_1, l_2, l_3, l_4. For instance, a twisted cubic with l_1, l_2

as chords, meeting l_3 once, but l_4 not at all, becomes a curve of degree $3\times 3-2-2-1-0$, *i.e.* 4. As a matter of fact, though it is immaterial for the present purposes to prove it, the lines l_1, l_2, l_3, l_4 are projective with l'_1, l'_2, l'_3, l'_4, and so the transformation may be considered to take place in one space.

The well known theorem that, *if one twisted cubic exists with four given lines as tangents, an infinite number exists,* is proved most easily by reciprocating with respect to any linear complex containing the four lines ; but if the transformation under discussion above is used we have, remembering that l_1, l_2, l_3, l_4 are projective with l'_1, l'_2, l'_3, l'_4, "If four lines are tangents to a cubic, a line can be drawn to touch the four conicoids each containing three of the lines ; and conversely." If one such line can be drawn, it is an easy matter to show that all the lines of a certain regulus possess this property. Mr. P. W. Wood has given* an analytical proof of the theorem in italics.

2. A few definitions and explanations are now necessary. The word "cubic" will denote "twisted cubic." In the following work we are usually considering the general case, and where the work does not apply to particular cases (*e.g.* where the cubics are degenerate) that fact is not necessarily stated. An axis of a cubic is the reciprocal of a chord, that is to say, a line which lies in two of its osculating planes. If there is an infinite number of tetrahedra with faces osculating T' and vertices lying on T, T and T' being two cubics, then T' is said to be inscribed in T, and T to be circumscribed about T'. We shall use the theorem that unless two cubics lie on a conicoid (which will never be the case in the subsequent work) they have only a finite number of common chords.

(*a*) We proceed to examine a useful method of representing one cubic by means of another. Take a cubic of reference T and project it from one of its points into a conic C on a plane Π. Then there is a projective correspondence between the chords of T and the lines in Π, the points on T which the chords connect being projected into the points on Π where the lines cut C. Consider now another cubic T_1. The chords of T that meet it will be projected into lines enveloping some rational curve, since there is just one chord for each point of T_1, in general. It is important to infer from this projective correspondence that the envelope cannot degenerate unless T meets T_1, supposing T and T_1 to be genuine cubics. If

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[* Cf. P. W. Wood, "The Twisted Cubic," *Cambridge Tracts*, 1913, p. 21.]

T_1 meets T in 1, 2, 3, 4 points, the envelope contains 1, 2, 3, 4 points as part of itself. So if the envelope splits up into simpler envelopes, all but one of these envelopes must be points.

We notice that just six tangents to the envelope pass through any point of C, since just six chords of T through a given point of T meet T_1. The rational envelope is accordingly of the sixth class. A double tangent will represent a chord of T that meets T_1 twice, that is to say, a common chord of T and T_1. Since the curve is rational, we should expect just ten distinct double tangents, and it is easy to show from particular cases that there are not less than ten, and so there are exactly ten, in the general case. This number may be inferred from Halphen's theorem. The important thing to notice is that the ten common chords are given by the double tangents of a genuine rational class sextic.

Similarly two cubics have ten common axes.

Now take four of the common chords of two cubics as $l_1 l_2 l_3 l_4$, and apply the transformation of § 1. The cubics become lines, and their other six common chords become six cubics having the two lines, and l_1, l_2, l_3, l_4, as chords. From which it follows easily that given six lines there are just six cubics which have them all as chords. It is important to observe that if two of the six cubics coincide, then the lines must satisfy some definite condition, which could be deduced (theoretically) from the condition that two of the common chords of two cubics should coincide. That is to say, if the six lines are given by coordinates, a definite function of these coordinates, which does not vanish identically, will vanish if two of the cubics coincide.

Similarly there are in general just six cubics which have all the six lines as axes.

It will be clear from the subsequent work that if two of the first set of six cubics coincide, so will two of the second set.

(*b*) There is another way of using the chords of T to represent a cubic T' on the plane Π. This time the chords of T which lie in an osculating plane of T' are considered. As before a sextic envelope is obtained in Π, though not now a rational one; this time we are more concerned with the ways in which this envelope can be degenerate than with its general properties. A double tangent to this envelope corresponds in this case to a chord of T, which is an axis of T'. It is important to know in what circumstances there can be an infinite number of such lines. We shall show that if T has an infinite number of chords which are axes of T', there are only three cases, viz.:

(i) T lies on a conicoid which is touched by the osculating planes of T' [the system of generators which are axes of T' being the same as of those which are chords of T].

(ii) T and T' are reciprocal with respect to a linear complex.

(iii) T is circumscribed about T' in accordance with our definition.

For if the sextic envelope in Π is to have an infinite number of double tangents, it must contain as part of itself a double envelope. This envelope must be of class 1, 2, or 3. When this envelope is a point, we have case (i) above. When it is a conic we have case (ii), as may be easily seen from the fact that the lines in Π which touch a conic correspond with the chords of T which belong to a linear complex. In the last case we perceive that every chord of T which lies in one osculating plane of T' lies in another also. From this fact it is easy to deduce that T' is inscribed in T. In order to investigate this case more thoroughly, we note that in the case above there is just one tetrahedron inscribed in T and circumscribed about T', of which any particular point of T is a vertex. The general set of tetrahedra whose vertices lie on T, having this property, is given by $a_\theta^4+\lambda b_\theta^4$, where λ varies and θ is a parameter determining the points of T. It is well known that the faces of such a set of tetrahedra osculate a cubic T', and it is easy to see that the tetrahedra may be determined by $a_\phi{}^4+\lambda b_\phi'{}^4$, where λ varies and ϕ is a parameter determining the osculating planes of T'. Moreover all the tetrahedra are self-conjugate with respect to a conicoid. The reciprocal symmetry of the relation between T and T' is thus made clear. Now if we consider the representation of T' by means of T on a plane, as above under § 2 (*b*), the chords of T which are axes of T' are projected into lines touching a class cubic in the plane. Corresponding to each tetrahedron circumscribed about T' and inscribed in T, there is a quadrangle whose vertices lie on the conic C in the plane which is the projection of T, all the six sides of which touch the class cubic. Suppose now that T'' too is inscribed in T, and that we find the class cubic corresponding to it also. These two class cubics will have nine common tangents, the projections of chords of T which are axes of T' and T''. It is thus proved that if T is circumscribed about T' and T'', nine out of the ten common axes of T' and T'' are chords of T. Suppose that these nine lines were also chords of another cubic T_1. Use T as cubic of reference, and represent T' and T'' as above, and T_1 by the representation of § 2 (*a*), as a rational class sextic. The nine common axes of T' and T'' are given by the nine common tangents of the class cubics. But since they are now being supposed to be chords of T_1, they must be double tangents of the sextic representing T_1. It is easy to see that this is impossible unless the class sextic degenerates into two cubics, which is a negligible case [see § 1 (*a*), above]. Hence given nine of the ten common axes of T' and T'', there is at most one cubic T which has them all as chords, and hence there are at most ten cubics circumscribed about two given cubics. It will be proved later that there are exactly ten [§ 4, below].

3. Let us now take six lines a, b, c, d, e, f, the six cubics $T_r (r, 1 \rightarrow 6)$ having the lines as chords, and the six cubics T'_r having the lines as axes. Consider the cubics inscribed in T_1. Since each of them has as axes an *infinite* number of chords of T_1, it only requires *one* condition to make a given chord one of that number. Since again there are ∞^6 cubics inscribed in T_1 it would appear to be possible to stipulate six chords of T_1, *e.g.* a, b, c, d, e, f, and find a cubic T' inscribed in T_1 and having the six chords as axes. We should accordingly expect that one, or more, of the cubics T'_r was inscribed in T_1. This will be proved to be the case, by means of another representation of cubics inscribed in a given cubic.

Correlate points of T_1 projectively with points of a twisted quartic Q in four dimensions. Then a cubic T' inscribed in T_1 determines a linear series of tetrahedra in T_1, and so on Q. Since it is a linear series in four dimensions, the hyperplanes containing the different tetrahedra all contain one fixed plane. (This may be seen by finding the plane from two of the hyperplanes, and then considering all hyperplanes containing the plane.) There is accordingly a projective correspondence between cubics inscribed in T_1 and planes in four dimensions. The chords of T_1 which are axes of T' correspond to the chords of Q which meet the plane corresponding to T'. Hence to find T' given six of these chords of T_1, we have simply to find a plane meeting six given chords of Q. Now given six lines in four dimensions, in general five planes may be drawn to meet them all,* and, although this is not the general case of six lines, we shall for the moment assume that just five planes can be drawn to meet six given chords of Q (see below). Hence there are five cubics inscribed in T_1 with a, b, c, d, e, f as axes. Thus five of the cubics T'_r are inscribed in T_1. Similarly five of the cubics T_r are circumscribed about each of the cubics T'_r. It follows that the two sets of cubics split up into six pairs,

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[* Cf., for instance, Richmond, "On the Figure of Six Points in a Space of Four Dimensions," *Quarterly Journal*, Vol. 31 (1899); Weitzenböck, "Proj. Geom. des R_4," *Wiener Ber.*, Vol. 121 (1912).]

that is to say T_r, T'_r (r, $1 \rightarrow 6$), so that T_r is circumscribed about every one of five T'_s ($s \neq r$) but not about T'_r.

The point left unsettled above is important inasmuch as most of the rest of the paper depends upon the theorem under consideration. In order to prove it, take six general lines, and show how to obtain a quintic equation giving the points of intersection of the five planes which meet them all with one of the lines. Then let the coordinates determining the lines assume special values so that the lines become chords of a twisted quartic, and suppose that the quintic equation (which will not vanish identically) has now n distinct roots. Finally, give still more special values to the coordinates so that the six chords meet in three pairs. The quintic equation cannot have more distinct roots in this than in the preceding case, and it is geometrically obvious that there are now just five distinct planes which meet all the lines.*

4. Seeing that the necessary spade work is now at an end, it is well to review the present state of our knowledge.

Given six lines a, b, c, d, e, f, there are six cubics, T_r, which have them as chords, and six cubics, T'_r, which have them as axes. T_r is circumscribed about T'_s, but not about T'_r. We have, moreover, a transformation of cubics with four given chords which may prove useful, and wish to verify now a suspicion that there are as many as and therefore just ten cubics circumscribed about two given cubics. See § 2 (b), above.

Consider the two cubics T'_5, T'_6 and their ten common axes. The cubics T_1, T_2, T_3, T_4 are all circumscribed about these two cubics; hence by § 2 (b) each of these four has as chords nine of the common axes of T'_5 and T'_6, and no two of them the same nine. These ten axes consist of a, b, c, d, e, f, and four others, which may be denoted by (123), (234), (314), (124), where (123) is a chord of T_1, T_2, and T_3, etc. Thus T_1, T_2, T_3 have a common chord apart from a, b, c, d, e, f, which is an axis of T'_5 and T'_6. Similarly T_1, T_2, T_3 have common chords apart from a, b, c, d, e, f, which are axes of T'_4, T'_6, and of T'_4, T'_5 respectively. Hence there is a line which is a chord of T_1, T_2, T_3 and an axis of T'_4, T'_5, T'_6; for, otherwise, T_1, T_2, T_3 must have three common chords apart from a, b, c, d, e, f, and T'_4, T'_5, T'_6 will similarly have three common axes apart from a, b, c, d, e, f. In this case it will be seen that T'_4 and T'_5 have altogether eleven common

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[* One plane through the three points of intersection of the pairs of lines, three planes of which one passes through one of the three points of intersection, and one plane through none of the three points of intersection.]

axes,* which is not the case. Hence there is a line (123) which is a chord of T_1, T_2, T_3 and an axis of T'_4, T'_5, T'_6. There are in all twenty lines of this sort, given by (rst), where r, s, t take different values from one to six and order is immaterial; (rst) is a chord of T_r, T_s, T_t, and an axis of the three T' cubics other than T'_r, T'_s, T'_t. It is not difficult to show that these lines are in general distinct.

It is worth while observing that, as far as we have seen, the only chords of T_1 which are axes of T'_1 are a, b, c, d, e, f themselves. This is verified by Halphen's theorem, for since $3\times1+1\times3=6$, six is the number of lines which in general are axes of one cubic and chords of another. T_1 and T'_1, in fact, may be chosen at random and uniquely determine the whole configuration.

A number of theorems may be written down from the existence of these twenty lines, such as that if seven lines are chords of two cubics they are chords of three and axes of three more. Some of these theorems will be mentioned in the next paragraph.

It is possible also to deduce now that the number of cubics circumscribed about two given cubics is exactly ten. For we have seen in the work above that there is a cubic having as chords any nine of the common axes of T'_5 and T'_6 which include a, b, c, d, e, f. Accordingly to find a cubic in the cases where one of a, b, c, d, e, f is left out of the ten axes, it is only necessary to choose as initial lines not a, b, c, d, e, f but a, b, c, d, e and (123) say, and consider the cubics which have them as chords. Hence we obtain one cubic for each set of nine out of the ten common axes of two cubics, and the ten cubics thus obtained are the only cubics circumscribed about the two cubics.

5. Some of our results can be obtained in a simpler manner by considering classes of quartic surfaces. The equation of a quartic surface can satisfy thirty-four linear conditions. Hence if we consider quartic surfaces through the six lines a, b, c, d, e, f, we can still impose four linear conditions upon the equation. Since we know twelve points of intersection of the cubic curve T_1 with any such quartic surface, we can by use of one more linear condition force the surface to contain T_1. In like manner we can use up two more conditions in making the surface contain T_2 and T_3. Hence there is a family of surfaces of the form $F_4+\lambda F'_4$ containing $a, b, c,$

[* In the case suggested as being the actual one, the ten common axes of T'_4, T'_5 consist of a, b, c, d, e, f, and the single ones belonging each to the triplets $T'_4T'_5T'_1$, $T'_4T'_5T'_2$, $T'_4T'_5T'_3$, $T'_4T'_5T'_6$. Any other possibility would involve more than ten common axes for T'_4, T'_5.]

d, e, f and T_1, T_2, T_3. But such a family must have a common curve of the sixteenth degree, and we have only found a curve of the fifteenth degree so far. Accordingly the quartic surfaces must have another line in common. But when a line is common to two quartic surfaces it is easy to show that it cuts the rest of their curve of intersection in just six points. Now a, b, c, d, e, f already cut the rest of the curve of intersection in six points each. Hence the common line does not meet any of them, and so must be a chord of each of the three cubics T_1, T_2, T_3. Thus if three cubics have six common chords they have seven.

By a similar process, or by using the transformation of § 1, with a, b, c, d as l_1, l_2, l_3, l_4, we obtain, "If seven lines are chords of two cubics they are chords of three."

The existence of the twenty lines (rst) is thus proved more easily than before, but the connection between the cubics T_r, T'_r is absent. We should of course obtain twenty lines from the cubics T'_r, *e.g.* $(r's't')$, but we should be unable to say that $(1'2'3') \equiv (456)$ as is actually the case.

It has been shown that a system of quartic surfaces $F_4+\lambda F'_4$ contains a, b, c, d, e, f, and (rst). It is worth while noticing that given a, b, c, d, e, f, the twenty lines (rst) are the only lines not meeting a, b, c, d, e, f which have this property. For suppose two quartic surfaces meet in l_1, l_2, l_3, l_4 and α, β, γ; then the rest of their curve of intersection must be three cubics with these seven lines as common chords. To prove this apply the transformation of § 1 with respect to l_1, l_2, l_3, l_4. The quartic surfaces remain quartic surfaces, and meet in l_1, l_2, l_3, l_4 and T_α, T_β, T_γ, the three cubics into which α, β, γ have been transformed. The remainder of their common curve is of the third degree, and must cut each of T_α, T_β, T_γ in six points, since before transformation the rest of the curve of intersection met α, β, γ in six points each. These conditions can only be satisfied if the curve splits up into three non-intersecting lines, all of them chords of T_α, T_β and T_γ. On reversing the transformation the required result is obtained.

It has been shown that there is a family of quartic surfaces containing T_1, T_2, T_3 and a, b, c, d, e, f. Now use the last condition, and make the surface contain T_4 as well. The resulting quartic contains a, b, c, d, e, f, (234), (341), (412), and (123). But these ten lines are the ten common axes of T'_5 and T'_6. We have accordingly proved that the ten common axes of two cubics lie on a quartic surface. This surface obviously contains the ten cubics circumscribed about T'_5 and T'_6. Each of the cubics has as chords all of the lines but one, and *vice versa*.

The fact that these ten lines lie on a quartic surface has, I believe, been proved by Reye.* This surface is the locus of points from which the six osculating planes to the two cubics touch a cone of the second degree. From this point of view it is quite easy to see that both the ten lines, and ten cubics lie on the surface, if their existence is assumed.

Cayley's Problem of the Seven Lines.

6. The bearing of the foregoing work on Cayley's problem of seven lines will now be shown. A quartic surface cannot be drawn to contain seven lines chosen at random. There is one coefficient too few. If six of the lines are fixed, the result must satisfy a condition in order for a quartic surface to contain them all. Cayley proved† that unless the seventh line met one of the six it would have to belong to a quartic complex. The problem is to find properties of this complex. The corresponding problem of five lines on a cubic surface leads to the result that the fifth line belongs to a quadratic complex. In this case the complex is known to split up into the two aggregates of lines meeting either one of the two lines meeting the original four. It was therefore conjectured that the complex arising from the quartic problem might be reducible. This is not in general the case. The following fact should be observed. The condition that five lines should lie on a cubic surface is precisely the same as the reciprocal condition that all the planes through any of the lines should touch a class cubic surface, or as we may say, the condition that the lines should lie on a class cubic surface. In both cases the condition is that there should be one line meeting all five lines, or else that two of the lines should meet. A corresponding property holds good for the quartic problem, that is to say, "If seven lines lie on a quartic surface they lie on a class quartic surface also." We shall begin by taking six general lines, that is to say, six lines that have no particular properties, and show how to deal with special cases afterwards. The quartic complex to which a line must belong, if it is to lie on a quartic surface through the lines without meeting any of them, will be examined. It will be proved identical with the quartic complex obtained in the reciprocal manner from the condition that the seven lines should lie on a class quartic surface. Note that the twenty lines (rst) which have the property doubly in that they all serve for a

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[* *Crelle's Journal*, Vol. 82 (1877), p. 79, § 39. In case the two cubics have a common osculating plane the locus is a cubic surface, Reye, *Math. Ann.*, Vol. 55 (1902), p. 257. If in place of two cubic curves we have a line and five transversals of this, the locus of a point from which these six lines are projected by planes touching a quadric cone is the cubic surface containing the lines, a fact which is used below, § 7.]

[† "A Memoir on Quartic Surfaces," *Collected Papers*, Vol. 7, p. 178.]

whole system of surfaces and not for one only, are identical in the two cases, and this might lead us to expect the complete coincidence of the complexes. Before beginning the actual proof, it is necessary to observe that all chords of any of the cubics T_r belong to the first complex (C), and all axes of any of the cubics T'_r belong to the second complex (C'). For it is always possible to draw a quartic surface through a cubic and any seven of its chords, since $13+7.3 = 34$. We thus know eighteen of the lines of C in any plane, to wit, the sides of the six triangles in which the cubics T_r meet the plane.

Consider now an osculating plane of T'_1. The other osculating planes of T'_1 cut this plane in lines enveloping a conic, and the cubics T_2, T_3, T_4, T_5, T_6 (circumscribed about T'_1) meet the plane in triangles circumscribed about this conic. Hence, since the sides of these triangles, that is to say, fifteen tangents of the conic, belong to C, a quartic complex, all tangents to the conic belong to C. It follows that all axes of T'_1, and so of the other cubics T'_r, belong to C. Similarly all chords of the cubics T_r belong to C'. Hence in any plane the complexes C and C' have twenty-four lines in common, and accordingly they coincide if they are not degenerate. Suppose C splits up into two quadratic complexes; then the chords of T_1 and T_2, say, will all belong to the same quadratic complex, which is not true, as the two cubics do not lie on a conicoid. Suppose then C splits up into a cubic and a linear complex. Then the chords of the cubics T_r must all belong to the cubic complex. Considering the osculating plane of T'_1 above, we note that if this is so, one of the sides of the triangle, in which T_1 meets the plane, will touch the conic enveloped by the axes of T'_1, unless all the three sides meet in a point, that is to say, unless T_1 touches the plane. But if there is in every osculating plane of T'_1 an axis of T'_1 which is a chord of T_1, there must be an infinite number of such lines altogether, and this is impossible, since T'_1 is not inscribed in T, and the lines a, b, c, d, e, f do not in general belong to a linear complex (see § 2 (b)). If, again, every osculating plane of T'_1 touches T_1, then every tangent of T_1 lies in an osculating plane of T'_1, and every tangent of T'_1 meets T_1. In such a case (if such a case exists), it is quite easy to show from the sextic of reference of § 2 (b), that there are not six distinct lines not belonging to a linear complex, which are chords of T_1 and axes of T'_1. This point is really trivial. Hence in the general case the complex C is not degenerate. The complexes C and C' accordingly coincide.

The theorem has been proved, starting with what we called "six general lines." Let us enunciate what this means in detail. It is as follows:—"Given six lines in space, which do not belong to a linear complex, and six distinct genuine cubics, which have them all as chords, then if a quartic surface can be drawn through these six lines and a seventh, a class quartic surface can also be drawn to contain the seven lines." The general theorem which we set out to prove will now be deduced. Express lines by means of four coordinates x, y, z, t (not homogeneous), by some such means as by taking their points of intersection with two planes. Then if seven lines are given by the suffixes 1 to 7, the condition that they should lie on a quartic surface, is either that two of the lines should meet, or that a certain rational integral function F_4 of the twenty-eight variables (x_r, y_r, z_r, t_r), $(r, 1 \rightarrow 7)$ should vanish. F_4 is of the fourth degree in the coefficients of any one line. Similarly F_4' may be found from the reciprocal problem. It has been shown that in a certain case F_4 is irreducible. It is accordingly not identically reducible. Let G denote the rational function which vanishes when any two of the lines meet, and let H_r, K_r, L_r $(r, 1 \rightarrow 7)$ respectively denote the rational functions which vanish when the six lines formed by leaving out the line with the suffix r possess the following respective properties:—

(1) belong to a linear complex,

(2) have one degenerate cubic of which they are chords,

(3) have not six distinct cubics of which they are chords;

then, if $G \neq 0$, and we can find one set H_r, K_r, L_r none of which vanish, we have proved that $F_4 = 0$ necessitates $F_4' = 0$. Hence F_4, which is irreducible, must divide the product $F_4' G \prod_1^7 H_r K_r L_r$ (see below). But F_4 cannot divide G or any of the factors H_r, K_r, L_r, since we have found a case in which F_4 vanishes, and neither G nor H_r, K_r, L_r (for one value of r) vanishes. F_4 must accordingly coincide with F_4', since both are of the fourth degree in the coefficients of any one line. Since the only other sufficient condition for the seven lines to lie on a quartic surface is that two of them should meet, which is the same in the reciprocal case, it follows that *if any seven lines at all lie on a quartic surface, then they lie also on a class quartic surface.*

In the course of the preceding proof a theorem (the truth of which is obvious), of rational integral functions of n complex variables is assumed, to wit, that if whenever F (irreducible) vanishes, Φ also vanishes, then F must divide Φ. This is the sort of theorem that the geometrician has a right to expect the analyst to prove for him. It belongs to the same class, as a theorem giving a rigorous proof of the properties of a (1, 1)-correspondence, or of the number of points of intersection of two curves.

7. Certain properties of the chords of a cubic T and the axes of a cubic T' inscribed in it will now be given.

(*a*) If T' is inscribed in T, the chords of T, and the axes of T', belong to an infinite number of tetrahedral complexes, one arising from each tetrahedron that has its vertices lying on T, and its faces osculating T'.

(*b*) The chords of T, and the axes of T', are the only common lines of any two of these complexes.

(*c*) If the chords of T, and the axes of T' belong to the same quadratic complex, then T' must necessarily be inscribed in T.

(For consider the lines of the complex through any point of T.)

(*d*) If there is one tetrahedron with vertices lying on T, and faces osculating T', and one chord of T, apart from the edges of this tetrahedron, which is an axis of T', then T' must be inscribed in T.

(This follows immediately from (*c*).)

(*e*) If two quadratic (necessarily tetrahedral) complexes contain all the chords of T, the remainder of their common lines consists of the axes of a cubic T' inscribed in T.

(*f*) If there is one tetrahedron with vertices lying on T, which is self-conjugate with respect to a certain conicoid, then there is an infinite number of such tetrahedra, and T is reciprocated with respect to this conicoid into a cubic T' inscribed in T, which osculates the faces of all the tetrahedra.

Two particular cases of the theorem of the last paragraph are worthy of mention.

(*a*) If five of the six lines a, b, c, d, e, f, are met by two lines, any line which lies on a quartic surface through these eight lines must meet one of them. There are certain exceptions to this, in the form of the lines of certain reguli. The quartic complex of the general case here degenerates into the two lines which meet the five (a, b, c, d, e), and the line f taken twice.

(*b*) If a, b, c, d, e meet a line, and a, b, c, d, f meet another line, the quartic complex degenerates into these two lines, and a tetrahedral complex. e and f are opposite edges of the tetrahedron of this complex. Geometry connected with this case is given below, in the proof of Mr. Grace's theorem on six lines meeting a single line.

The theorem which we are going to prove may be stated thus :*

" l_1, l_2, l_3, l_4, l_5, l_6 are six lines meeting a single line l. l'_1 is the line of the Double Six containing l_2, l_3, l_4, l_5, l_6 which does not meet any of them. l'_2, l'_3, l'_4, l'_5, l'_6 are similarly found. The lines l'_1, l'_2, l'_3, l'_4, l'_5, l'_6 all meet another line l'."

It will appear from the proof that the configuration is reciprocated into itself with respect to a certain conicoid, each line denoted without a dash being reciprocated into the corresponding line with a dash. It is clear from the transformation of § 1, or otherwise, that there is just one non-degenerate cubic curve T, which has l_1, l_2, l_3, l_4, l_5, l_6 as chords.† It is also clear that any non-degenerate cubic which has l_1, l_2, l_3, l_4, l_5 as chords has l'_6 as a chord also. (This is obvious if the cubic is regarded as part of the curve of intersection of a cubic surface containing l_1, l_2, l_3, l_4, with the cubic surface through l_1, l_2, l_3, l_4, l_5, l'_6). Hence T has as chords l'_1, l'_2, l'_3, l'_4, l'_5, l'_6. Similarly there is a cubic T' with the twelve lines l_r and l'_r as axes. It will now be shown that T is circumscribed about T', and that l_r, l'_r are opposite edges of a tetrahedron with faces osculating T' and vertices lying on T.

It is well known that the lines l_1, l_2, l_3, l_4, l_5 and l are projected into six lines touching a conic, from any point on the cubic surface containing l_1, l_2, l_3, l_4, l_5. We shall here use the converse as well (which is not difficult to prove), namely, that if the lines are projected into lines touching a conic from a point in space, then that point lies on the cubic surface. It follows that from any point of T the seven lines l, l_1, l_2, l_3, l_4, l_5, l_6 are all projected into lines touching a conic, and that the points of T are the only points possessing the property, except for the points of l. Similarly, the osculating planes of T' meet l, l_1, l_2, l_3, l_4, l_5, l_6 in points lying on a conic, and are the only planes that have this property, except for planes that contain l.

Let λ be the line other than l which meets l_1, l_2, l_3, l_4. Then λ meets l'_5 as well. Suppose that T meets l'_5 in A and B, l_5 in C and D. Then we shall show that the plane ABC osculates T', and it will follow by

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[* See "Circles, Spheres, and Linear Complexes," by J. H. Grace, B.A., Fellow of St. Peter's College, *Transactions of the Cambridge Philosophical Society*, Vol. 16 (1898), pp. 153–190.]

[† In applying the transformation of § 1, for the case of six lines having a common transversal, the transformations of two of these six lines, by means of cubic surfaces containing the other four, are degenerate cubic curves, each containing as part of itself that common transversal of these other four which does not meet the two transformed lines. The remaining parts of these degenerate cubic curves are conics, and these have only one common chord, the intersection of their planes. But another proof may be given, as below.]

symmetry that the other faces of the tetrahedron $ABCD$ osculate T'. For, since the six lines l_1, l_2, l_3, l_4, l, λ, and the cubic T, form the complete intersection of two cubic surfaces, they meet any plane in nine points lying on two cubic curves. If then three of these points lie on a line the other six must lie on a conic. But the plane ABC meets T and λ in four points of which three are on the line l'_5. Hence the points in which this plane meets l_1, l_2, l_3, l_4, l, and the point C, lie on a conic. But C is on l_5. Hence ABC meets l, l_1, l_2, l_3, l_4, l_5 in six points on a conic. But ABC contains l'_5, which lies on a cubic surface containing l, l_1, l_2, l_3, l_4, l_5. Hence it meets these six lines in six points on a conic. It follows that the plane ABC meets l and all the lines l_r in seven points on a conicoid, and so osculates T'. In a similar way ABD osculates T', and, by interchanging l and λ, it can be shown that ACD and BCD osculate T'. Hence $ABCD$ is a tetrahedron with vertices lying on T and faces osculating T'. It immediately follows from § 7 (d), since there are other lines which are chords of T and axes of T', that T is circumscribed about T'.

l_r, l'_r are the opposite edges of tetrahedra circumscribed about T' and inscribed in T. But all these tetrahedra are self-conjugate with respect to a conicoid. There is therefore a conicoid with respect to which l_r reciprocates into l'_r. l will reciprocate into l', meeting all the lines l'_r. Also, as l_1, l_2, l_3, l_4, l'_5, l'_6 are met by a single line, so will be l'_1, l'_2, l'_3, l'_4, l_5, l_6. The lines l_1, l'_1, l_2, l'_2, etc., are such that, if one of each pair be taken, the six resulting lines are either met by a single line, or else form one half of the lines of a Double Six. There are thirty-two lines such as l which meet six of these twelve lines. These thirty-two lines are all the lines which are common to four quadratic complexes. If the lines l_r, l'_r are projected into a plane from a point of T, then the twelve lines touch a class cubic [see § 2 (b)], and any six of them which are the projections of lines meeting a single line touch a conic. The . . . .

[The MS. ends here.]

REMARKS ON MR. WAKEFORD'S PAPER

By H. F. Baker.

An attentive reader of Mr. Wakeford's very remarkable paper will wish to place the results which he obtains into relation with those of the existing literature dealing with the same matters. As this literature is extensive, and Mr. Wakeford's references are of the scantiest, the reader may be glad of the following remarks. The paper may be said to contain: (*a*) a consideration of a particular birational transformation of space, already dealt with by Hesse, Cremona, Noether, and Cayley; (*b*) two particular theorems apparently both first given by Cremona, of which the second is dealt with in an imperfect way by Cayley, also without references; namely, that two cubic curves of general position have ten common chords, and that six lines of general position are chords of six cubic curves; (*c*) the theory of the relations of the six cubic curves having six lines as chords, and the six cubic developables having these lines as axes, with application to the problem considered by Cayley of the condition necessary for seven lines, in order that they should lie on a quartic surface. Mr. Wakeford's contribution to the latter problem appears to be new, and the writer possesses letters from Mr. Wakeford showing that he valued it highly. But the relations of the six cubics and six developables are in intimate connexion with many papers written by Reye, dealing with *apolarity*. (*d*) A proof, which adds a most interesting additional fact, of Mr. Grace's theorem as to the new line deducible from six arbitrary lines having a common transversal. The account given by Mr. Wakeford appears somewhat hurried, and may well have been written out when he was on active service (the book containing the whole paper was found in his kit-bag). The following remarks, which are in many ways incomplete, are arranged under these four heads.

(*a*) Mr. Wakeford's account does not bring out that the corresponding points are conjugate to one another in regard to three quadric surfaces; and this does not seem to be mentioned by Cayley [*Coll. Papers*, Vol. 7 (1869), p. 234].

Let l, m, l', m' be any four lines of which no two intersect; take the two common transversals of these four, and upon each of these the two points which are harmonically conjugate as well in regard to l and l' as in regard to m and m'. With the four points so obtained as points of reference, the equations of the two transversals can be supposed to be $x = 0$, $y = 0$, and $z = 0$, $t = 0$. Choosing the ratio of y to x, and of t to z, properly, the equations of the lines l, m can then be supposed to be respectively, $y = px$, $t = qz$, and $y = x/p$, $t = z/q$; and thence the equations of l' and m' will, respectively, be $y = -px$, $t = -qz$ and $y = -x/p$, $t = -z/q$. Now consider the cubic surface represented, for any constant values of the ratios $A : B : C : D$, by the equation

$$\begin{vmatrix} A, & B, & -C, & -D \\ x, & p^2y, & z, & q^2t \\ p^2x, & y, & q^2z, & t \\ py, & px, & qt, & qz \end{vmatrix} = 0;$$

for all values of $A : B : C : D$ it is at once seen that this contains the four given lines l, m, l', m'. Regarding these ratios as variable the equation thus represents the general cubic surface containing these four lines. Further, if this equation be written in the form

$$Ax'+By'+Cz'+Dt' = 0,$$

this being a definition of the ratios $x' : y' : z' : t'$ in terms of $x : y : z : t$, we have the three identities

$$x \,.\, x'+p^2y \,.\, y'-z \,.\, z'-q^2t \,.\, t' = 0,$$

$$p^2x \,.\, x'+y \,.\, y'-q^2z \,.\, z'-t \,.\, t' = 0,$$

$$py \,.\, x'+px \,.\, y'-qt \,.\, z'-qz \,.\, t' = 0;$$

thus the two points, (x, y, z, t) and (x', y', z', t'), are conjugate to one another in regard to each of the three quadric surfaces

$$x^2+p^2y^2 = z^2+q^2t^2, \quad p^2x^2+y^2 = q^2z^2+t^2, \quad pxy = qzt,$$

and therefore, also, in regard to any quadric surface whose equation is a linear function of the equations of three quadrics. This shows that $x : y : z : t$ are precisely the same functions of $x' : y' : z' : t'$ as are these of the former.

It can be verified that these formulæ lead to

$$\frac{y'-px'}{t-qz} = \frac{t'-qz'}{y-px} = (p^2-q^2)(pqyt-xz)+(p^2q^2-1)(pyz-qxt),$$

and that the last expression, equated to zero, represents the quadric surface containing the three lines

$$(y=-px,\ t=-qz),\ (y=p^{-1}x,\ t=q^{-1}z),\ (y=-p^{-1}x,\ t=-q^{-1}z).$$

By changing p, q respectively into

$$(-p, -q) \text{ or } (p^{-1}, q^{-1}), \text{ or } (-p^{-1}, -q^{-1}),$$

we obtain three other such equations. The points where a line meets any one of the four quadric surfaces, each containing three of the four given lines, are thus transformed to the points where the corresponding cubic curve meets that one of the four lines which does not lie on that quadric surface.

For references to the transformation see Noether, *Math. Ann.*, Vol. 3 (1871), p. 548, who refers to Hesse (*Crelle*, Vol. 49), Steiner (*Crelle*, Vol. 53; *u. Monatsber. Ber. Ak.*, 1856), Geiser (*Crelle*, Vol. 69), Cremona (*Crelle*, Vol. 68).

(*b*) *The number of common chords of two cubic curves in space, in general position*, is given as ten by Cremona, *Crelle*, Vol. 60 (1862), p. 192, whose proof supposes one of the two cubics to degenerate into a conic and a line meeting this. The number is a particular case of that of the chords common to any two curves in space, also there given by Cremona, which is found in Cayley, *Coll. Papers*, Vol. 5 (1863), p. 171. The number may also be deduced, as Mr. Wakeford remarks, as a particular case of Halphen's formula for the number of lines common to two congruences (see, for instance, Jessop, *The Line Complex*, 1903, p. 259). But Halphen's formula appears more interesting when interpreted as a theorem for the number of points of intersection of two surfaces lying on a quadric fourfold in five dimensions (Bertini, *Geometria Proj. degli Iperspazi*, 1907, pp. 140, 141), being then a continuation of that for the number of common points of two curves lying on a quadric surface in three dimensions.

We may obtain the number of common chords of two curves by the Principle of Correspondence. For simplicity we suppose one at least of the curves to be rational. Let the curves be C_m, C_n, of respective orders m and n, the curve C_m being rational; let r, s be the number of chords of these curves, respectively, which pass through an ordinary point of space. To any point P, of the curve C_m we can make correspond $4rs$ points P', of the same curve, by drawing through P the s chords of the curve C_n, and, from each extremity of every such chord, drawing the r chords of the curve C_m, taking P' to be an extremity of one of these chords. The number of positions of P which give rise, in this way, to the same position of P', is to be found by remarking that the intermediate point Q, of the curve C_n, arises by the fact that the line $P'Q$ is a common generator of the cones, of vertex P', projecting from P' the two curves C_m, C_n, so that to P' correspond $(m-1)n$ positions of Q; while P is to be found from Q by the fact that QP is a common generator of the cones, of vertex Q, projecting from Q the two curves C_n, C_m, so that to Q correspond $(n-1)m$ positions of P. Thus between P, P' there is a correspondence of indices $[(m-1)n(n-1)m, 4rs]$. There are therefore (C_m being rational) $(m-1)n(n-1)m+4rs$ positions of P for which the corresponding P' coincides with it; when this is so, however, the chord from P to C_n coincides with a chord of C_m, and the two ends of this chord are two of the coincidences in question, each arising twice over according as the point Q of the chord of C_n is at either end of this. The number of common chords is therefore

$$\tfrac{1}{4}(m-1)n(n-1)m+rs.$$

For the case when C_m, C_n have h common points, the number of common chords not passing through these points is stated by Cremona, *Crelle*, Vol. 60 (1862), p. 192, to be

$$\tfrac{1}{4}mn(m-1)(n-1)+rs+\tfrac{1}{2}h(h-1)-h(m-1)(n-1).$$

The proof of the theorem for the number of lines common to two congruences by interpretation in five dimensions, is as follows: Any line (in three dimensions) is represented by a point in five dimensions, and the aggregate of all the lines in three dimensions is represented by all the points of a certain quadric fourfold in five dimensions. This quadric fourfold we denote by Ω. All the lines passing through any point of three dimensions are represented by the points of a plane, lying entirely on Ω; this we may call a plane of the first kind; all the lines lying in any plane in three dimensions are represented also by the points of a plane, lying entirely on Ω, which we may call a plane of the second kind. As is clear from the three dimensions, any two planes of Ω of the first kind have a point in common, as also have any two planes of the second kind. But a plane of the first kind has not in general a point common with a plane of the second kind; if it have, then these two have an infinity of points in common, lying on a line. A congruence of lines (l, l'), in three dimensions, of which l lines pass through an arbitrary point, and l' lines lie in an arbitrary plane, is represented, in five dimensions, by a surface lying entirely on Ω, having l points common with any plane of Ω of the first kind, and l' points common with any plane of Ω of the second kind. This reminds us then of a curve lying on a quadric surface in three dimensions, having α points common with every generator, of the surface, of the first kind, and α' points common with every generator of the second kind. It is known that two such curves of a quadric, (α, α') and (β, β'), have $\alpha\beta'+\alpha'\beta$ points common. Similarly, in five dimensions, two surfaces (l, l') and (m, m'), lying on Ω, have $lm+l'm'$ points common. And the result is capable of generalisation to space of any *odd* number of dimensions, the formula being derived from the two pairs of component indices in a different way according as the number of dimensions of the space is $\equiv 3$ or $\equiv 1$ (modulus 4).

To deduce the result for the space S_5 from that for the space of three dimensions S_3, let O denote any point of Ω; the tangent S_4 of Ω at O meets Ω in ∞^3 points, lying on ∞^2 lines, and lying on ∞^1 planes of each kind, all passing through O. Let σ denote any definite S_4 lying in the S_5. The planes of Ω through O, on the tangent S_4, meet σ each on a line; two planes of the same kind through O do not meet again, and give two skew lines on σ; two planes of different kind having O in common have a line in common, which meets σ in a point. The intersection of σ with the tangent fourfold at O lies on a S_3; in this, therefore, we have two systems each of ∞^1 skew lines, any line of either system meeting every line of the other system in a point; these two systems of lines, therefore, are those of a quadric surface lying in S_3. A surface on Ω, meeting the planes of Ω, of the two kinds, respectively, in l and l' points, which we denote by $F_2^{l,\,l'}$, when projected from O on to the space of four dimensions σ, becomes a surface therein; and this intersects the S_3 arising from the tangent fourfold of Ω at O, in a curve, lying on the quadric surface above spoken of, and meeting every generator of this quadric, of one system, in l points, and every generator of this, of the other system, in l' points. That $F_2^{l,\,l'}$ is a surface involves that, in S_5, it meets any threefold in a finite number of points; projected on to σ it gives a surface meeting a plane in the same number of points, that is a surface of the same order; in S_5, a threefold is determined, passing through O, by two planes of Ω of different kinds, meeting on a line through O; these planes meet σ in two lines containing, respectively, l and l' points of the surface, in σ, obtained by projection of $F_2^{l,\,l'}$, and these lines lie in a plane of σ. The surface obtained by projection of $F_2^{l,\,l'}$, in σ, is thus of order $l+l'$. If then we have, on Ω, two surfaces $F_2^{l,\,l'}$ and $F_2^{m,\,m'}$, their projections on σ have $(l+l')(m+m')$ points in common. Now consider how many of these arise from merely "apparent" intersections of $F_2^{l,\,l'}$ and $F_2^{m,\,m'}$, that is, by intersections of σ with lines through O which meet $F_2^{l,\,l'}$ and $F_2^{m,\,m'}$ in different points. Such a line will have three distinct intersections with Ω, and, therefore, lie entirely upon it; it will thus be on the tangent fourfold of

Ω at O, and will meet σ on the quadric surface above described lying in σ. The number of apparent intersections is therefore that of the intersections of two curves (l, l') and (m, m'), lying on this quadric, namely, is $lm'+l'm$. The number of actual intersections of $F_2^{l,\,l'}$ and $F_2^{m,\,m'}$ is therefore

$$(l+l')(m+m')-(lm'+l'm) \quad \text{or} \quad lm+l'm'.$$

This is the number given by what is known as Halphen's theorem.

That there exist six cubic curves having six given lines of general position as chords is proved by Cremona in the paper referred to, *Crelle*, Vol. 60 (1862), p. 191, as follows. If a, b, c, d, e be any five lines of general position, and a variable plane through e meet the other four lines, respectively, in A, B, C, D, the point of intersection, P, of the lines AB and CD describes a cubic curve having the five given lines each as a chord. The two ends of the chord lying on the line a form a pair of points on this cubic curve, which, with the two ends of the chord lying on the line b, define an involution of pairs of points of the curve; there is another involution similarly determined by the two pairs of points which are the ends of the chords c and d. The two ends of the chord on the line e are the pair of points common to these two involutions; for the transversal from P to a, b meets e, as does the transversal to c, d. Conversely, if five chords of a cubic curve a, b, c, d, e, be such that the chord e gives the pair of points common to the two involutions which are given, respectively, by the two pairs of chords a, b and c, d, then the plane of the two transversals, drawn from an arbitrary point of the curve, one of them to the chords a, b, and the other to the chords c, d, contains the line e. The proposition reciprocal to the foregoing is, that if a, b, c, d, and f be any five lines of general position, and, from a variable point of the line f, there be drawn the transversal to the lines a, b, and also the transversal to the lines c, d, then the plane of these two transversals describes a cubic developable of which the lines a, b, c, d and f are axes. In particular, in this reciprocal proposition, if a, b, c, d and f be chords of a cubic curve, it is clear, from what is said above, that the two planes of this developable, one through each end of the chord f of this cubic curve (other than the two planes of the developable which intersect in f), meet in the chord, e, of the cubic curve determined as giving the pair common to the two points involutions on the curve determined by a, b and by c, d. For the construction for such a plane, when we start from an end of the chord f, is that by which we may arrive at the chord e from any point of the cubic curve. The chord e is thus an axis of the cubic developable determined, as above, by a, b, c, d, and f. If, with f, we take the pairs a, c and b, d, we shall obtain a different developable, and, as axis of this, a different chord, e', of the cubic curve. The planes of this developable which meet in e' will however also pass through the ends of the chord, f, of the cubic curve. So for the pairs b, c and a, d.

Hence, if a, b, c, d, f, g be six given lines of general position, a cubic curve having them as chords will have as chord the line e which gives the common pair of the two involutions determined thereon by the chords a, b and by the chords c, d; and this line e will be an axis common to the two cubic developables determined as above by the two pairs of chords a, b and c, d, taken first with f, and then with g. The planes of these developables intersecting in the line e will determine four points of the cubic curve, two on the line f and two on the line g; when these points are found the curve can be constructed, for instance by axial pencils of planes through three of the chords a, b, c, d. Thus, to find the cubic curves having the six given lines a, b, c, d, f, g, as chords, we consider the ten common axes of the two cubic developables (a, b, c, d, f) and (a, b, c, d, g), of which however the four a, b, c, d do not determine points of the cubic curve. The other six lead each to such a cubic curve as desired, having the six given lines as chords. There are then six such curves.

Mr. Wakeford's solution leads to this conclusion in a much briefer way. By transformation with cubic surfaces passing through the lines a, b, c, d, the lines f, g become two cubic curves having, beside a, b, c, d, six other common chords say p, q, r, s, t, u; the transformations of these six chords are the six cubic curves desired, having a, b, c, d, f, g as chords. The study of the relations between the two figures would seem to be an interesting problem. Cremona's construction is interesting, too, in connexion with Mr. Wakeford's treatment of Mr. Grace's theorem.

It is supposed that the chords of the cubic curve to be constructed, which are given, are lines of general position. As examples of cases where this is not so we remark: (1) when the six given lines are such that every five of them have a common transversal, there is an infinite number (∞^4) of cubic curves having these lines as chords, all lying on the cubic surface which then exists containing the six lines; (2) when the six given lines, say $a_1, \ldots, a_6$, are such that there is a common transversal, b_1, of a_2, a_3 and a_4, a_5, a_6, and a common transversal, b_2, of the five a_3, a_1, a_4, a_5, a_6, and also a common transversal, b_3, of a_1, a_2, a_4, a_5, a_6, there is not in general any proper cubic curve having the six given lines as chords. There is a degenerate cubic curve consisting of the lines b_2, b_3 and the line joining the points (a_2, b_3), (a_3, b_2); there is also a degenerate curve consisting of the lines b_2, b_3 and the line of intersection of the planes $[a_2, b_3]$, $[a_3, b_2]$. And there are four other degenerate curves of similar descriptions. When a_1, a_2, a_3, b_1, b_2, b_3, and a_4, are given, it is possible to choose b_5, b_6, however, so as to have case (1). Lastly, (3), if the six given lines have one common transversal, there is, as Mr. Wakeford remarks, only one proper cubic curve having them all for chords. In this case, there are ∞^2 degenerate curves, each consisting of the common transversal and a conic which meets this and meets the six given lines.

(c) *In regard to apolarity.*—When a quadric locus f contains the vertices of a tetrahedron which is self-polar in regard to a quadric envelope ϕ, we shall say that f is *out-polar* to ϕ, and ϕ is *in-polar* to f. When every quadric locus which contains a cubic curve is out-polar to a quadric envelope ϕ, we shall say that the cubic curve is out-polar to ϕ. A sufficient condition for this is that four points of the cubic curve should be the vertices of a tetrahedron self-polar in regard to ϕ. Similarly a cubic developable will be said to be in-polar to a quadric locus f when every quadric envelope containing the planes of the developable is in-polar to f. When an algebraic series of sets of four points upon a cubic curve is such that one of the sets is determined by the assignment of any one of its four points, the series is given, by the variation of λ, from an equation of the form $U+\lambda V=0$, where U, V are two quartic polynomials in the parameter by which the points of the curve may be represented. In this case the tetrahedra formed by the sets of four points are all self-polar in regard to a quadric, and the faces of these tetrahedra are planes of a cubic developable. The quadric, regarded as an envelope, is then in-polar to the cubic curve, and regarded as a locus is out-polar to the cubic developable. We shall speak of the cubic curve as out-polar to the cubic developable. Mr. Wakeford speaks of the cubic developable, or rather of the cubic curve of which the planes of the developable are the osculating planes, as being inscribed to the cubic curve. For these matters reference may be made to Reye, *Crelle*, Vol. 82 (1877); cf. also *Crelle*, Vol. 77 (1874). One particular result may be remarked, as of importance below: when a cubic curve is out-polar to a quadric envelope reducing to two points, these points are on a chord of the cubic and harmonic in regard to the curve.

Now suppose that we start with four linearly independent quadric loci f_1, f_2, f_3, f_4. These are all out-polar to a linear aggregate of six quadric envelopes $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6$, which is determined by f_1, f_2, f_3, f_4. Conversely the linear system of loci f_1, f_2, f_3, f_4 is determined when $\phi_1, \ldots, \phi_6$ are given. The theory of the relations of this system and aggregate is considered by Reye, *Crelle*, Vol. 82 (1877), p. 75. In the system of loci there are quadric cones, each of the form

$$c_1f_1+c_2f_2+c_3f_3+c_4f_4 = 0,$$

∞^2 in number, and there are plane pairs, finite in number. Among the aggregate of envelopes there are conics, ∞^4 in number, and there are point-pairs, ∞^2 in number. Consider, first, three of the quadric loci, say f_1, f_2, f_3. Take the point P', which is the intersection of the polar planes, in regard to these three loci, of another point P. As P moves along an arbitrary line, P' will describe a cubic curve. The polar plane of P in regard to the fourth quadric f_4, meets this curve in three points. There is thus a correspondence (1, 3) between two points of this curve. We are able to infer that there are four positions of P upon the line for which the polar planes of P, in regard to the *four* quadrics f_1, f_2, f_3, f_4, meet in one point. The locus of points P, for which this property holds, is thus a quartic surface $J(f)$, containing pairs of corresponding points P, P'. And each of these points is a vertex of a quadric cone belonging to the linear system of point quadrics f_1, f_2, f_3, f_4. The pair of corresponding points is also, clearly, a point pair of the linear aggregate of quadric envelopes $\phi_1, \ldots, \phi_6$, in-polar to the system $f_1, \ldots, f_4$. A particular case of a quadric cone is a pair of planes; thus the line of intersection of every pair of planes of the system $c_1f_1 + \ldots + c_4f_4 = 0$ lies on the quartic surface $J(f)$; conversely every line lying on this surface is the intersection of such a pair of planes, for every point of such a line is a vertex of a cone of the system. The theory is considered by Cayley, *Coll. Papers*, Vol. 7 (1870), p. 168.

To obtain the number of lines lying on the surface $J(f)$ we may, therefore, enquire what is the number of plane pairs out-polar to six quadric envelopes. If we suppose that this number is the same as for six quadric envelopes which all reduce to double points, the number will be that of the pairs of planes containing six points, namely *ten*. The number can, however, be obtained as a consequence of a theorem which is otherwise of importance here. It can in fact be shown that *there are two cubic developables which are both in-polar to all the quadric loci of the system* $c_1f_1 + \ldots + c_4f_4 = 0$. This means, as we have explained, that the three quadric envelopes, whose common planes (other than those through a common line) constitute one of these cubic developables, are all in-polar to every quadric locus of the system. The theorem is given by Reye, *Crelle*, Vol. 82 (1877), p. 78, § 38; the developables T_5', T_6' of Mr. Wakeford's paper (§ 4) are a case. By this theorem, then, six particular quadric envelopes are obtained (three from each of the cubic developable), and the system $c_1f_1 + \ldots + c_4f_4 = 0$ may thence be defined as that of the quadric loci out-polar to these six envelopes. Assuming the theorem for a moment, and remarking that a quadric locus consisting of a pair of planes out-polar to a cubic developable will intersect in an axis of the developable, and be harmonic in regard to the planes of the developable passing through this axis, we see that every plane pair of the system (cf) has for intersection one of the ten common axes of the two developables, so that the system (cf) has at most ten plane pairs. Conversely, through every axis common to the two cubic developables can be drawn a definite pair of planes, harmonic in regard to the planes of each developable passing through this axis; this pair of planes will then be out-polar to both developables, and therefore be a surface of the system (cf). There are then just ten plane pairs belonging to the system (cf), and just ten lines on the surface $J(f)$.

To prove the theorem assumed, that there are two cubic developables in-polar to all the surfaces (cf), we take the reciprocal theorem, that there are two cubic curves out-polar to an aggregate of four quadric envelopes $\gamma_1\psi_1+\ldots+\gamma_4\psi_4=0$. For this purpose we suppose the aggregate to be given by four linearly independent point pairs. Then the theorem is: *given four lines and on each of them an involution of pairs of points, two cubic curves exist having these lines as chords, meeting each in a pair of the involution determined thereon.* Let a, b and c, d be the given lines, which we arbitrarily divide into two pairs; through any cubic curve having these lines as chords can be described two quadric surfaces, one to contain the chords a, b, and the other to contain the chords c, d; these quadrics will intersect further in a chord of the curve, which we denote by p. Conversely, if p were given, the curve could be obtained as the residual intersection of the quadrics (a, b, p) and (p, c, d). A point of the cubic curve on the chord a, being on the quadric (p, c, d), is on a line meeting these last three lines; thus the curve meets the line a on the two transversals of the four lines a, p, c, d. Consider the quadric surface $(a\,;\,c, d)$, and the generators of this of the system other than these lines; by the involution supposed to be given on the line a, the generators of this regulus are arranged in pairs in involution. By the condition for the intersection of the cubic curve with the line a, the line p is, thus, to cut the quadric $(a\,;\,c, d)$ on two generators forming a pair of the involution of generators of this regulus. But, when the generators of one system of a quadric surface are arranged in involution, the lines cutting the quadric in points lying on pairs of this involution are the lines of a linear complex (as is shown below). Thus, the line p, in consequence merely of the fact that the cubic curve cuts the line a in pairs of a given involution thereon, belongs to a definite linear complex. By considering similarly the quadric surfaces $(b\,;\,c, d)$, $(a, b\,;\,c)$, $(a, b\,;\,d)$, we see that p is defined as a line common to four linear complexes. Without entering into further enquiry as to the generality of these complexes, we may therefore infer that there are two lines p, and hence two cubic curves, as stated.

To prove that the lines, meeting the generators of one system of a quadric in points lying on pairs of a given involution of these generators, are the lines of a linear complex, it may be sufficient to remark that lines, with this property, which lie in a plane, intersect a conic in this plane in pairs of points of an involution on this conic, and so pass through a point; while lines, with this property, which pass through a point, each lie on two tangent planes of an involution of such planes of a quadric cone, and so lie in a plane. Analytically, for the quadric $xy = zt$, the generators $x = \lambda z$, $t = \lambda y$, and $x = \lambda' z$, $t = \lambda' y$, belong to an involution, if

$$a\lambda\lambda' + b(\lambda + \lambda') + c = 0,$$

with constant values of $a : b : c$. The lines meeting such pairs belong to the linear complex given by

$$al + b(n + n') + cl' = 0;$$

this contains one system of generators of the quadric $xy + zt = 0$, given by

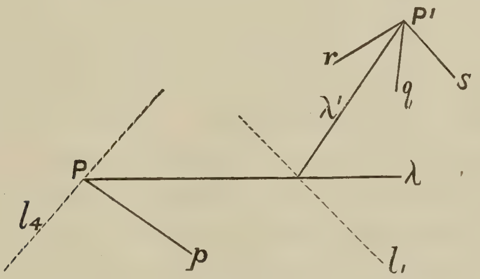
$$l = 0, \quad l' = 0, \quad n + n' = 0.$$

It is thus proved that there are two cubic developables both in-polar to all the quadric loci of the system $c_1 f_1 + \ldots + c_4 f_4 = 0$. In particular, these are in-polar to all the quadric cones of this system, one of which has a vertex at any point of the quartic surface $J(f)$. Thus, the three planes of either of these cubic developables which pass through any point of this surface, form a *three-face* which is self-polar in regard to the quadric cone having this point as vertex. The three planes of the other developable through this point have the same property. Therefore these six planes touch a quadric cone. And the quartic surface $J(f)$ can thus be obtained as the locus of a point such that the six planes of two cubic developables, which pass through this point, touch a quadric cone, a result due to Reye (see the references given in the text, above).

The quartic surface $J(f)$ contains also ten cubic curves, each the locus of the points corresponding to the points of any one of its ten lines, taken in respect to any three surfaces of the system (cf); in particular, by taking the correspondence in regard to any three of the plane pairs of this system, we see that the cubic curve corresponding to any one of the lines has the other nine lines as chords (Cayley, *Coll. Papers*, Vol. 7, p. 168). The surface may be identified as containing any six of the ten lines and four of these cubic curves, those namely having these six lines as chords.

At this point we can make contact with Mr. Wakeford's paper. Suppose that six lines of general position are given. There are, we have seen, six cubic developables having these lines as axes. Choose, arbitrarily, any two of these six developables, say ϵ and ζ. Each will be determined by the aggregate of the tangent planes common to three quadric envelopes. Thereby six quadric envelopes arise. To these there is a system of quadric loci, out-polar thereto, of which four are linearly independent. From this system arises a quartic surface; this quartic surface contains the ten common axes of the two cubic developables ϵ and ζ, and, in particular, the six given lines; and a plane pair belonging to the system of quadric loci may be found as the two planes, through any one of the six given lines, which are harmonic in regard to the pair of planes, of both of the two developables ϵ and ζ, intersecting in this line. The quartic surface will contain four of the six possible cubic curves having the six given lines as chords, namely those which correspond, in respect of the system of quadric loci, to the four other lines of the quartic surface; it will also contain six other cubic curves, each having five of the six given lines as chords.

To make still clearer the identification of the figure considered with that considered by Mr. Wakeford, we may give a direct proof that every one of the ten cubic curves of the quartic surface is out-polar to both the developables ϵ and ζ. Let l_4, l_1, l_2, l_3 be any four of the ten axes common to the developables ϵ and ζ; let the planes, of the system of quadric loci, passing through these be respectively denoted by (α_4, β_4), (α_1, β_1), (α_2, β_2), (α_3, β_3). Let P be any point of the line l_4; through P there passes one plane, say p, of the developable ϵ, beside the two intersecting in l_4. Conversely, any plane, p, of the developable ϵ, meets l_4 in a point P. Join P to the lines l_1, l_2, l_3, respectively, by the planes λ, μ, ν; let λ' be the plane through l_1 harmonically conjugate to λ with respect to α_1 and β_1, and, similarly, μ' the plane through l_2 harmonically conjugate to μ with respect to α_2 and β_2, and ν' the plane through l_3 harmonic to ν with respect to α_3



and β_3. Let P' be the intersection of the planes λ', μ', ν'; and let q, r, s be the planes of the developable ϵ which pass through P'. By the construction, P' is the point of the quartic surface corresponding to P, and describes a cubic curve as P describes the line l_4. Let t, u be the planes of the developable ϵ which intersect in l_4. The six planes of ϵ, denoted by p, q, r, s, t, u, are touched by four linearly independent quadric envelopes; four such envelopes, however, are the three which determine ϵ, together with all the planes through the point-pair P, P'; and each of these four is in-polar to the system of quadric loci, founded, for instance, on the four plane pairs (α_1, β_1), (α_2, β_2), (α_3, β_3), (α_4, β_4), by what we have seen. Wherefore, every quadric envelope containing the six planes p, q, r, s, t, u, being a linear function of the four particular envelopes which have been specified, is equally in-polar to the system of quadric loci. In particular, the envelope, consisting of all the planes through the point-pair (p, r, s) and (q, t, u), is so in-polar, and so on. The four planes p, q, r, s are thus entirely symmetrical to one another, and either of them determines the other three by a construction similar to that by which q, r, s have been deduced from p, the planes t and u remaining the same. But an algebraic series of sets of four planes, belonging to the developable ϵ, of which any set is equally well determined by any one of its four planes, forms a series of tetrahedra all self-polar in regard to the same quadric, and the locus of intersection of three planes of the set is a cubic curve. The quadric is at once out-polar to ϵ, and in-polar to the cubic curve, and this curve is out-polar to ϵ. Thus it appears that the cubic curve which is the locus of the point P' is out-polar to the developable ϵ. By similar reasoning it is out-polar to the developable ζ. And thus every one of the ten cubic curves is out-polar both to ϵ and ζ. This is the theorem referred to.

It would, however, be interesting to pursue the matter further, and determine the quadric which arises as the intermediary between the system of quadric loci and the developable ϵ, which is not the same, presumably, as that arising similarly for the developable ζ.

It now appears that, if six lines be given, by choosing out, arbitrarily, two of the cubic developables which have these for axes, we can obtain fifteen quartic surfaces (cf. Cayley, *Coll. Papers*, Vol. 7, p. 170, § 93), each containing the six given lines and four others, each containing four of the cubic curves which have the given lines as chords, as well as six other cubic curves, the ten cubic curves on any surface being all out-polar to the two developables which identify the surface. Moreover, of the four cubic curves in question, having the given lines as chords, any three have another common chord, one of the four remaining lines of the quartic surface. The total number of lines thus arising, beside the original six, is twenty; through each of these pass three of the quartic surfaces. Each quartic surface is a locus of vertices of cones belonging to a system of quadric loci, of which four are linearly independent; in all, in the complete figure, there arise twenty quadric loci, each out-polar to three cubic developables, from the six, which have the six given lines as axes.

(*d*) If we are given a line, t, here called the transversal, and five other lines meeting this, but not meeting one another, say a, b, c, d, e, then a cubic surface exists containing the six lines; thereon the lines a, b, c, d, e are five lines of one row of a double six of lines; we denote the remaining line of this row by f'. There are thirty-six double sixes, and seventy-two such rows of six lines; no two such rows have five lines in common. There are upon the surface seventy-two families of cubic curves, all those of one family meeting the lines of one row of a double six each in two points; thus every cubic curve upon the surface meeting each of a, b, c, d, e in two points has f' for a chord. A cubic curve belonging to a particular family can be drawn through two arbitrary points of the surface. The line f' can be constructed independently of the cubic surface; every four of the lines a, b, c, d, e has another transversal beside t; these five transversals all meet another line; this is the line f'. To every one of these statements there is a reciprocal. A class cubic envelope exists containing the lines t, a, b, c, d, e, in regard to which there exists a line f', constructed in reciprocal manner, and therefore coinciding with f'. There are cubic developables, whose planes belong to those of the class cubic, having each of a, b, c, d, e as axes, and of these f' is also an axis; and so on.

Now consider six skew lines a, b, c, d, e, f having a common transversal t; leaving aside the line f, which, in this connexion, we may speak of as the omitted line, there is, as above, a cubic surface containing the other lines, and, upon this, the line f', which, in this connexion, we may call the completing line. This cubic surface is met by the line f in two points, not upon the transversal t; through these points can be drawn, upon the surface, a cubic curve having a, b, c, d, e as chords, which also has f' as chord. The same cubic curve is evidently obtained if any other of the lines a, b, c, d, e, f is taken as omitted line, for a cubic curve lies entirely upon a cubic surface which contains five of its chords, and the curve is identified by its intersections with the omitted line; this curve has therefore six chords such as f', these being the completing lines for the six cubic surfaces constructed in the way explained. The theorem under consideration relates to these six completing lines. Denote this cubic curve by θ. Reciprocally, there is a unique cubic developable, Θ, having $a, b,$

c, *d*, *e*, *f*, and these same six completing lines, as twelve axes. The points of the cubic curve θ are those points of space from which the seven planes to the lines *t*, *a*, *b*, *c*, *d*, *e*, *f* touch a quadric cone; the planes of the developable Θ are the planes meeting these seven lines in points of a conic; of these statements, made by Mr. Wakeford, a direct proof is given below.

Taking the cubic surface which omits the line *f*, any plane through the completing line f', meets the original transversal *t*, the five lines *a*, *b*, *c*, *d*, *e*, and the cubic curve, each in one point not on this completing line; these points lie on a conic, the residual intersection of this plane with the cubic surface. If the plane be taken to pass through either of the points in which the cubic curve θ meets the line *f*, the plane will meet the seven lines, other than f', namely *t*, *a*, *b*, *c*, *d*, *e*, *f*, in points lying on a conic, and will therefore be a plane of the cubic developable Θ, of which the two planes passing through the axis f' are thus constructed. Reciprocally, considering the cubic envelope which omits the line *f*, and the planes of this which pass through any point of the completing line f' (these being reciprocal to the points of the cubic surface lying on any plane through f'), these planes touch a quadric cone; of such planes are the planes, from the point, containing the lines *t*, *a*, *b*, *c*, *d*, *e*, and the plane, of the cubic developable Θ, through the point (other than the two planes of this developable which contain the axis f'). If, in particular, the point taken on the line f' be one of the two points in which this line is met by the planes, of the developable Θ, through the axis *f*, these being, like all planes through this line planes of the cubic envelope, the point will be one from which the planes to the seven lines *t*, *a*, *b*, *c*, *d*, *e*, *f* all touch a quadric cone; thus, this point, and, so, both the points in which the planes, of the cubic developable Θ, through *f*, meet the line f', lie on the cubic curve. We have thus shown that (1) the planes joining the completing line (f') to the two points where the cubic curve θ meets the omitted line (*f*), (2) the planes joining the omitted line to the two points where this cubic curve meets the completing line, are all planes of the cubic developable Θ. By taking each of the lines *a*, *b*, *c*, *d*, *e*, *f* in turn as omitted line, we thus have six tetrahedra inscribed in the cubic curve, whose faces are planes of the cubic developable. Now if two tetrahedra have their vertices on a cubic curve there is a quadric surface in regard to which they are both self-polar, and conversely, a cubic curve which contains seven of the vertices of two tetrahedra self-polar in regard to a quadric, contains the remaining vertex. Taking then the quadric surface which has two of the six tetrahedra above as self-polar, the cubic developable, which contains the eight faces of these, is identified as being the polar reciprocal of the cubic curve in regard to this quadric surface. The developable is thus in-polar to the curve, and the other four tetrahedra are self-polar in regard to the quadric. The lines a', b', c', d', e', f' are thus the polar lines, respectively, of a, b, c, d, e, f; they are therefore met by another line, which is the polar line of the original transversal t, in regard to the quadric. This completes the proof of the Grace-Wakeford theorem.

Mr. Wakeford proceeds somewhat differently. The completing line is a common transversal of the transversals of every four of the selected lines; let l_1, l_2, l_3, l_4, l_6 denote the selected lines, the omitted line being l_5, and the completing line l'_5; let λ be the second transversal, beside the original transversal of l_1, l_2, l_3, l_4, so that λ meets l'_5. It we take as selected lines, however, l_1, l_2, l_3, l_4, and l_5, we obtain a completing line l_6, which is equally met by λ. The line λ thus meets l_1, l_2, l_3, l_4, l'_5, and l'_6, and we have a double six of lines

$$\begin{array}{cccccc} l_1 & l_2 & l_3 & l_4 & l'_6 & l_5 \\ m_1 & m_2 & m_3 & m_4 & l & \lambda \end{array}$$

lying on the cubic surface determined by l_1, l_2, l_3, l_4, l'_6, and λ. Regarding the six lines l_1, l_2, l_3, l_4; l'_5, l'_6, with their transversal λ, as fundamental, and taking the same cubic curve and cubic developable as before, the line l_5 appears as the completing line when l'_5 is the omitted line. Wherefore the two planes through the completing line l_5, one to each of the points where the cubic curve meets the omitted line l'_5, are planes of the cubic developable.

Now consider the theorem assumed in the proof, which we take in its reciprocal form: *if a conic meet a line and six other skew lines which meet this, the plane of the conic is that of a cubic developable having the six lines as axes.*

On the common transversal take two arbitrary points (t), (z); through these draw two arbitrary planes not containing the transversal, as well as two arbitrary planes containing the transversal, thus obtaining a tetrahedron $txyz$, of which the vertices (x) and (y) are those not on the transversal.

To represent the points of an arbitrary line meeting the transversal tz, we may take two points $(p, q, r, 0)$ and $(p, q, 0, s)$ on this; then any point of the line is $(p, q, \rho r, \sigma s)$, where ρ, σ are variable, subject to $\rho+\sigma = 1$. A conic meeting the transversal lies on two cones, of equations of the forms

$$z(Ax+By)+ ax^2+ 2hxy+ by^2 = 0,$$

$$t(Ax+By)+a_1x^2+2h_1xy+b_1y^2 = 0,$$

which both contain the transversal, and touch along this. Denoting $ax^2+2hxy+by^2$ and $a_1x^2+2h_1xy+b_1y^2$ respectively by $f(x, y)$ and $f_1(x, y)$, and $f(B, -A)$, $f_1(B, -A)$ respectively by f and f_1, the quadric surface

$$(zf_1-tf)(Ax+By)+f(x, y)f_1-f_1(x, y)f = 0$$

contains the curve of intersection of the two cones, but breaks up into $Ax+By = 0$ and another plane; this latter is then that of the conic. *After division by* $Ax+By$ *the left side is homogeneously of the third degree in* A, B, a, h, b, a_1, h_1, b_1; it is in fact capable of the form

$$zf_1-tf+x\frac{af_1-a_1f}{A}+y\frac{bf_1-b_1f}{B}.$$

Now the condition that this conic should meet the line, meeting the transversal tz, whose points are given, with varying ρ, σ (subject to $\rho+\sigma = 1$), by $(p, q, \rho r, \sigma s)$, is that the equations

$$\rho r(Ap+Bq)+f(p, q) = 0, \qquad \sigma s(Ap+Bq)+f_1(p, q) = 0,$$

should lead to the same values for ρ, σ; this is merely that $F = 0$, where

$$F = Ap+Bq+\frac{1}{r}f(p, q)+\frac{1}{s}f_1(p, q) = 0,$$

and this is a single linear homogeneous equation for the ratios of A, B, a, h, b, a_1, h_1, b_1. If the conic is to meet six such given lines meeting the original transversal, these eight coefficients will each be expressible as a linear function of a parameter θ. The plane of the conic then appears in a form in which its coefficients are cubic functions of the parameter θ. This proves that the planes are those of a cubic developable.

Further, the conditions for the plane of the developable to contain the line $(p, q, \rho r, \sigma s)$ are at once seen to be $\omega = 0$, $\Omega = 0$, where

$$\omega = \frac{1}{r}f+\frac{1}{s}f_1, \quad \Omega = AB+\frac{1}{r}(pBa+qAb)+\frac{1}{s}(pBa_1+qAb_1);$$

and these are such that

$$ABF+pq\omega = (Ap+Bq)\Omega.$$

The condition for the plane to contain the point $(0, 0, r, -s)$, where the line meets the transversal, is $\omega=0$ alone. When $Ap+Bq$ does not vanish, the condition $\omega = 0$, with $F = 0$, leads to $\Omega = 0$; and, as Ω is of the second degree in A, B, a, b, a_1, b_1, this leads to two planes containing the line. When $Ap+Bq = 0$, the conditions $\omega = 0$, $F = 0$ are identical, and do not involve $\Omega = 0$; the corresponding plane does not contain the line. We thus see that each of the six lines is an axis of the developable.

We may in fact use this analysis to dispense with any previous knowledge of the cubic surface. If the variable plane be required only to meet the transversal and *five* of the lines in points lying on a conic, the A, B, a, h, b, a_1, h_1, b_1, will thereby be determined as homogeneous linear functions of three parameters, which we may think of as coordinates in a fixed plane, ϖ. The coordinates of the variable plane,

$$\xi = \frac{af_1 - a_1 f}{A}, \quad \eta = \frac{bf_1 - b_1 f}{B}, \quad \zeta = f_1, \quad \tau = -f,$$

correspond then to four cubic curves in the fixed plane, ϖ, having six common points; these are the five common points of the curves $f = 0$, $f_1 = 0$, other than the common double points of these, together with this point counted once. Thus we see that the variable plane is that of a cubic envelope, which is at once seen to contain the transversal and the five lines.

Appendix.—The following are transcripts of four letters sent to the present referee by Mr. Wakeford. The last is inscribed simply Trin. Coll., Sunday—and the date is uncertain. It was probably written just after Mr. Wakeford's leaving the hospital, when he came home with his first wound. The first three are written from Badajoz Barracks, Wellington Lines, Aldershot (xvii Leicestershire, 7th Service Battalion), and are of respective dates 10 Nov. 1914, 17 Jan. 1915, 27 Jan. 1915. The first gives a summary of the results of the present paper. There was reason to fear that this paper was lost, for some considerable time.

(1) "Dear ——, I am taking the liberty of writing to you, since you once encouraged me to come to you about Geometry, and I shall not be up at Cambridge this term. In the Long Vacation I was considering twisted cubics with 6 given lines as chords. This led me on to consider Cayley's problem as to the condition that 7 lines should lie on a quartic surface. I proved that if 7 lines lie on a quartic surface, then a surface of class four can be drawn touched by all planes through any of the seven lines. The relation is, in fact, reciprocal. I enclose a list of the results. If you could let me know if much work has been done before on them, or if you think any of them interesting, I should be very grateful. The theorem about the 7 lines on a quartic surface seems to be true, not only in the general case, but in every case under the sun.—Yours faithfully, E. K. Wakeford (Trin. Coll., Camb.)."

"Results.

"(1) Given six lines there are *six* t cubics T_r $(r, 1 \rightarrow 6)$ which have them as chords, and six t cubics T'_r, so that the six lines lie in two osculating planes of each of them.

"(2) An ∞ of tetrahedra can be described with vertices on T_r and faces osculating T'_s $(r \neq s)$.

"(3) There are ${}^6C_3$ lines, *e.g.* (136), so that (136) is a chord of T_1, T_3, T_6, and lies in two osculating planes of T'_2, T'_4, and T'_5. These 20 lines are the only lines not cutting the original six, such that two quartic surfaces pass through the original six, and any one of the 20 lines.

"(4) If three cubics have 6 common chords they have seven. If 7 lines are common chords of 2 cubics they are chords of a third cubic also (these can be obtained from considering systems of quartic surfaces). The 10 lines which lie in two osculating planes of both of two twisted cubics, lie on a quartic surface; *E.g.* the six lines and (123), (234), (341), (412), (obtained, I think, by Reye, probably in another way).

"(5) It is clear that a quartic can be drawn through any seven chords of a twisted cubic $(13+21 = 34)$, *e.g.* through the original six lines and a chord of T_1. By considering an osculating plane of T'_1, we show that any line lying in two osculating planes of T'_1, and sim. of T'_2, etc., belongs to the quartic complex of lines which are such that a quartic can be drawn through the original six lines and the line in question. From this it follows immediately that the reciprocal property is true for all such lines, *i.e.* that a class quartic surface can be drawn touched by any plane through any of the six original lines and the line in question. Hence, *if a quartic can be drawn through seven lines, one can be drawn through their reciprocals with respect to a conicoid or linear complex.*"

(2) "Dear ——, I wrote to you last term about some mathematical results, and you were good enough to say that you would let me hear in a few days. As I have heard nothing since, I imagine that you may have forgotten about it, and am therefore writing again, in case you should have time to give me any information about the work.—Yours very truly, E. K. Wakeford."

(3) "Dear ——, I hope I have not seemed impatient about the cubics; I was very glad to get your letter. I have not written all the proofs out yet, in fact I have not written any out since September. I

shall do so now and in case they are published, shall feel very grateful for your kind offer about proofs. I have not read Cayley's paper upon quartic surfaces unless it is the one in which he deals with the "7 line" problem, and proves that given 6 lines the 7th belongs to a quartic complex. I don't suppose I shall leave for the Front till the end of March.—Yours very truly, E. K. Wakeford."

(4) "Dear ——, I got the MS. quite safely, and have collected a certain number of references. I will write the paper out shortly and send it in. Possibly I shall first do the work on Canonical Forms I mentioned to you. The theorem $F^m \equiv \Sigma FF'$ I found proved by Hilbert in *Göttinger Nachrichten*, 1892. I wonder that it has not been brought more into prominence as it seems to be important.—Yours very truly, E. K. Wakeford."

[The concluding portion of the paper on "Canonical Forms", here referred to, has been edited by Mr. Grace, *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1920), p. 403.]

RELATION BETWEEN APOLARITY AND CLEBSCH'S MAPPING OF THE CUBIC SURFACE ON A PLANE

By WILLIAM P. MILNE.

[Received December 3rd, 1921.—Read December 15th, 1921.]

1. *Introduction.*

In a paper "On the Geometry whose Element is the 3-Point of a Plane" (*Transactions of the American Mathematical Society*, Vol. 5, No. 4, pp. 467–476, October 1904), Morley obtained the following result:—

If a pencil of cubic curves circumscribe two triangles and be apolar to each of them, the curves intersect in three further points, which form a triad apolar to every member of the pencil.

In the *Proceedings of the Edinburgh Mathematical Society*, Vol. 30, Session 1911–12, I obtained the same result (being unaware of Morley's paper until my attention was called to it by Mr. H. W. Richmond), and have frequently employed it as fundamental in my subsequent investigations into the applications of apolarity to the theory of cubic curves. In both papers the method of establishing the theorem was entirely algebraic and shed little light on the real essence of the problem. In conversation with Mr. Grace, it seemed desirable to obtain a proof which would remove this defect, and Mr. Richmond suggested that as the vertices of two triangles were involved, giving six points in all, it might be possible to utilise Clebsch's method of mapping out the cubic surface on a plane (*Crelle*, Vol. 65, 1866), and then to regard every plane section of the cubic surface as corresponding to a cubic curve through the six base-points of the plane. Acting on this suggestion, I have been able to prove the theorem synthetically and also to obtain several new results with regard to the geometry of the 27 lines on the cubic surface, particularly with reference to Steinerian Trihedra (Steiner, "Über die Flächen dritten Grades," *Crelle*, Vol. 53, 1857).

2. *Synthetic Proof of the Fundamental Theorem.*

Let the cubic surface Π be mapped out on the plane π according to Clebsch's method (*Crelle*, Vol. 65, 1866), so that a (1, 1)-correspondence exists between the points of the surface Π and the plane π. Let A_1, A_2, A_3, A_4, A_5, A_6 be the six base-points of the plane π to which correspond respectively on the surface Π the six lines a_1, a_2, a_3, a_4, a_5, a_6, which are known to form half of a double-six. Then we know that a (1, 1)-correspondence exists between the points of a cubic curve S through the base-points and the corresponding points of the plane cubic Σ in which the surface Π is cut by the plane σ that corresponds to S.

We shall now establish a result that will be required later on:—

If a (1, 1)*-correspondence exists between the points of two cubic curves* Γ *and* Γ', *and if* ABC *and* DEF *be two co-apolar triangles on* Γ, *then the corresponding points* $A'B'C'$ *and* $D'E'F'$ *constitute two co-apolar triangles on* Γ'.

Suppose for convenience that the two cubics Γ and Γ' are regarded as lying in the same plane. Let P, Q be any two points on Γ, and P', Q' the corresponding points on Γ'. Let the chords PQ, $P'Q'$ meet Γ, Γ' in the third points U, V' respectively. Let a variable chord through U meet Γ in the points L, M, and let L', M' be the corresponding points on Γ'. Then the rays corresponding to ULM through V' are $V'L'$ and $V'M'$. Hence a (2, 2)-correspondence exists between the rays through U and those through V'. The locus of the intersection of corresponding rays through U and V' is therefore a binodal quartic having nodes at U and V'. Let, further, the lines UPQ and $V'P'Q'$ meet in O, and consider a ray through U very near in position to UPQ. This ray meets the binodal quartic at the two points where it is intersected by the two corresponding rays through V'. Hence, in the limit UPQ meets the quartic in two coincident points at O. Similarly $V'P'Q'$ meets the quartic in two coincident points at O. The quartic is therefore trinodal having the third node at O. Consider next one of the four tangents drawn from U to Γ. In this case the points L and M coincide, and hence $V'L'$ and $V'M'$ coincide. The ray ULM therefore touches the quartic. Hence the four rays from U to Γ each touch the quartic curve. But in the case of a trinodal quartic only two tangents can be drawn from a node to the curve, and consequently the above (2, 2)-correspondence in which the rays $V'L'$, $V'M'$ corresponding to ULM are separate and distinct must degenerate into the special case of $V'L'$ and $V'M'$ being coincident. Thus a (1, 1)-correspondence exists between the rays through U and those through V'. To the rays, therefore, $U[A, B, C, D, E, F]$ of Γ will correspond the rays $V'[A', B', C', D', E', F']$ of Γ' respectively. It follows that if the pencil $U[ABC]$ be apolar to $U[DEF]$, then the pencil $V'[A'B'C']$ must be apolar to $V'[D'E'F']$. Hence if ABC and

DEF be two triangles subtending at every point U of Γ pencils which are apolar to one another, then must $A'B'C'$ and $D'E'F'$ be two triangles subtending pencils at every point V' of Γ' which are apolar to one another. Consequently, if ABC and DEF be co-apolar triangles of Γ, then $A'B'C'$ and $D'E'F'$ must be co-apolar triangles of Γ'. The required result is thus established, and it is evident that it is not necessary to regard the two cubics Γ and Γ' as being coplanar, provided that a (1, 1)-correspondence exists between the points of each. It follows as an immediate corollary to the above result that *if ABC be a triangle inscribed in Γ and apolar to it, then $A'B'C'$ is a triangle inscribed in Γ and apolar to it.*

It is also to be noted that in virtue of the (1, 1)-correspondence between the rays through U and those through V', the cross-ratio of the tangents from U to Γ is equal to that of the tangents from V' to Γ'. The two cubics Γ and Γ' have therefore the same absolute invariant and must be expressible by the same elliptic functions. They must, in fact, be identical projectively, and corresponding points have parameters whose sum or difference is constant. The condition that ABC shall be apolar to Γ involves only the differences of the parameters of A, B, C, which proves at once that $A'B'C'$ must also be apolar to Γ from a somewhat different standpoint (*Proc. London Math. Soc.*, Ser. 2, Vol. 18, pp. 375–384).

Consider now a cubic curve S lying in the plane π and passing through the base-points. Let, further, S be apolar to $A_1A_2A_3$. The points of S will correspond on Π to the points of a plane cubic Σ in which the surface is cut by a plane σ. The condition that S be apolar to $A_1A_2A_3$ is linear in the coefficients of S, and hence in the coefficients of σ. The plane σ corresponding to the plane cubic S will therefore always pass through a fixed point F_{123}. Now the points A_1, A_2, A_3 of S correspond to the points of Σ in which the lines a_1, a_2, a_3 intersect the plane σ, and hence, in virtue of the (1, 1)-correspondence existing between the plane cubics S and Σ, we see that the points in which the lines a_1, a_2, a_3 intersect σ form an inscribed apolar triangle with respect to Σ. We therefore have the following result:—

If in the case of a cubic surface three lines of the same system of a double-six intersect a plane in three points forming an apolar triangle with respect to the plane cubic which is the section of the plane and the surface, the plane always passes through a fixed point.

We next proceed to find the position of the point F_{123}. Let the line A_rA_s, joining two base-points of π, meet the cubic curve S again in the point C_{rs}. Then to the point C_{rs} on S corresponds on Σ the point of intersection of the line c_{rs} and the plane σ where c_{rs} is that line on the cubic surface which is the line of intersection of the planes $a_r b_s$ and $a_s b_r$. Now we know that since $A_1 A_2 A_3$ is an apolar triad on S, the points C_{15}, C_{25}, C_{35} obtained by projecting $A_1 A_2 A_3$ through A_5 on to the curve S again is also an apolar triad with respect to S. Hence we deduce that if the plane σ cut the three lines a_1, a_2, a_3 in three points forming an apolar triad on Σ, then also must σ cut c_{15}, c_{25}, c_{35} in three points forming an apolar triad on Σ; and similarly with regard to c_{16}, c_{26}, c_{36}. Now consider a plane section σ through the line c_{14}, such that c_{15}, c_{25}, c_{35} cut this section in three points forming an apolar triad on Σ. In this case, the line c_{14} and a conic constitute the plane cubic of intersection Σ, whose meeting points with c_{15}, c_{25}, c_{35} consist of the points where c_{25}, c_{35} cut c_{14} together with the point in which c_{15} cuts the conic. The condition that this triad of points shall be apolar to the section Σ consisting of the line c_{14}, and the above conic is easily shown to be that the points $c_{14} c_{25}$ and $c_{14} c_{35}$ shall be conjugate with respect to this conic. The points $c_{14} c_{26}$ and $c_{14} c_{36}$ must similarly be conjugate with regard to the same conic. Hence the required plane is easily found thus:—

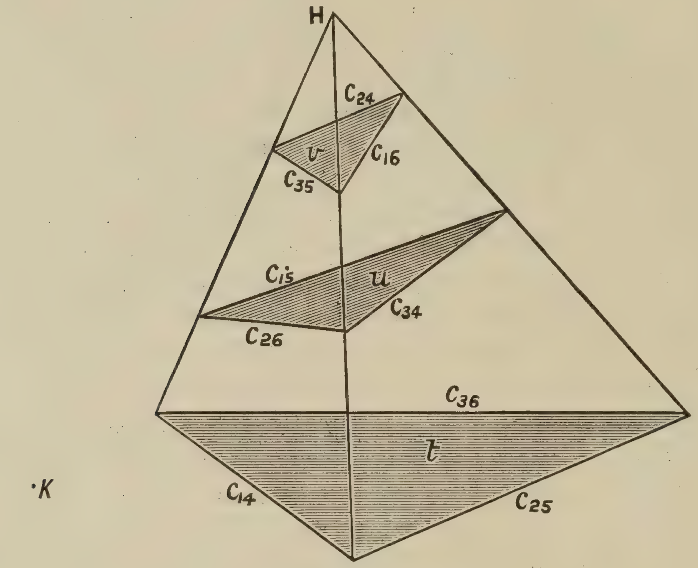
If a plane a be drawn through c_{14} cutting the cubic surface in a conic having $c_{14} c_{25}$ and $c_{14} c_{35}$ (or what is tantamount to the same thing $c_{14} c_{26}$ and $c_{14} c_{36}$) as conjugate points, this plane will pass through the fixed point F_{123}.

Let the planes β, γ through c_{25}, c_{36} respectively be similarly defined as cutting the surface in conics having $c_{25} c_{14}$, $c_{25} c_{34}$ (or $c_{25} c_{16}$, $c_{25} c_{36}$) and $c_{36} c_{14}$, $c_{36} c_{24}$ (or $c_{36} c_{15}$, $c_{36} c_{25}$) as respective pairs of conjugate points. Then F_{123} is the point of intersection of the three planes a, β, γ thus defined. Let us also define the point F_{456} as being the point common to all planes σ whose points of intersection with a_4, a_5, a_6 constitute a triangle apolar to Σ. As before, the plane $c_{14} F_{456}$ will cut the cubic surface in a conic having $c_{14} c_{25}$, $c_{14} c_{26}$ or ($c_{14} c_{36}$, $c_{14} c_{35}$) as conjugate points, and similarly for the planes $c_{25} F_{456}$ and $c_{36} F_{456}$.

Consider next in the subjoined diagrammatic table the determinantal system of lines, formed by rotating the suffixes 1, 2, 3 of the first column in one direction and 4, 5, 6 in the opposite direction, so as to obtain the second and third columns:—

$$\begin{array}{c|ccc|}
 & t & u & v \\
x & c_{14} & c_{26} & c_{35} \\
y & c_{25} & c_{34} & c_{16} \\
z & c_{36} & c_{15} & c_{24}
\end{array}$$

They constitute the lines of intersection of two complementary Steinerian Trihedra, the three planes x, y, z of one trihedron being defined by the lines in the rows, and the three planes t, u, v by the lines in the columns. Let x, y, z meet in the Steinerian Vertex H and t, u, v in K. We proceed to show that F_{123}, F_{456} harmonically separate H, K.



Consider, therefore, the pencil of cubic surfaces $tuv+\rho xyz=0$ through the nine lines of intersection of the above pair of Steinerian Trihedra. Plainly a (1, 1)-correspondence exists between each of the pencils of planes α, β, γ and the members of the pencil of cubic surfaces. But when the trihedron tuv is the member of the pencil of cubic surfaces under consideration, it is evident that α, β, γ each become coincident with the plane t. Hence, the locus of the point $F_{123}\equiv\alpha\beta\gamma$ is a straight line. Also, in the case of that cubic surface of the pencil consisting of the trihedron xyz, it is plain that α, β, γ coincide with x, y, z respectively. Hence the straight line which is the locus of F_{123} must pass through H. We deduce therefore the following proposition :—

The locus of the point F_{123} is the straight line joining the vertices H, K of the complementary Steinerian Trihedra xyz and tuv.

Similarly F_{456} lies also on this line. Hence :—

All cubic curves S of the plane π passing through the base-points and *having* $A_1A_2A_3$ *and* $A_4A_5A_6$ *as inscribed apolar triangles, correspond to the sections of the cubic surface* Π *by planes* σ *passing through the vertices of a pair of complementary Steinerian Trihedra.*

Now it is evident that any plane section of σ through H and K cuts the cubic surface Π in a plane cubic having H, K as corresponding points on its Hessian curve. Hence the line HK cuts the cubic curve Σ in three collinear points constituting an apolar triad with respect to Σ (*Proc. London Math. Soc.*, Ser. 2, Vol. 9, p. 239). But the pencil S of cubic curves lying in the plane π and passing through the vertices of the two triangles $A_1A_2A_3$ and $A_4A_5A_6$ apolar to each of them, have in common three further points which correspond to the three points of intersection of the line HK with the cubic surface Π, since HK is the axis of the pencil of planes σ that correspond to the pencil of plane cubics S. Further, since the three intersections of the line HK with the surface Π constitute an apolar triad on Σ, the section of σ with Π, we obtain the following result, in virtue of the fact that a (1, 1)-correspondence exists between the points of S and those of Σ :—

The three further points of intersection of the pencil of cubic curves S *circumscribing the triangles* $A_1A_2A_3$ *and* $A_4A_5A_6$, *and apolar to each of them, constitute a triad apolar to every member of the pencil of cubic curves.*

The fundamental theorem is thus established.

Also, F_{123} *and* F_{456} *harmonically separate* H *and* K.

This is plain from the fact that the two planes through c_{14} cutting Π in two conics having $c_{14}c_{25}$, $c_{14}c_{35}$ and $c_{14}c_{25}$, $c_{14}c_{26}$ respectively as conjugate points, harmonically separate x and t.

Finally we have the following property :—

The two planes joining F_{123} *and* F_{456} *to* c_{14} *are the two planes that harmonically separate both* x, t *and the two plane sections of* Π *through* c_{14} *whose conics touch* c_{14}.

AN ANALYTICAL TREATMENT OF THE 3-BAR CURVE

By F. V. MORLEY.

(*Communicated by* G. H. HARDY.)

[Read June 9th, 1921.—Received July 24th, 1921.]

Introduction.

The theory of the 3-bar curve has been recently enriched by the theorem of Colonel Hippisley* and the comprehensive memoir of Dr. Bennett.† There seems, however, to be room for a treatment of the curve itself from the point of view of inversive geometry.‡ Such is the following account. It falls into three parts.

1. The vector treatment of the general 3-bar curve; in which the algebra leads up to the keystone property of the transformation of the curve into itself, given by Dr. Bennett.

2. The mapping of the general curve, in which Darboux's treatment of the jointed quadrilateral is developed with detail and is applied to the curve itself.

3. The special cases of the rational and degenerate curves, whose consideration is apparently novel.

SECTION 1.—*The Vector Treatment of the General* 3-*Bar Curve.*

1.1. Darboux§ was the first to point out the analytical handling of the jointed quadrilateral by means of the two fundamental vector identities which express the closure condition. If we call the lengths of the sides in order of size $\rho_1\rho_2\rho_3\rho_4$ and their directions $t_1t_2t_3t_4$ (where $t = e^{i\theta}$, θ being the angle made with any fixed line), then these conjugate identities are

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1919), pp. 136 ff.

† *Ibid.*, Vol. 20 (1920), pp. 59 ff.

‡ Inversive plane geometry, as opposed to projective plane geometry, assumes infinity to be a point. For the name and a sketch of the method of analysis here used, see Prof. Morley, *Trans. Amer. Math. Soc.*, Vol. 8 (1907), p. 14.

§ *Bull. des Sciences Mathématiques* (1879), pp. 109 ff.

$$\rho_1 t_1+\rho_2 t_2+\rho_3 t_3+\rho_4 t_4 = 0,$$

$$\rho_1/t_1+\rho_2/t_2+\rho_3/t_3+\rho_4/t_4 = 0,$$

or, symbolically,

$$(\rho t) = 0, \tag{1.11}$$

$$(\rho/t) = 0. \tag{1.12}$$

When one bar of the jointed quadrilateral is fixed, say ρ_4, there is still the one degree of freedom which gives 3-bar motion. Let us make

$$\rho_4 = 1, \quad t_4 = -1,$$

which is most simply done by calling the ends of the bar 0 and 1. A point x rigidly attached to the traversing bar may then be named from the base point 0 as

$$x_0 = \rho_1 t_1 + k\rho_2 t_2, \tag{1.13}$$

where k is a complex constant. But, named from the point 1, x will be

$$x_1 = \rho_2 t_2(k-1)-\rho_3 t_3,$$

using the identity (1 . 11) in what we may call the non-homogeneous form

$$\rho_1 t_1+\rho_2 t_2+\rho_3 t_3 = 1. \tag{1.14}$$

It is thus suggested that x is best named symmetrically, from base point anywhere, as

$$x = c_1\rho_1 t_1+c_2\rho_2 t_2+c_3\rho_3 t_3, \tag{1.15}$$

where the three c's are complex constants.

Consider now each c as a point of the plane. Since the identity (1.14) holds, we may write (1 . 15) as

$$x-c_3 = (c_1-c_3)\rho_1 t_1+(c_2-c_3)\rho_2 t_2.$$

This gives the naming of x with respect to c_3, and must therefore be identified with (1 . 13) when $c_3 = 0$. That is,

$$c_1-c_3 = 1, \quad c_2-c_3 = k.$$

When c_3 is 0, c_1 is the point 1, and there is a third point

$$c_2 = k,$$

the vertex of a triangle on c_3c_1 similar to that attaching x to the traversing bar. Hence the triple generation theorem of Roberts,[*] for this third point c_2 plays a part equivalent to either of the other two in the relation (1 . 15). And as the non-homogeneous relation

$$(\rho t) = 1 \qquad \text{(1 . 14, symbolically written)}$$

implying its conjugate $(\rho/t) = 1,$

determines 3-bar motion, so the additional equation (linear in t and x)

$$x = (c\rho t) \qquad \text{(1 . 15, symbolically written)}$$

determines the 3-bar curve.

1 . 2. This permits us to name the various vectors in Cayley's figure of the triple generation,[†] as in our Fig. 1. We see that the three arrangements of linkworks are all of different types, corresponding to Darboux's classification of quadrilaterals.[‡] Namely, if the sides of the triangle $c_1c_2c_3$ are $\lambda_1\lambda_2\lambda_3$, where

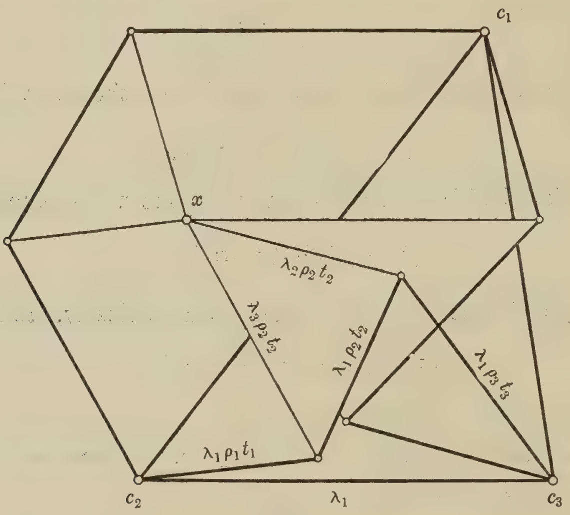

FIG. 1.

$$\lambda_1 < \lambda_2 < \lambda_3,$$

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[*] *Proc. London Math. Soc.*, Ser. 1, Vol. 7 (1876), p. 17.

[†] *Ibid.*, p. 136; cf. Clifford, *Kinematic*, p. 149.

[‡] *Loc. cit.* A wider classification of the jointed quadrilateral, according as the arms may turn completely round their pivots or only swing through an arc, is given in Burmester, *Kinematik*, Leipzig, 1888, pp. 283 ff. It does not concern us here.

then the 3-bar on λ_1 has the bars in order $\rho_1\rho_2\rho_3$; that on λ_2 in order $\rho_2\rho_3\rho_1$; and that on λ_3 in order $\rho_3\rho_1\rho_2$.

Furthermore, we see that the equations

$$x = (c\rho t), \tag{1.15}$$

$$1 = (\rho t), \tag{1.14}$$

the first linear in t and x, the second linear in t, provide us with a generalization of the 3-bar problem into that of putting a triangle of given shape with vertices on three fixed circles. For the vertices may be taken

$$a_1+r_1\tau_1, \quad a_2+r_2\tau_2, \quad a_3+r_3\tau_3,$$

and the condition that a point x be affinely attached to the triangle is

$$\begin{aligned} x-a_1-r_1\tau_1 &= k_2(a_1+r_1\tau_1-a_2-r_2\tau_2) \\ &= k_3(a_1+r_1\tau_1-a_3-r_3\tau_3). \end{aligned}$$

This gives two relations; one linear in x and τ, and the second (formed by subtracting the right-hand sides) linear in τ. This last equation is of form

$$1 = b_1\tau_1+b_2\tau_2+b_3\tau_3,$$

where the b's are given numbers, and in general complex; say $b_i = \rho_i\beta_i$. The equation is then

$$1 = \rho_1\beta_1\tau_1+\rho_2\beta_2\tau_2+\rho_3\beta_3\tau_3,$$

so that on taking $t_i = \beta_i\tau_i$, we have the standard form

$$1 = (\rho t).$$

The equation linear in x and τ similarly reduces to the form of (1.15), and we have the theorem *that a point affinely attached to a triangle of given shape traces a 3-bar curve when the vertices of the triangle move each on a fixed circle.*

Under this more general problem falls the 3-bar problem, of placing a rigid bar on two fixed circles. The latter is the special case when the triangle of given shape degenerates into a base of constant length and two parallel sides.

The equations so far used, (1.15) and (1.14), have been non-homogeneous. But the same relations in homogeneous form, each in four terms,

$$(c\rho t) = 0, \tag{1.21}$$

$$(\rho t) = 0, \tag{1.11}$$

indicate a mutuality between the four c's. In the triple generation figure (Fig. 1) one c is taken specially, and called x. But there will be a complete linkage in which three triangular plates are pivoted at each of the four c's; the set of three triangles pivoted at c_1 being all similar to the triangle $c_2c_3c_4$, and so on.

1.3. Let us put into the present notation an important fact about the 3-bar curve, referred to in the Introduction as having been proved by Colonel Hippisley. Namely, that if a point x be reflected in the sides of a triangle $c_1c_2c_3$, and a point y can be found at *given* distances from the three images $x_1x_2x_3$, then x and y move on the same 3-bar curve. That is, Colonel Hippisley focuses attention on pairs of points x, y such that the image of x in, say, c_3c_1, is at a given distance from y. Obviously such a relation is mutual.

Let x and x_2 be images in c_3c_1, and let us for simplification take $c_1c_2c_3$ as *turns*, or points on the base circle of radius unity. Then (writing $\bar{x}$ for the conjugate of x)

$$x_2+\bar{x}c_3c_1 = c_3+c_1,$$

and

$$|y-x_2| = |y+\bar{x}c_3c_1-c_3-c_1| = \text{constant}, \tag{1.31}$$

is the relation demanded.

Now to any point on the 3-bar curve given by

$$x = (c\rho t),$$

it is natural to pair the point, also on the curve,

$$y = (c\rho/t).$$

To identify this pairing with Colonel Hippisley's pairing it is necessary to show that x and y so defined satisfy the relation (1.31). From this definition of x and y,

$$x+c_3c_1\bar{y} = \rho_2t_2\left(\frac{c_3c_1}{c_2}+c_2-c_3-c_1\right)+c_3+c_1,$$

using (1.14); so that

$$x+c_3c_1\bar{y}-c_3-c_1 = \rho_2t_2\frac{(c_1-c_2)(c_3-c_2)}{c_2},$$

which is of constant length.

1.4. The relation (1.31) between the Hippisley pairs may be written

$$(x+\bar{y}c_3c_1-c_3-c_1)(y+\bar{x}c_3c_1-c_3-c_1) = k_2,$$

say. There will be three such equations. Multiplying out the left-hand side, we have

$$xy+(x\bar{x}+y\bar{y})c_3c_1-(x+y)(c_3+c_1)+\bar{x}\bar{y}\,c_3^2c_1^2-(\bar{x}+\bar{y})c_3c_1(c_3+c_1)$$
$$+(c_3+c_1)^2=k_2.$$

By choice of unit point $c_1c_2c_3$ may be taken as the roots of

$$t^3-s_1t^2+s_2t-1=0, \qquad (1.41)$$

where the discriminant is positive.* Then $\bar{s}_1=s_2$, and $s_3=1$ or

$$c_3c_1=1/c_2\,;$$

and as
$$c_3+c_1=s_1-c_2,$$

we may substitute above and get

$$(x+y)c_2^3+[xy+\bar{x}+\bar{y}-(x+y)s_1]c_2^2+[x\bar{x}+y\bar{y}-(\bar{x}+\bar{y})s_1]c_2+x\bar{x}$$
$$+(s_1-c_2)^2=k_2.$$

Now we get no further by keeping the right-hand side simply k_2. But if a, b be a selected pair of points in the desired mutual relation, we can let the right-hand side be

$$k_2\equiv(a+\bar{b}c_3c_1-c_3-c_1)(b+\bar{a}c_3c_1-c_3-c_1).$$

Multiplying this out and collecting terms, we shall have three equations of the type

$$(x+y-a-b)c^3+[xy-ab+\bar{x}+\bar{y}-\bar{a}-\bar{b}-(x+y-a-b)s_1]\,c^2$$
$$+[x\bar{x}+y\bar{y}-a\bar{a}-b\bar{b}-(\bar{x}+y-\bar{a}-\bar{b})s_1]\,c+(\bar{x}\bar{y}-\bar{a}\bar{b})=0. \qquad (1.42)$$

This must be consistent with (1.41), which means first that

$$\bar{x}\bar{y}-\bar{a}\bar{b}=-(x+y-a-b),$$

giving the two conjugate relations

$$xy-ab+\bar{x}+\bar{y}-\bar{a}-\bar{b}=0, \qquad (1.43)$$

$$\bar{x}\bar{y}-\bar{a}\bar{b}+x+y-a-b=0, \qquad (1.44)$$

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* Here the discriminant is $(s_1\bar{s}_1-9)^2-4\,(3s_1-\bar{s}_1^2)(3\bar{s}_1-s_1^2)$. When this is negative the roots of (1.41) are one point on the unit circle and two points inverse to it.

and secondly that

$$x\bar{x}+y\bar{y}-a\bar{a}-b\bar{b}-(\bar{x}+\bar{y}-\bar{a}-\bar{b})s_1 = s_2(x+y-a-b). \qquad (1.45)$$

1.5. The relations (1.43) and (1.44) may be written

$$xy+\bar{x}+\bar{y} = \bar{A},$$

$$\bar{x}\bar{y}+x+y = A.$$

These two give a one-to-one correspondence, mutual or involutory, between x and y. Thus the whole curve has a transformation into itself, in which the Hippisley pairs figure as correspondent points. The singular points of this transformation are found by expressing y in terms of x,

$$y = \frac{x-A+\bar{A}\bar{x}-\bar{x}^2}{x\bar{x}-1}.$$

The transformation is such that when x is on the base circle ($x\bar{x} = 1$), y is in general infinite; but the singular points, for which y is indeterminate, are given by

$$x^3-Ax^2+\bar{A}x-1 = 0.$$

The conditions of this transformation suggest the simple transformation by isogonal conjugates with respect to three points on the base circle. For three such points a_i whose symmetrical functions are σ_1, σ_2, σ_3, the transformation by isogonal conjugates is most easily expressed

$$x+y+\sigma_3\bar{x}\bar{y}-\sigma_1 = 0.$$

And so we have the important theorem of Dr. Bennett, *that there is a transformation (by isogonal conjugates) sending the curve into itself.* The singular points are such that

$$\sigma_3 = 1, \qquad \sigma_1 = A.$$

The points a_i are not the points c_i; but when the discriminant in σ is positive they too are on the base circle. They are (as we shall see in the next section) the double points of the 3-bar curve, and the familiar relation of Cayley is here expressed as

$$s_3 = \sigma_3.$$

1.6. The Hippisley pairs are picked out of all pairs of the transformation by the condition (1.45). To make this factorable, add $2s_1s_2$ to each side and write

$$(x-s_1)(\bar{x}-s_2)+(y-s_1)(\bar{y}-s_2) = \text{constant};$$

and, since $s_2 = \bar{s}_1$, we have Dr. Bennett's theorem, *that the sum of the squares of the distances of x and y from s_1 (the orthocentre of the triangle c_i) is constant.* That is, we may write

$$|x-s_1|^2+|y-s_1|^2 = 2k^2.$$

This is linear in k^2; hence, by varying k, we have a pencil of sextics with the same points c_i and a_i.

When x and y are equidistant from s_1, they are on a circle of fixed radius k and centre s_1. There are on this circle three pairs of corresponding points, intersections of the circle and its transform—a circular quartic (speaking projectively) with double points at a_i. The 3-bar curve is *uniquely* determined by such a quartic, as well as by the circle and the three pairs of points on it.

The Hippisley construction takes the three images x_i of x in the sides of the triangle c_i, and with these as centres describes circles with given radii. In general these circles will not meet in a point. They may do so, and then we have this point y, paired with x in the involution. The question rises as to there being more than one point y corresponding to x. Geometrically this occurs when the circles meet in two points; that is, when their centres are on a line. But the three centres x_i are on a line only when x is on the circumcircle. Such a point x having two correspondents y must be a double point. Hence, as has been stated, the double points of the 3-bar curve $(a_1 a_2 a_3)$ must be on the base circle. The two correspondents y to each node x will show up only for real intersections of the Hippisley circles; but when they show up they are images in the Wallace line of the node, and on the opposite side of the nodal triangle (see Fig. 3).

1.7. The three equations

$$1 = (\rho t),$$

$$x = (c\rho t),$$

$$\bar{y} = (\bar{c}\rho t),$$

are three equations linear in t_i; hence t_i is linear in x, $\bar{y}$. And substituting in the remaining identity

$$1 = (\rho/t),$$

we have a cubic equation between x, $\bar{y}$; and this is another entry of

Dr. Bennett's remarkable theorem, *that the antipoints of Hippisley pairs lie on a cubic curve.*

An equation of the 3-bar curve is suggested by the homogeneous equations

$$(\rho t) = 0,$$

$$(c\rho t) = 0.$$

If we eliminate t_1, we get

$$-(c_2-c_1)\rho_2 t_2 = (c_3-c_1)\rho_3 t_3 + (c_4-c_1)\rho_4 t_4,$$

$$-(\bar{c}_2-\bar{c}_1)\rho_2/t_2 = (\bar{c}_3-\bar{c}_1)\rho_3/t_3 + (\bar{c}_4-\bar{c}_1)\rho_4/t_4.$$

If from these we eliminate t_2 by multiplication, we obtain

$$(c_2-c_1)(\bar{c}_2-\bar{c}_1)\rho_2^2 = (c_3-c_1)(\bar{c}_3-\bar{c}_1)\rho_3^2 + (c_4-c_1)(\bar{c}_4-\bar{c}_1)\rho_4^2$$
$$+(c_3-c_1)(\bar{c}_4-\bar{c}_1)\rho_3\rho_4\, t_3/t_4 + (\bar{c}_3-\bar{c}_1)(c_4-c_1)\rho_3\rho_4\, t_4/t_3.$$

Calling the distance from c_2 to $c_1 = \delta_{12}$, and $t_3/t_4 = \lambda$, we have a quadratic

$$A\lambda^2 + B\lambda + C = 0,$$

where

$$A = (c_3-c_1)(\bar{c}_4-\bar{c}_1)\rho_3\rho_4,$$

$$B = -\delta_{12}^2\rho_2^2 + \delta_{13}^2\rho_{\cdot} + \delta_{14}^2\rho_4^2,$$

$$C = (\bar{c}_3-\bar{c}_1)(c_4-c_1)\,\rho_3\rho_4.$$

If we had eliminated t_2 first, and then t_1, we should have had a similar quadratic

$$A'\lambda^2 + B'\lambda + C' = 0,$$

with similar expressions for A', B', C'. The eliminant of these two quadratics may be written

$$-CAB'^2 - C'A'B^2 + (CA' + C'A)\,BB' = (CA' - C'A)^2,$$

where

$$CA = \delta_{13}^2\delta_{14}^2\rho_3^2\rho_4^2,$$

$$C'A' = \delta_{23}^2\delta_{24}^2\rho_3^2\rho_4^2,$$

and by virtue of multiplying the following identity and its conjugate

$$(c_1-c_2)(c_3-c_4) = (c_1-c_3)(c_2-c_4) - (c_1-c_4)(c_2-c_3),$$

we have $\delta_{12}^2\delta_{34}^2 = \delta_{13}^2\delta_{24}^2 + \delta_{14}^2\delta_{23}^2$

$$-(c_1-c_3)(\bar{c}_1-\bar{c}_4)(\bar{c}_2-\bar{c}_3)(c_2-c_4) - (\bar{c}_1-\bar{c}_3)(c_1-c_4)(c_2-c_3)(\bar{c}_2-\bar{c}_4).$$

Hence

$$-(CA' + C'A) = (\delta_{12}^2\delta_{34}^2 - \delta_{13}^2\delta_{24}^2 - \delta_{14}^2\delta_{23}^2)\,\rho_3^2\rho_4^2.$$

The result of elimination must be symmetrical ; hence we need find only the coefficients of, say ρ_1^4, ρ_2^4, $\rho_1^2\rho_2^2$. These are respectively

$$-\delta_{12}^4\delta_{13}^2\delta_{14}^2\rho_3^2\rho_4^2,$$

$$-\delta_{12}^4\delta_{23}^2\delta_{24}^2\rho_3^2\rho_4^2,$$

$$-\delta_{12}^4(\delta_{12}^2\delta_{34}^2-\delta_{13}^2\delta_{24}^2-\delta_{14}^2\delta_{23}^2)\rho_3^2\rho_4^2.$$

The factor $-\delta_{12}^2\rho_3^2\rho_4^2$ may be dropped, and the result written

$$\overset{4}{\Sigma}\,\delta_{12}^2\delta_{13}^2\delta_{14}^2\rho_1^4+\overset{6}{\Sigma}\,\delta_{12}^2\,(\delta_{12}^2\delta_{34}^2-\delta_{13}^2\delta_{24}^2-\delta_{14}^2\delta_{23}^2)\,\rho_1^2\rho_2^2=0.$$

This equation reduces conveniently when the four c's are in special configurations. As, for instance, when the four c's are on a circle, by Ptolemy's theorem, we have

$$\delta_{12}\delta_{34}+\delta_{13}\delta_{42}+\delta_{14}\delta_{23}=0,$$

$$\delta_{12}^2\delta_{34}^2-\delta_{13}^2\delta_{24}^2-\delta_{14}^2\delta_{23}^2=2\delta_{13}\delta_{14}\delta_{23}\delta_{42},$$

and we have the form of the equation which is given by Dr. Bennett for this case

$$\overset{4}{\Sigma}\,\delta_{12}\delta_{13}\delta_{14}\,\rho_1^2=0.$$

In terms of the points themselves, when on the base circle, this equation is

$$\overset{4}{\Sigma}\,(c_1-c_2)(c_1-c_3)(c_1-c_4)\,\rho_1^2/c_1=0.$$

SECTION 2.—*The 3-Bar Curve as an Elliptic Curve.*

2 . 1. In previous papers on the 3-bar curve little has been said about its actual appearance.* For a full discussion of the jointed quadrilateral the fundamental identities (1 . 11) and (1 . 12) must be solved. The 3-bar curve may then be expressed by (1 . 15) or (1 . 21), and studied for itself. Darboux† pointed out the solution of the identities in terms of elliptic functions. We shall here put into the Weierstrass notation the formula which Darboux took from Jacobi. Let us start with the function

$$\frac{\sigma_1 u}{\sigma u}=\xi_1 u=\sqrt{(\wp u-e_1)}, \tag{2.11}$$

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* For some beautiful diagrams of the 3-bar curve see the Atlas of Burmester's *Kinematik*, Leipzig, 1888, especially Figs. 346–353. They do not, however, emphasize the distinctions here desired.

† *Loc. cit.*

where $\xi_1 u$ is in effect Jacobi's function $1/\mathrm{sn}\, u$. This function is elliptic with periods $2\omega_1$ and $4\omega_2$, for it changes sign when for u we write $u+2\omega_2$. But the product of any even number of factors

$$\xi_1(u-a)\,\xi_1(u-b)\,\ldots$$

is elliptic with the standard periods $2\omega_1$, $2\omega_2$; for the changes of sign are cancelled. Hence

$$fu = \xi_1(u-a)\,\xi_1(u-b)\,\xi_1(u-c)\,\xi_1(u-d)$$

is elliptic with the standard periods. It has simple poles at a, b, c, d; the residue at a is

$$\xi_1(a-b)\,\xi_1(a-c)\,\xi_1(a-d).$$

By the theorem that the sum of the residues in any cell is zero, we have immediately the formula

$$\begin{aligned} &\xi_1(a-b)\,\xi_1(a-c)\,\xi_1(a-d) \\ &+\xi_1(b-a)\,\xi_1(b-c)\,\xi_1(b-d) \\ &+\xi_1(c-a)\,\xi_1(c-b)\,\xi_1(c-d) \\ &+\xi_1(d-a)\,\xi_1(d-b)\,\xi_1(d-c) = 0. \end{aligned} \tag{2.12}$$

We write u in place of d, and put this result in the form

$$\sum^{3} \frac{\xi_1(a-b)\,\xi_1(a-c)}{\xi_1(u-b)\,\xi_1(u-c)} = 1. \tag{2.13}$$

Now* $$[\wp v - e_1][\wp(v+\omega_1)-e_1] = (e_1-e_2)(e_1-e_3) = k_1^2. \tag{2.14}$$

Writing $v = -\frac{\omega_1}{2}+u$, we have

$$\xi_1\left(\frac{\omega_1}{2}-u\right)\xi_1\left(\frac{\omega_1}{2}+u\right) = k_1. \tag{2.15}$$

Or we may write $$\xi_1(u)\,\xi_1(u+\omega_1) = -k_1. \tag{2.16}$$

Using this last in the form

$$\frac{1}{\xi_1(u-a+\omega_1)} = -\frac{\xi_1(u-a)}{k_1},$$

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* Harkness and Morley, *Introduction to the theory of analytic functions*, p. 261.

we may put $u+\omega_1$ for u in (2.13), and write it

$$\sum^{3} \xi_1(a-b)\,\xi_1(a-c)\,\xi_1(u-b)\,\xi_1(u-c) = k_1^2. \tag{2.17}$$

2.2. We may now identify equations (2.13) and (2.17) with the identities fundamental to the 3-bar linkwork. In

$$\rho_1 t_1+\rho_2 t_2+\rho_3 t_3 = 1,$$

$$\rho_1/t_1+\rho_2/t_2+\rho_3/t_3 = 1,$$

let us write

$$\left.\begin{aligned} k_1\rho_1 &= \xi_1(a-b)\,\xi_1(a-c) & k_1/t_1 &= \xi_1(u-b)\,\xi_1(u-c) \\ -k_1\rho_2 &= \xi_1(b-c)\,\xi_1(b-a) & -k_1/t_2 &= \xi_1(u-c)\,\xi_1(u-a) \\ k_1\rho_3 &= \xi_1(c-a)\,\xi_1(c-b) & k_1/t_3 &= \xi_1(u-a)\,\xi_1(u-b) \end{aligned}\right\}, \tag{2.21}$$

and determine u, a, b, c so that the relations hold.

The relation (2.15) may be put in the form

$$\xi_1\left(\frac{\omega_1}{2}+i\eta\right)\xi_1\left(\frac{\omega_1}{2}-i\eta\right) = k_1,$$

where η is real. If the factors are conjugate, k_1 is necessarily the positive root of (2.14). We may then put

$$\xi_1 u = \xi_1\left(\frac{\omega_1}{2}+i\eta\right) = \sqrt{k_1}.t,$$

$$\xi_1 \bar{u} = \xi_1\left(\frac{\omega_1}{2}-i\eta\right) = \sqrt{k_1}/t,$$

and may again take $\sqrt{k_1} = (e_1-e_2)^{\frac{1}{4}}(e_1-e_3)^{\frac{1}{4}}$,

to be positive.

We are here concerned with self-conjugate networks, and hence with two types—the rectangular and the rhombic. (Fig. 2.) In the rectangular

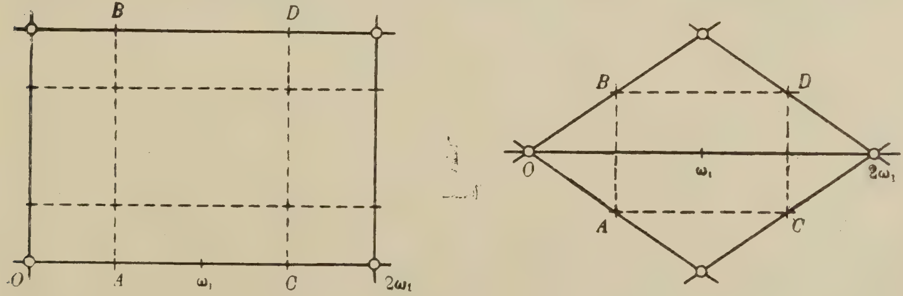


FIG. 2.

case, when u traces a vertical line through $\omega_1/2$, the function $\xi_1 u$ traces a circle. If a, b, c are pure imaginaries, then it follows that the expressions for ρ_1, ρ_2, ρ_3 are real, for $\xi_1(i\eta)$ is a pure imaginary. But to determine u, a, b, c in the rhombic case it is better to proceed as follows. Let u trace a horizontal line through $\omega_2/2$. Then

$$\bar{u} = u+\omega_1-2\omega_2$$

and
$$\xi_1\bar{u} = -\xi_1(\bar{u}+2\omega_2) = -\xi_1(u+\omega_1).$$

From the relation now true

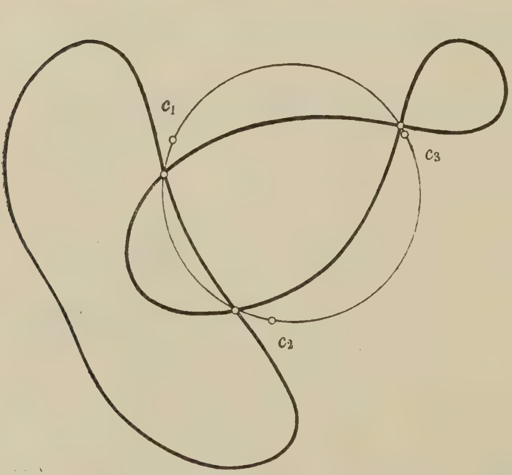
$$\xi_1 u \,.\, \xi_1\bar{u} = k_1,$$

we may proceed as before. If b is real, $u-b$ still moves on the horizontal line through $\omega_2/2$; here then a, b, c may be real.

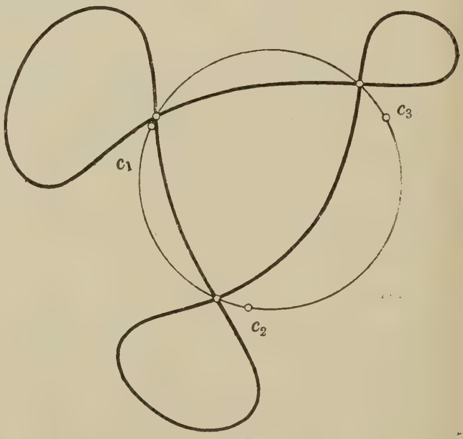
2.3. We are now in a position to discuss the appearance of the 3-bar curve, for the problem is reduced to one of mapping. The map-equation of the 3-bar curve is

$$x = \sum^3 c_1 \frac{\xi_1(a-b)\,\xi_1(a-c)}{\xi_1(u-b)\,\xi_1(u-c)}. \qquad (2.31)$$

We shall be concerned with only a single cell of the network (Fig. 2). In the rectangular case, when u traverses the vertical lines through $\omega_1/2$ and $3\omega_1/2$ of the fundamental cell, x will trace the appropriate 3-bar curve. Here the two paths AB and CD are distinct, and the map in the plane of x will consist of two corresponding parts—a *bipartite* curve. In the case of the rhombic network, as u traverses the two lines BD and AC, x will



Biparti



Unipartite.

trace its 3-bar curve; but here the point D is congruent with A, and hence the passage of u along BD, AC is continuous. BD and AC are in effect one line. The map in the x-plane will in this case be a *unipartite* curve.

The Hippisley pairs on the 3-bar curve were obtained in § 1.3 as the points corresponding to the parameters t and $1/t$. In terms of the elliptic parameter they correspond to u and $u+\omega_1$, and for the bipartite curve the transformation by isogonal conjugates amounts to an interchange of u between the points AB and CD. Hence the transformation maps one part into the other.

When u is on the horizontal lines through $\omega_2/2$ and $3\omega_2/2$ in the rectangular case, or on the paths AB, CD in the rhombic, the map consists of 3-bar curves orthogonal and confocal with those discussed.

2.4. The map-equation (2.31) defines x as an elliptic function of the third order, with poles

$$u = a+\omega_1, \quad u = b+\omega_1, \quad u = c+\omega_1.$$

The conjugate equation is

$$\bar{x} = \sum^{3} \bar{c}_1 \frac{\xi_1(a-b)\,\xi_1(a-c)}{\xi_1(u-b+\omega_1)\,\xi_1(u-c+\omega_1)}. \qquad (2.41)$$

When a curve is given by a map-equation, say

$$x = f(z),$$

the path of z being given, the values of z which make x infinite may or may not make infinite all values of

$$\bar{x} = \bar{f}(\bar{z}).$$

Those finite values of $\bar{x}$ corresponding to the z's which make x infinite, we shall call the *poles* of the curve. Suppose the curve given in the self-conjugate form

$$\phi(x, \bar{x}) = 0.$$

This we regard as the coincidence case of

$$\phi(x, \bar{y}) = 0,$$

where equally

$$\phi(\bar{x}, y) = 0.$$

Then x and y are images in the curve, and those finite values of x which make y infinite, viz. those finite points which are images of infinity, are the poles. This variation of language from the commonly used "singular foci" of projective geometry seems advantageous in any inversive argument.

In the 3-bar curve given by the map-equations above, when $u = a + \omega_1$, $x \to \infty$, and $\bar{x} = \bar{c}_1$; and equally when $u = a$, $\bar{x} \to \infty$, and $x = c_1$. Consequently *the points* $c_1 c_2 c_3$ *are the poles of the curve.*

In the classification of inversive curves the 3-bar is an elliptic *bicubic*, or curve of order 6. For a bilinear function of x and $\bar{x}$ becomes, on substitution from (2.31) and (2.41), an elliptic function of order 6; and hence the curve may have three image-pairs in common with a circle, or may be met by a circle in at most six points. Speaking projectively, the curve is met by a line in six points; for a linear function of x and $\bar{x}$ is on substitution equally an elliptic function of order 6. From this point of view the curve is then an elliptic *tricircular sextic.*

Section 3.—*The Rational and Degenerate 3-Bar Curves.*

3.1. The underlying network shows that there are two types of the elliptic 3-bar curve. A convenient classification by analogy comes from the fundamental homogeneous identities (1.11) and (1.12). Darboux* suggested that if the t's are homogeneous coordinates of a point in space, then these equations represent the join of a plane with a cubic surface; that is, a planar cubic curve may be taken as the curve of positions for a 3-bar linkwork.

The condition for this cubic to be rational is (if $\rho_i \neq 0$)

$$\pm \rho_1 \pm \rho_2 \pm \rho_3 \pm \rho_4 = 0.$$

For a physical quadrilateral we may take ρ_i positive, and for convenience in order of size. Then, if

$$\rho_4 \neq \rho_1 + \rho_2 + \rho_3,$$

the only combination which can be zero is

$$-\rho_1 + \rho_2 + \rho_3 - \rho_4.$$

Call this ρ. This single combination can be positive, negative, or zero. When

$$\rho > 0,$$

the 3-bar curve is bipartite, corresponding to the bipartite cubic. For let the linkwork be such that the least and greatest sides shall be opposite.

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* *Loc. cit.*

If
$$\rho_1 = 0$$
and
$$\rho_2 + \rho_3 > \rho_4,$$
there will be no motion. If ρ_1 is small, and
$$\rho_2 + \rho_3 > \rho_1 + \rho_4,$$
there will be a small closed motion on each side of the pivot line, giving the bipartite curve. Hence we have the analogies

Cubic curve.	Shape of 3-bar.	3-bar curve.	Coefficients.
Bipartite	Bipartite sequence	Bipartite	$\rho_2 + \rho_3 > \rho_1 + \rho_4$
Unipartite	Unipartite sequence	Unipartite	$\rho_2 + \rho_3 < \rho_1 + \rho_4$
Rational	Circumscriptible	Rational	$\rho_2 + \rho_3 = \rho_1 + \rho_4$
Double point	Collapse	Extra d.pt. (4)	
Conic and line	Contraparallelogram and parallelogram; or kites	Biquadratic and circle	$\rho_1 = \rho_2, \quad \rho_3 = \rho_4$
2 d.pts.		2 extra d.pts. (5)	
3 lines	Rhombus	3 circles	$\rho_1 = \rho_2 = \rho_3 = \rho_4$
3 d.pts.		3 extra d.pts. (6)	

3.2. To handle the rational curve we may let either the rectangular or the rhombic network degenerate. Taking the first case, with the periods $2\omega_1$ and $2\omega_2$ changing to π and ω respectively, we have
$$\xi_1 u = \sqrt{(\wp u - e_1)} = \cot u,$$
the sign being such that
$$\lim_{u \to 0} u \xi_1 u = 1.$$
Now the formula (2.15) is
$$\cot\left(\frac{\pi}{4} - u\right) \cot\left(\frac{\pi}{4} + u\right) = k_1, \tag{3.21}$$
so that
$$k_1 = 1.$$
The formula (2.12) is
$$\begin{aligned} &\cot(a-b)\cot(a-c)\cot(a-d) \\ +&\cot(b-a)\cot(b-c)\cot(b-d) \\ +&\cot(c-a)\cot(c-b)\cot(c-d) \\ +&\cot(d-a)\cot(d-b)\cot(d-c) = 0, \end{aligned} \tag{3.22}$$

and finally the parametric equation (2 . 31) for the 3-bar curve is

$$x = \sum^{3} c_1 \frac{\cot(a-b)\cot(a-c)}{\cot(u-b)\cot(u-c)}. \tag{3 . 23}$$

3 . 3. More convenient than this is the algebraic form. For

$$\cot(a-b) = i\frac{e^{2ia}+e^{2ib}}{e^{2ia}-e^{2ib}}.$$

Let then $\alpha = e^{2ia}, \quad \beta = e^{2ib}, \quad \gamma = e^{2ic}, \quad \lambda = e^{2iu},$

and $\cot(a-b) = i\dfrac{\alpha+\beta}{\alpha-\beta}, \quad \cot(u-a) = i\dfrac{\lambda+\alpha}{\lambda-\alpha},$

so that (3 . 23) becomes

$$x = \sum^{3} c_1 \frac{\alpha+\beta}{\alpha-\beta}\cdot\frac{\alpha+\gamma}{\alpha-\gamma}\cdot\frac{\lambda-\beta}{\lambda+\beta}\cdot\frac{\lambda-\gamma}{\lambda+\gamma}. \tag{3 . 31}$$

Since a, b, c, were pure imaginary, α, β, γ are real. And since now

$$u = \frac{\pi}{4} + i\eta,$$

$$\lambda = ie^{-2\eta},$$

a pure imaginary. Thus

$$\bar{\lambda} = -\lambda.$$

To $\pm\lambda$ correspond the Hippisley pairs on the rational bicubic.

The extra double-point must go into itself under the isogonal transformation. But the fixed points of the transformation are the four incentres of the triangle formed by the singular points. Hence *the extra double-point of a rational 3-bar curve must be at one of the incentres of the triangle formed by the other three double-points.* It corresponds to the values $\lambda = 0$ and $\lambda = \infty$. The extra double-point is then

$$x = \sum^{3} c_1 \frac{\alpha+\beta}{\alpha-\beta}\cdot\frac{\alpha+\gamma}{\alpha-\gamma}. \tag{3 . 32}$$

If it were desirable to study the rational curve in detail, we should express (3 . 31) in partial fractions as

$$x = \frac{A}{\lambda+\alpha} + \frac{B}{\lambda+\beta} + \frac{C}{\lambda+\gamma},$$

where $A/\alpha + B/\beta + C/\gamma = 0.$

We may invert with respect to the double-point, getting a rational quartic whose behaviour is known.

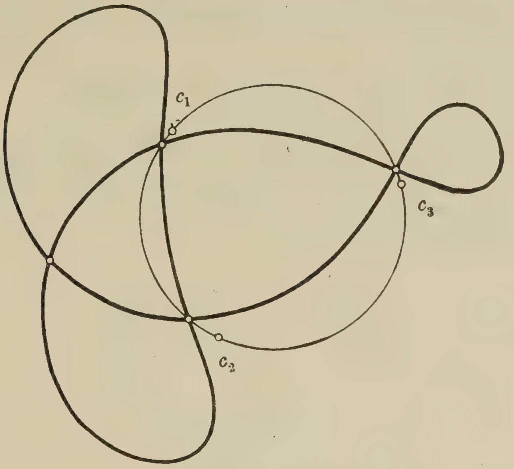

FIG. 4.—Rational 3-Bar Curve.

The identity (2.13) is now in the form

$$\sum^{3} \frac{\alpha+\beta}{\alpha-\beta}\cdot\frac{\alpha+\gamma}{\alpha-\gamma}\cdot\frac{\lambda-\beta}{\lambda+\beta}\cdot\frac{\lambda-\gamma}{\lambda+\gamma}=1. \tag{3.33}$$

This, when cleared of fractions, is a cubic in λ. But again it is obviously true, for it is true when

$$\lambda = \pm\alpha,\ \pm\beta,\ \pm\gamma,\ 0,\ \infty,\ \ldots .$$

This equation and its conjugate

$$\sum^{3} \frac{\alpha+\beta}{\alpha-\beta}\cdot\frac{\alpha+\gamma}{\alpha-\gamma}\cdot\frac{\beta+\lambda}{\beta-\lambda}\cdot\frac{\gamma+\lambda}{\gamma-\lambda}=1, \tag{3.34}$$

are the solutions of the 3-bar motion when the rational or collapse condition holds.

3.4. If we force the two conditions to be true

$$-\rho_1+\rho_2+\rho_3-\rho_4=0,$$

$$-\rho_1+\rho_2-\rho_3+\rho_4=0,$$

which give

$$\rho_1=\rho_2, \quad \rho_3=\rho_4,$$

we have a type of 3-bar motion for which the associated cubic has two double-points; that is, breaks up into a conic and a line. The quadrilateral is in this case either

(1) parallelogram or contraparallelogram, for which the equal bars are opposite;

(2) kite,* for which the equal bars are adjacent.

But by constructing the triple generation figure when one 3-bar is a contraparallelogram, the other two are seen to be kites. Hence the motion in each case is the same. Let us analyse case (2). The identities are here (if we fix ρ_4 with ends 0 and 1)

$$\rho_1(t_1+t_2) = 1-t_3,$$

$$\rho_1(t_1+t_2) = t_1 t_2\left(1-\frac{1}{t_3}\right),$$

from which
$$1-t_3 = \frac{t_1 t_2(t_3-t_1)}{t_3},$$

and either (*a*) $t_3 = 1$ or

$$(b)\ \ t_3 = -t_1 t_2.$$

In (*a*) we have collapse of the bar ρ_3 upon the fixed bar, and a point P attached to the traversing bar ρ_2 will describe a circle about one pole, named here 0. In case (*b*) consider the plane of the traversing bar, say the x-plane, as sliding over the fixed plane of the pivots, say the y-plane. The naming of P in the y-plane is

$$y = \rho_1 t_1 + \rho_1 t_2 k.$$

If we write the name of P in the x-plane as

$$x = \rho_1 k,$$

so as to make the unit of length the same in both planes, the equation of motion (or equation connecting the two names of the point) is

$$y = \rho_1 t_1 + t_2 x,$$

or, in terms of t_1,
$$y = \rho_1 t_1 + \frac{1-\rho_1 t_1}{\rho_1 - t_1}\, x. \qquad (3.41)$$

For given x this is a biquadratic with a node.

---

* Using "kite" for both the "kite" and "spearhead" of Kempe's nomenclature (A. B. Kempe, *How to Draw a Straight Line*, 1877).

For the x-centrode $D_t y = 0$,

giving the equation

$$(\rho_1^2-1)x = \rho_1(\rho_1-t_1)^2. \tag{3.42}$$

This is a limaçon, with focus $x=0$ and double-point $x=\rho_1$. The y-centrode is another limaçon

$$(\rho_1^2-1)y = \rho_1^2-2\rho_1 t_1-\rho_1^2 t_1^2, \tag{3.43}$$

with focus $y=1$ and double-point $y=0$.

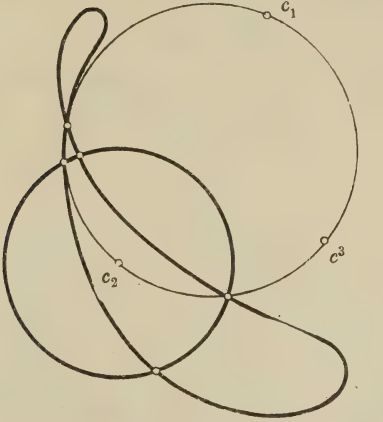

FIG. 5.—3-Bar Curve with Five Double-Points.

Now in the case (1) of contraparallelogram motion it is geometrically obvious that the centrodes are either ellipses or hyperbolas according to whether the short or the long side of the contraparallelogram is fixed.* *There is then equivalence between the rolling of two equal conics and the rolling of two related limaçons.* Either will produce a degenerate bicubic consisting of a biquadratic and a circle. The degeneration of the bicubic corresponds to the degeneration of the associated cubic into a conic and a line.

3.3. If the three conditions are satisfied

$$-\rho_1-\rho_2+\rho_3+\rho_4 = 0,$$

$$-\rho_1+\rho_2-\rho_3+\rho_4 = 0,$$

$$-\rho_1+\rho_2+\rho_3-\rho_4 = 0,$$

---

* See Sylvester, *Nature*, Vol. 12 (1875), p. 214 footnote; also Burmester, *loc. cit.*, and Figs. 359–367.

then the linkwork must be such that

$$\rho_1 = \rho_2 = \rho_3 = \rho_4.$$

The arrangement of each 3-bar is rhombic. The associated cubic breaks up into three lines. The 3-bar bicubic breaks up into three circles with centres c_1, c_2, c_3, radii $\lambda_1, \lambda_2, \lambda_3$ respectively. The configuration is that of a triangle (the three extra double-points) with its mid-points (the poles) and feet of perpendiculars (the three double-points on the base circle). The base circle is the Feuerbach or nine-point circle. (Fig. 6.)

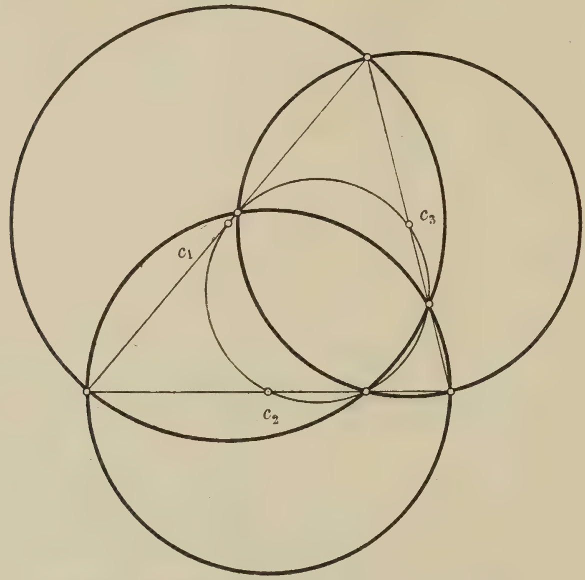

FIG. 6.—3-Bar Curve with Six Double-Points.

INTEGRATION OVER THE AREA OF A CURVE AND TRANSFORMATION OF THE VARIABLES IN A MULTIPLE INTEGRAL

By W. H. YOUNG.

[Read December 9th, 1920.]

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PART I.—*Introduction.*

1. In a previous communication to the Society* I laid the foundations of a theory of the transformation of the variables in a multiple integral analogous to that known to hold good in the case of a single variable, and I stated general conditions, valid not only in the case there explicitly considered, but also, as was implicitly stated, when the integrand is of a general nature.

I confined myself however in the paper in question, for reasons some of which are there given, to the discussion of the formula for an area,† a formula which constitutes at once a particular case of the formula of transformation‡ and the corner stone of the theory. On the one hand, in the passage from the straight line to higher space, additional insight is required into the concepts of area, volume, and the like; and on the other, as will appear from the present paper, a new concept, namely, that of integration over what may be called *a directed area, volume*, or *hypervolume*, is needed for the proper elaboration of the complete theory.

In a subsequent communication to the Royal Society,§ however, I attacked the question, in so far as it was possible to do so without introducing this innovation. It was, of course, necessary to make the restriction that the integrals which occur, *both in the final result and in the course of the reasoning*, should be integrals in the accepted sense of the word, and that, accordingly, the areas should be ordinary simple‖ areas. With this restriction I was able to show that *the formula of transformation* holds good, whenever *the formula for an area* does so.

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* W. H. Young, "On a Formula for an Area," *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1919), pp. 339–374.

† $A = \iint_A dx\,dy = \int_a^c \int_b^d \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv.$

‡ $\iint_C f(x, y)\,dx\,dy = \int_a^c \int_b^d f\{x(u, v), y(u, v)\} \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv.$

§ "On Change of the Variables in a Multiple Integral," *Proc. R. S.* (A), Vol. 96, pp. 82–91.

‖ Throughout the paper the term "simple curve" implies a curve devoid of multiple points, and "simple area" means the area of a simple curve, and is therefore an area in the elementary sense.

It is, however, a fundamental requirement of analysis that we should be spared, as far as possible, the necessity of proving that a restriction which plays no part in the final result, holds good in the interim.

Now even if the area of integration is an ordinary simple area, it is inconvenient to have to prove the fact. Moreover, it may very well happen that though the limits of integration of one of the two integrals, whose equivalence is in question, are such as to render the area of integration of the other a simple area, this is not so for all the range of possible limits which are bound to occur during our discussion.

In the present paper we are able to dispense with the restriction in question, replacing it by one which is in point of fact fulfilled whenever the conditions obtained in my earlier paper for the validity of *the formula for an area* hold good.* In particular *the formula of transformation*

$$\iint_C f(x, y)\, dx\, dy = \int_a^c \int_b^d f\{x(u, v),\ y(u, v)\} \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv,$$

is valid, for all bounded measurable functions $f(x, y)$, *provided that the transformation*

$$x = x(u, v), \quad y = y(u, v),$$

which changes the fundamental rectangle $(a, b;\ c, d)$ *in the* (u, v)*-plane into the curve* C *in the* (x, y)*-plane, is such that the derivates of* x *and* y *with respect to* u *and* v *are all bounded;*† *or more generally,*‡ *if* $x(u, v)$ *and* $y(u, v)$ *are integrals*§ *with respect to each variable separately, while their derivates with respect to each variable separately are numerically less than summable functions of the variable of differentiation alone.*||

The curve C may or may not be a "simple" curve, but if "simple" the integral is the ordinary integral, and if non-simple must be understood in the extended sense given to the term "integral" in the present paper.

2. The definition which I have adopted of integration over the area of

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* Exception must be made of the more general condition given in the second part of (iii) in the enunciation of Theorem VI, *loc. cit.*, p. 365, for which the validity of the formula of transformation is not demonstrated below.

† Theorem 4 of "A Formula for an Area," p. 361.

‡ Theorem 5, *loc. cit.*, p. 362.

§ Absolutely convergent Lebesgue integrals.

|| A still more general set of conditions, of which these are special cases, is explicitly discussed in § 24 below. This corresponds to Theorem VI, *loc. cit.*, p. 365, first set of conditions.

a curve is analogous to that I have given for directed area itself, the latter being in fact a special case of the former.

We take the accepted definition of the integral of a continuous function over a triangle, and hence, by summation, get the integral over any polygon, simple or not. The integral over the area of the curve is then defined in terms of the integral over an inscribed polygon, namely as the limit, if finite and unique, of the integrals over polygons inscribed in the curve—the vertices of each polygon being taken on the curve in a prescribed sense—when the lengths of the sides tend simultaneously to zero in any manner whatever.

It is convenient to suppose the function to be continuous in the first instance, and to use the method of monotone sequences, in order to ascend to the corresponding definition of the integral of any bounded function.

It should be remarked that in that part of our theory of the transformation of the variables in a multiple integral which deals with ordinary integration (*i.e.* when the region of integration is simple), no more is required than the notion of the integral of a continuous function over a non-simple area. The method of monotone sequences enables us to pass from a continuous function to any bounded function without altering the region of integration. For the rest it is found that unbounded functions which possess integrals over a directed area constitute a generalisation of functions which are summable over an ordinary area, but that neither of the two classes is included in the other.

It is evident indeed, looking at the matter more, for the moment, from the point of view of *multiplicity*, that a directed area may be regarded as partly positive and partly negative. Clearly, then, it might happen that a function which was positive only might possess an integral over the area, and might (defined, as this is, as the limit of the integral of a bounded function over the whole area) fail to possess one when certain of its values are made zero or negative, a circumstance which cannot arise in the case of integration over an ordinary area.

The simplicity of our theory would appear to be largely due to the fact that a set of zero plane content possesses *areal content* zero with respect to our curve. *Areal content* is here defined as the integral over the area of the curve, of the function which is unity at the points of the set and zero elsewhere. This coincides therefore, when the curve is simple, with the plane content of the part of the set inside the curve. Hence it follows that *two bounded functions which have the same Lebesgue double integral have the same integral over the area of our curve*, and the same is true for unbounded functions, in so far as the integrals exist.

3. As regards the appropriateness of our definitions of area and of integration over an area, various considerations suggest themselves. We remark first that, in a bounded portion of the plane, curves may be drawn whose areas are infinite, and others which cannot be said to possess an area, finite or infinite, the term *area* being understood in the sense which I have given. Neither of these possibilities occurs, however, in the case of a curve C of the type which we may call *semi-rectifiable*, one defined by equations of the form

$$x = x(u), \quad y = y(u),$$

where $x(u)$ *and* $y(u)$ *are both continuous, and at least one of them,* $y(u)$, *has bounded variation.*

But although in the detailed work of the present paper the sufficient conditions obtained for the existence of the integral of a function $f(x)$ over the area of a curve C, and the transformation of the variables in such an integral entail the existence of the area of C, this is no part of the general definition. Indeed, even when no number can be adjoined to the curve, which can rightly be termed its *area*, it will in general happen that not merely a particular function, but even whole classes of functions possess *integrals over the area of the curve* according to our definition, or rather, as we may now logically express it, *two-dimensional integrals over the curve.* That it would be a spurious limitation to demand as a necessary condition for such integration over a curve C the existence of the area of the curve is evident, when we recall the fact that ordinary integration between infinite limits is virtually equivalent to integration with respect to a function which has not bounded variation.

We have next to remark that our definitions virtually give us a carefully selected means of securing an area for our curve, when more directly geometrical methods would fail. We shall understand this better if we introduce the *multiplicity* of a point with respect to the curve. This is the number of complete anticlockwise revolutions of a half-ray joining the point (x, y) to a variable point which traces out the curve in its prescribed sense. In the case of a simple Jordan curve, the multiplicity $m(x, y)$ is zero outside the curve and inside is unity, the sign being $+$ or $-$ according to the sense in which the curve is supposed described. More generally, if the curve be a Jordan curve, cutting itself, but doing so only a finite number of times, $m(x, y)$ has, at any point (x, y) not lying on the curve, a definite value, which is zero, or one of a finite number of integers, positive or negative. In particular this is the case when the curve is a polygon, or a Jordan curve which is *rectifiable*,[*] or *semi-rectifiable*, and has only a finite number of multiple points. The area as I have defined it, then exists, and may, as is shown below, be identified with the integral of the multiplicity function $m(x, y)$ over any sufficiently large rectangle which includes within it all the points of the curve.

This equivalence suggests that this double integral might be adopted as the definition not only in these cases but also when the curve has an infinite number of multiple points. It leads us to the naive conception that the curve determines portions of the plane, or, more generally, sets of points, which are to be considered as occurring a certain positive or negative number of times; and we are tempted to calculate the area of the curve by summing algebraically their areas, or plane contents, prefixing to each the corresponding positive or negative integer. Apart, however, from the tacit assumption here made that points at which the multiplicity is fractional, or irrational, or infinite, or completely indeterminate may be left out of consideration, it is evident that such a definition suggests no obvious rule for the summation of such a series, containing as it does an infinite number of terms. The idea of order does not occur in it, and before such definition could be used, it would be necessary to prove that the order of summation was immaterial, so that any order taken at random gave an absolutely convergent series.

It is evident likewise, that if, without entering into actual geometrical considerations, we define the area as

$$\int_a^c \int_b^d m(x, y)\, dx\, dy,$$

or the integral of a function $f(x, y)$ as

$$\int_a^c \int_b^d m(x, y)\, f(x, y)\, dx\, dy,$$

we should either be tacitly requiring that the integral still existed when the integrand was made positive everywhere, or else we should find ourselves involved in the difficult domain of non-absolutely convergent integrals.

The difference between the two definitions is vital. The one we have adopted contains the idea of order, and the other does not. It is a strong argument in favour of our definition, and of the consideration of directed areas in general, which can only be logically treated with the aid of the idea of order, that in higher space, even in ordinary space of three dimensions, considerations of order are, as we have shown elsewhere, inevitable, if we are to discuss satisfactorily even the ordinary entities such as the area of an ordinary curved surface, in which, as such, questions of sign do not appear to have any place.

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[*] When both $x = x(u)$ and $y = y(u)$ are continuous functions of u, with bounded variation.

Similar considerations of course arise when it is a question, not of area, but of integration over an area. It is evident that the definition of the latter must follow the same lines as that of the former. Here indeed the idea of order suggests itself so far still more naturally, that even in ordinary one-dimensional integration of the Stieltjes type, all depends on our ordering our variable of integration suitably by means of a subsidiary independent variable.

The fact that our definition satisfies the desideratum of maintaining the well-known and fundamental formula for the expression of a two-dimensional integral over the area of a curve as a contour integral round the perimeter, serves both to give an extended meaning to this analytical formula and also to secure the equality of the integral with the sum of the integrals over any directed areas into which the area of integration may be divided, or, more correctly speaking, disintegrated.

From the point of view of analysis then our definitions of area and integral mark an advance. They give an extended meaning and scope to formulæ already in use, so that the work founded on them is of value, even if, for one reason or another, it should be thought undesirable to investigate more closely their geometrical significance. This fact is of peculiar importance in the generalisations to higher dimensions. There is no occasion, for example, to make our analytical investigations subsidiary to the question, whether, and in what degree, simple generalisations exist in higher space for Jordan's theorem about a plane curve, and other theorems connected with it.

PART II.—*The Integration of Continuous Functions over the Area of a Curve.*

4. We have now to show that a closed curve C being defined by equations of the form

$$x = x(u), \qquad y = y(u) \quad (u_0 \leqslant u \leqslant U),$$

where one of these functions at least, say $y(u)$, has bounded variation, and both are continuous—so that the curve is what we have called *semi-rectifiable*—a continuous function $f(x, y)$ has an integral over the area of C, whether C

be simple or not. At the same time, we shall show that our definition, when C is simple, agrees with the usual one.

The method adopted corresponds closely to that employed in proving the existence of the area, and its identification, when the curve is simple, with the ordinary area. The formula is the following :—

$$\iint_C f(x, y)\, dx\, dy = \int_{u_0}^{U} F[x(u), y(u)]\, dy(u),$$

where $$f(x, y) = \frac{\partial F(x, y)}{\partial x},$$

or more precisely, $F(x, y)$ is the indefinite integral* of $f(x, y)$ with respect to x, y being constant. We shall refer to this as *the contour integral formula.*

It will be remarked that we adopt the usual notation $\iint_C f(x, y)\, dx\, dy$ for integration over the area of the curve C, the symbols being now given an extended meaning.

5. We begin by defining the integral of a continuous function $f(x, y)$ over the area of a polygon P, not necessarily simple, at the same time proving what we have called *the contour integral formula* for this case.

Following the analogy above referred to, imagine the polygon P disintegrated into triangles Δ' in any convenient manner. For example, we may join the vertices of the polygon to any origin O in its plane. The sides of these triangles will have senses perfectly determined by the sense of the polygon P, these senses being such that each side of the polygon occurs once only in its proper sense, while each line introduced occurs twice, once in one sense and once in the opposite sense, these being such that the sides of each triangle Δ' have the same sense, anticlockwise or clockwise, as we follow them continuously all round the triangle. Each of the triangles Δ' will have a definite sign, $+$ if the sense in its perimeter is anticlockwise, and $-$ if clockwise. For instance, the triangle whose vertex is O and base $\overrightarrow{AB}$, one of the sides of the polygon, will have the sign $+$ or $-$ according as the perimeter described in the sense $\overrightarrow{ABOA}$ is anticlockwise or clockwise round the area of the triangle.

The integral of our continuous function $f(x, y)$ over each such triangle will be the ordinary double integral with the sign of the triangle prefixed, and, by the usual reasoning,* is expressible by our contour integral round the perimeter.

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* The lower limit of integration may be considered to be an arbitrary continuous function of y.

The sum of all such integrals, each with proper sign, is defined to be the integral of $f(x, y)$ over the polygon. That this definition is independent of the particular mode in which the polygon is disintegrated into triangles follows from the expression we thus obtain for it as a contour integral. The sum of the contour integrals round the triangles will in fact involve twice the part due to integration along any of the sides introduced, once positively and once negatively. We are thus left with only the perimeter of the polygon as contour of integration, so that *the contour integral formula is at once seen to be true for any polygon simple or not.*

Moreover, since there is, in dealing with a polygon, no distinction between the coordinates, we have three forms of the contour integral, which may be written

$$\iint_P f(x, y)\, dx\, dy = \int_P F(x, y)\, dy = -\int_P G(x, y)\, dx = \tfrac{1}{2}\int \{F dy - G dx\},$$

where $$F(x, y) = \int_\alpha^x f(x, y)\, dx, \quad G(x, y) = \int_\beta^y f(x, y)\, dy,$$

α and β being arbitrary continuous functions of y and of x respectively, which do not affect the result as we integrate all round the perimeter of the polygon.

6. We here digress to remark that the integral over a polygon can also, as is obvious in the case of a triangle, be expressed by the ordinary double integral

$$\int_a^c \int_b^d m(x, y)\, f(x, y)\, dx\, dy,$$

over any rectangle containing the polygon.

In fact, when we disintegrate the polygon into triangles Δ', each Δ' must, as it were, be cut out of a separate sheet; so that the triangles, some of which may be positive, and some negative, are to be regarded as superposed in part, wherever they have a common point. The bounding lines of the triangles form on the plane on which they lie a pattern or patchwork. Each patch Δ'', as we may call one of the simple polygonal parts thus isolated, has superposed to it, say, $m+\lambda$ positive triangles and

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* Rearranging the terms in the approximating double summation.

λ negative triangles. If therefore we disintegrate the integral

$$\iint_{\Delta'} f(x, y)\, dx\, dy$$

for each of these triangles into a sum of integrals for the patches forming it, and then add these sums together, so as to obtain another expression for the integral over the polygon, the patch Δ'' will be involved $m+\lambda$ times positively and λ times negatively, so that, on the whole, it will be involved m times in the final summation.

This integer m is equal to the multiplicity $m(x, y)$ of any point (x, y) inside the patch Δ'' with respect to the polygon, since the multiplicity is, in virtue of its definition, unaltered by the introduction of the disintegrating lines, and is therefore the sum of the multiplicities with respect to the individual triangles Δ'.

We have thus the expression

$$\iint_P f(x, y)\, dx\, dy = \Sigma m \iint_{\Delta''} f(x, y)\, dx\, dy = \Sigma \iint_{\Delta''} m(x, y) f(x, y)\, dx\, dy$$

$$= \int_a^c \int_b^d m(x, y)\, f(x, y)\, dx\, dy,$$

since $m(x, y)$ is zero at a point (x, y) outside every one of the non-overlapping patches Δ'', or, which is the same thing, outside every one of the overlapping triangles Δ'. Here $(a, b\,;\, c, d)$ is any rectangle containing the polygon and all the auxiliary lines of our figure.

7. The integral over the area of a closed curve,

$$x = x(u), \quad y = y(u) \quad (u_0 \leqslant u \leqslant U),$$

equally whether this is simple or not, is next defined as the unique limit, *when such a unique limit exists and is finite*, of the integrals over polygons, inscribed at successive stages in the curve in the customary manner, with vertices,

$$[x(u_1), y(u_1)],\ [x(u_2), y(u_2)],\ \ldots,\ [x(u_m), y(u_m)] \quad (u_1 < u_2 < \ldots < u_m),$$

in order on the curve, and sides whose lengths tend, as the stages advance, to zero as limit.

It immediately follows from our definition that *the integral of the sum of two functions is the sum of their integrals, and that, if a continuous function $f(x)$ lies in value between two continuous functions $g(x)$ and $h(x)$, the integral of $f(x)$ over our curve lies between those of $g(x)$ and $h(x)$.*

We assume now that the curve C is semi-rectifiable, *i.e.* such a Jordan curve[*] as was contemplated in § 2, defined by the equations

$$x = x(u), \quad y = y(u) \quad (u_0 \leqslant u \leqslant U),$$

in which $x(u)$, $y(u)$ *are continuous and* $y(u)$ *has bounded variation.* It will then be seen that *the integral of the continuous function* $f(x, y)$ *over the area of* C *exists and is expressed by the usual contour integral*

$$\int_{u_0}^{U} F\{x(u), y(u)\}\, dy(u).$$

Let (ξ, η) be a point on the polygon P, and let the vertices be

$$\{x(u_1), y(u_1)\}, \ \{x(u_2), y(u_2)\}, \ \ldots, \ \{x(u_m), y(u_m)\}.$$

By what has been already proved

$$\iint_P f(x, y)\, dx\, dy = \int_P F(\xi, \eta)\, d\eta = \sum_{i=1}^{m} \int_{y(u_i)}^{y(u_{i+1})} F(\xi, \eta)\, d\eta,$$

where $$y(u_{m+1}) = y(u_1).$$

Now $F(x, y)$ is defined independently of the polygon, being an indefinite integral of $f(x, y)$ with respect to x, and like its integrand $f(x, y)$, it is a continuous function of (x, y). Therefore, given any positive ϵ, we can determine ρ so that the oscillation of $F(x, y)$ in any circle of radius $\leqslant \rho$ is $< \epsilon$. Then

$$\left| \sum_{i=1}^{m} \int_{y(u_i)}^{y(u_{i+1})} F(\xi, \eta)\, d\eta - \sum_{i=1}^{m} [F\{x(u_i), y(u_i)\}\{y(u_{i+1}) - y(u_i)\}] \right|$$
$$\leqslant \epsilon \sum_{i=1}^{m} |y(u_{i+1}) - y(u_i)| \leqslant \epsilon B,$$

provided the sides of the polygon are all less than ρ in length, and B is the total variation of $y(u)$.

But the summation on the left tends, as the lengths of the segments (u_i, u_{i+1}) tend to zero, to a unique limit, namely

$$\int_{u_0}^{U} F\{x(u), y(u)\}\, dy(u).$$

Thus the integral of $f(x, y)$ over the polygon has, as the segments (u_i, u_{i+1})

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[*] Not necessarily simple. Throughout the remainder of the paper the letter C will be used for such a curve.

tend to zero, one or more limits, which differ from $\int_{u_0}^{U} F\{x(u),\ y(u)\}\, dy(u)$ by at most ϵB. Since ϵ is as small as we please, these limits must all coincide. Therefore, by definition,

$$\iint_C f(x,y)\,dx\,dy = \int_{u_0}^{U} F\{x(u),\ y(u)\}\, dy(u).$$

This proves the existence of our integral, and shows that it is expressed by the contour integral formula.

8. If both $x(u)$ and $y(u)$ are functions of bounded variation the curve is *rectifiable*, and we may, not only use all three forms of the contour integral

$$\int_{u_0}^{U} F(x,y)\,dy(u) = -\int_{u_0}^{U} G(x,y)\,dx(u) = \tfrac{1}{2}\int_{u_0}^{U} \{F(x,y)\,dy(u) - G(x,y)\,dx(u)\},$$

but also, introducing the length of the arc of the curve from a fixed origin as new variable, obtain the forms

$$\int F(x,y)\,\frac{dy}{ds}\,ds = -\int G(x,y)\,\frac{dx}{ds}\,ds = \tfrac{1}{2}\int \left\{F(x,y)\,\frac{dy}{ds} - G(x,y)\,\frac{dx}{ds}\right\} ds,$$

the integration being performed all round the curve. Indeed, now the derivates of x and y with respect to s are numerically $\leqslant 1$, so that x and y are integrals. Therefore taking s as our parameter u, we may, by the known theorems, replace $dy(u)$ and $dx(u)$ by $\frac{\partial y}{\partial s}\,dx$ and $\frac{\partial x}{\partial s}\,ds$. Again, in this case, $\frac{\partial y}{\partial s}$ and $-\frac{\partial x}{\partial s}$ exist as differential coefficients, except for a set of values of s of content zero, and are equal to the direction cosines l, m of the outward* normal. Thus we may write the contour integrals the forms

$$\int F(x,y)\,l\,ds = \int G(x,y)\,m\,ds = \tfrac{1}{2}\int \{lF + mG\}.$$

Each of these may be used for $\iint_C f(x,y)\,dx\,dy$ over the area of the rectifiable curve.

These forms reduce when $f(x,y) = 1$, to the familiar expressions for the area.

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* That is the position into which the tangent in the progressive direction, determined by u increasing, comes when rotated backwards, *i.e.* clockwise through a right angle.

9. *We can now identify our integral with the usual double integral, in the case when C is a semi-rectifiable simple Jordan curve.*

Since the integral, as I have defined it, has been shown to exist, we may approximate to it by means of the simple polygon P of § 3 above. The area of the ring between the curve and polygon may, since the curve is known now to have zero plane content, be supposed less than ϵ. When we divide up the plane into small rectangles of sides $\Delta x \Delta y$, the sum of those which contain the ring may then be taken to be less than ϵ. But the integral of a continuous function $f(x, y)$ over a simple polygon is the unique limit of a double summation

$$\Sigma\Sigma f(x, y)\, \Delta x \Delta y$$

over those rectangles $\Delta x \Delta y$ which contain points inside or on the polygon. The same is true for the integral (usual double integral) over a simple curve, provided the double summations then have a unique limit.

Since the polygon P and the curve C both lie in the ring, it follows that the small rectangles involved in the approximations to the two integrals are the same, except for certain of them, which are among those containing the ring. Thus, if B denote the upper bound of $|f(x, y)|$, the two double summations differ by less than $B\epsilon$.

Since this is true, however small the sides of P may be, the second of these double summations differs from the integral of $f(x, y)$ over C, as defined by me, by at most $B\epsilon$. Since ϵ is as small as we please, the integral over C, defined in the old-fashioned way, exists and is equal to that defined in my manner.

This proves the statement made at the beginning of the present article.

At the same time, using the multiplicity function $m(x, y)$, which is now 1 or -1 inside the curve, according to its sense, and zero outside, we have, as for a general polygon, the expression

$$\iint_C f(x, y)\, dx\, dy = \int_a^c \int_b^d f(x, y)\, m(x, y)\, dx\, dy,$$

when the semi-rectifiable curve C is devoid of multiple points, and, as in § 6, we may extend this to the case when the curve has a finite number of multiple points.

Part III.—*The Integration of Non-Continuous Bounded Functions over the Area of a Curve.*

10. In the preceding articles we have assumed the integrand to be a continuous function, and we appear to have obtained an absolutely satisfactory definition of the integral of such a function over a directed area. The question at once arises how far the reasoning employed depended on this hypothesis and how far it can be extended.

We see at once that we can do so, if the curve is a polygon, simple or not, the integrand being any bounded measurable function.

Indeed, the existence of the integral over every triangle and the validity of the contour integral formula having been supposed established,* the same follows over every polygon, since we only add a finite number of integrals over triangles, without any limiting process.

The argument of § 7, however, by which we pass from polygons to a curve, depends essentially on the integral $F(x, y) = \int f(x, y)\, dx$ being a continuous function of (x, y), and, if this condition is satisfied, remains valid for every bounded function $f(x, y)$, since the integral of such a function over a polygon has been satisfactorily defined. Thus we see at once that *we may extend our results, if $f(x, y)$ is a bounded function, and its indefinite integral with respect to x is a continuous function of (x, y).* The theorems will be precisely the same as if the functions were continuous, and in particular the definition will agree with the usual one when the curve is simple.

11. The above reasoning certainly fails, as it stands, for bounded functions in general, and, *a fortiori* for unbounded functions in general, even when they are of such a type as to possess an integral over every rectangle in the simple plane, *i.e.* to be *summable.* It will be seen, on the other hand, that *the method of monotone sequences enables us to give an unexceptional definition of the integral of any bounded measurable function.*

The fact that the method of definition suggested in the preceding article has only a limited application, and that, as appears from the contour integral formula, it agrees with the definition by monotone sequences in all cases which come under it, is a reason for adopting the latter definition, and we shall do so in the remaining part of the paper.

12. Let $f_1(x, y)$, $f_2(x, y)$, ..., $f_n(x, y)$, ... be any bounded monotone

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* By the method of monotone sequences, or otherwise.

sequence of functions for which the contour integral formula has been shown to hold, and let $f(x, y)$ be the limiting function. Then, integrating term-by-term with respect to x, and denoting the integrals by large letters, with appropriate indices when required,

$$\operatorname*{Lt}_{n\to\infty} F_n(x, y) = \operatorname*{Lt}_{n\to\infty} \int_a^x f_n(x, y)\,dx = \int_a^x f(x, y)\,dx = F(x, y)$$

(where a is an arbitrary continuous function of y); and, since $f_n(x, y)$ is a bounded function of (x, y, n), so is $F_n(x, y)$, so that $F_n\{x(u), y(u)\}$ describes a bounded sequence with $F\{x(u), y(u)\}$ as limit. We may therefore integrate term-by-term with respect to the function $y(u)$ of bounded variation, and obtain

$$\operatorname*{Lt}_{n\to\infty} \int_{u_0}^{U} F_n\{x(u), y(u)\}\,dy(u) = \int_{u_0}^{U} F\{x(u), y(u)\}\,dy(u).$$

But the integral on the left is equal to

$$\iint_C f_n(x, y)\,dx\,dy,$$

which has accordingly a unique limit, and this limit is

$$\int_{u_0}^{U} F\{x(u), y(u)\}\,dy(u),$$

which is our usual contour integral round the curve C.

Moreover, as has been shown in the theory of integration by means of monotone sequences, the uniqueness of the limit, and the value of the integral which expresses its value, are independent of the particular sequence $f_1(x, y)$, $f_2(x, y)$, ..., and depend only on the function $f(x, y)$ itself.

Thus, *defining the integral of the limiting function* $f(x, y)$ *as the limit of the integral of the constituent* $f_n(x, y)$ *of the defining sequence, when* $n\to\infty$, we obtain a faultless definition of the integral of any bounded l- or u-function, such a function being the limit of a sequence of continuous functions. Our definition leads moreover to the usual contour integral.

Hence, taking the constituents to be l- and u-functions, the sequence leads to the integrals of bounded lu- and ul-functions and their expression by the contour integral.

13. Since the right- or left-hand upper derivate of a continuous func-

tion is a *lu*-function, and the lower derivates are *ul*-functions, we see that we have the following theorem :—

THEOREM.—*If* $\xi(x, y)$ *be a continuous function of* (x, y) *and also an integral with respect to* x *for each fixed value of* y, *and if* C *be a semi-rectifiable Jordan curve (not necessarily simple), defined by the equations*

$$x = x(u), \quad y = y(u),$$

where $x(u)$ *is continuous and* $y(u)$ *continuous and of bounded variation with respect to* u, *then*

$$\iint_C \frac{\partial \xi}{\partial x}\, dx\, dy = \int_{u_0}^{U} \xi\{x(u),\, y(u)\}\, dy(u),$$

where the double integral is an integral over the directed area in accordance with our definition, and the single integral is a contour integral round the perimeter.

14. Proceeding as in the last article but one, we might now establish the validity of the representation of our integral as a contour integral for any bounded function which can be obtained by monotone or other bounded sequences. We thus at the same time prove the existence of the integral of such a function over the area of any semi-rectifiable curve. We prefer, however, to pass at once from *lu*- and *ul*-functions to any and every bounded summable function, by the argument I have already used in a like connection.

It will be easily seen that, the integral being defined by monotone sequences, as in § 12, a function $f(x, y)$ which lies between two bounded *lu*- or *ul*-functions $g(x, y)$ and $h(x, y)$ having the same integral, will, if it belongs to a class of functions whose integrals are defined, have again the same integral. *Without recurring to monotone sequences, except for the definitions in the case of l-, u-, lu-, and ul-functions, we may therefore define the integral of any bounded measurable function* $f(x, y)$ *as equal to the common value of the integrals of a bounded ul-function* $g(x, y)$ *and a bounded lu-function* $h(x, y)$, *differing only at a set* S *of content zero, and having therefore the same Lebesgue double integral and the same integral over the area of our curve.* That we have, in fact, such functions g and h,

$$g(x, y) \leqslant f(x, y) \leqslant h(x, y),$$

and that their Lebesgue double integrals are the same, are known theorems in the theory of integration, but that their integrals over our directed area are also equal must now be proved.

The difference of these two auxiliary functions $h(x, y) - g(x, y)$ is a bounded lu-function which is zero almost everywhere. Let $\phi(x, y)$ be such a function, then, by § 12,

$$\iint_C \phi(x, y)\, dx\, dy = \int_{u_0}^{U} \Phi\{x(u), y(u)\}\, dy(u),$$

where $$\Phi(x, y) = \int_a^x \phi(x, y)\, dx.$$

Under the integration sign y is a constant, while the integrand is never negative and it is zero except at the points of the set S. But, since S is of zero plane content, the section of S by the line $y =$ constant is of content zero, except for a set of values of y of content zero, which we may call the set E_y. Thus, if y does not belong to the set E_y, $\Phi(x, y)$ is zero for all values of (x, y).

Let E_u be the set on the u-line whose image in the y-line is the set E_y of content zero. *Then* $\Phi\{x(u), y(u)\}$ *is zero for all values of* u, *provided* u *does not belong to the set* E_u.

Now, by hypothesis, $y(u)$ is a function of u of bounded variation, but, without loss of generality, we may assume that $y(u)$ is an integral. For, if it is not, we only have to replace the variable u by an auxiliary variable σ, defined as the arc of the rectifiable curve

$$Y = y(u),$$

$$X = u.$$

We shall therefore assume $y(u)$ to be an integral. Then, by a known theorem,* *the set* E_u, *whose corresponding set* E_y *is of zero content, consists of a sub-set of zero content and a sub-set at every point of which* dy/du *is zero.*

Thus $\Phi\{x(u), y(u)\}\, dy/du$ is zero almost everywhere, and therefore, since y is an integral,

$$0 = \int_{u_0}^{U} \Phi\{x(u), y(u)\} \frac{dy}{du}\, du = \int_{u_0}^{U} \Phi\{x(u), y(u)\}\, dy(u) = \iint_C \phi(x, y)\, dx\, dy,$$

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* C. de la Vallée Poussin, 1914, *Cours d'Analyse*, 3rd ed., Vol. 1, p. 281, § 268. For an analogous theorem, not included in de la Vallée Poussin's, see my paper "On Functions of Bounded Variation," *Quarterly Journal*, Vol. 42 (1910), § 22, p. 80. The proof there given requires but slight modification to prove a general theorem embracing both these.

This proves that the integral over our area of a bounded lu-function $\phi(x, y)$, which is zero almost everywhere, is zero.

Hence, writing $\phi(x, y) = h(x, y) - g(x, y)$,

since
$$\iint_C \phi(x, y)\,dx\,dy = \iint_C h(x, y)\,dx\,dy - \iint_C g(x, y)\,dx\,dy,$$

it follows that
$$\iint_C h(x, y)\,dx\,dy = \iint_C g(x, y)\,dx\,dy,$$

as was stated at the beginning of the present article. *This justifies the definition of the integral of $f(x, y)$, where*

$$g(x, y) \leqslant f(x, y) \leqslant h(x, y),$$

as equal to the integral of $g(x, y)$ or $h(x, y)$.

Also, since the contour integral formula holds for $h(x, y)$ and $g(x, y)$,

$$\int_{u_0}^{U} H\{x(u),\, y(u)\}\,dy(u) = \int_{u_0}^{U} G\{x(u),\, y(u)\}\,dy(u).$$

But since $G(x, y)$, $F(x, y)$, $H(x, y)$ are the integrals of $g(x, y)$, $f(x, y)$, and $h(x, y)$ with respect to x, from a to x,

$$G(x, y) \leqslant F(x, y) \leqslant H(x, y),$$

and therefore

$$\int_{u_0}^{U} G\{x(u),\, y(u)\}\,dy(u) = \int_{u_0}^{U} F\{x(u),\, y(u)\}\,dy(u) = \int_{u_0}^{U} H\{x(u),\, y(u)\}\,dy(u),$$

since the extreme integrals have been already proved equal. Thus

$$\iint_C f(x, y)\,dx\,dy = \int_{u_0}^{U} F\{x(u),\, y(u)\}\,dy(u).$$

We have now defined the integral over the area of our curve C (not necessarily simple, but semi-rectifiable) when the integrand is any bounded function $f(x)$ having a Lebesgue integral, and shown that it obeys the contour integral formula.

15. We immediately obtain the fundamental result that *two bounded functions which have the same Lebesgue double integral (that is, which agree, except at a set of plane content zero), have the same integral over the area of our curve C.*

In fact, if the two functions $f(x, y)$ agree, except at a set of content zero, so do the two pairs of functions $g(x, y)$ and $h(x, y)$ for which this result has been already proved.

We have also the theorem that *a bounded function which is zero except at a set of plane content zero, has the integral zero over the area of our curve C.*

16. In particular, if the function $\lambda(x, y)$ is unity at the points of a set S of plane content zero, its integral over the area of our curve C is zero. That is,

$$0 = \int_{u_0}^{U} \Lambda\{x(u),\, y(u)\}\, dy(u) = \iint_C \lambda(x, y)\, dx\, dy. \tag{1}$$

If the curve defines a *multiplicity function* $m(x, y)$ as is the case if it is a polygon, a simple Jordan curve, or the sum of a finite number of such, we may, whether S has content zero or not, regard the integral

$$\iint_C \lambda(x, y)\, dx\, dy = \int_a^c \int_b^d m(x, y)\, \lambda(x, y)\, dx\, dy, \tag{2}$$

as the "content" of the part of the set inside the curve, the word "content" being used in an extended sense, each point being counted with its proper multiplicity. Without interpreting the result (1) in the general case, we may still use the expression *"areal content" with reference to the curve C for the integral* $\iint_C \lambda(x, y)\, dx\, dy$, and state the theorem equivalent to the equation (1), *that any set of points whose plane content is zero with respect to the simple plane has areal content zero with reference to our semi-rectifiable curve* C_1.

PART IV.—*The Integration of Unbounded Functions over the Area of a Curve.*

17. *Now let* $f(x, y)$ *denote any unbounded summable function of* (x, y).

First suppose $$f(x, y) \geqslant 0.$$

Let $$f_1(x, y) \leqslant f_2(x, y) \leqslant \dots \to f(x, y)$$

be any monotone increasing sequence of bounded functions having $f(x, y)$ as limit. Then, by § 15,

$$\iint_C f_n(x, y)\, dx\, dy = \int_{u_0}^{U} F_n\{x(u),\, y(u)\}\, dy(u), \tag{1}$$

for all values of n.

Also, by the theory of integration with respect to a function of bounded variation,

$$\underset{n\to\infty}{\mathrm{Lt}} \int_{u_0}^{U} F\{x(u),\, y(u)\}\, dy(u) = \int_{u_0}^{U} F\{x(u),\, y(u)\}\, dy(u), \tag{2}$$

provided the latter integral exists.*

But if one side of (1) has a unique limit, so has the other and this limit is independent of the particular sequence, $f_1, f_2, \ldots$ chosen. Thus the existence of $\int_{u_0}^{U} F\{x(u),\, y(u)\}\, dy(u)$ entails that of $\iint_C f(x, y)\, dx\, dy$ and *vice versa*. Moreover these two integrals are then equal, as follows from (1) when we proceed to the limit. Thus we have proved that the formula of transformation

$$\iint_C f(x, y)\, dx\, dy = \int_a^c \int_b^d F\{x(u),\, y(u)\}\, dy(u) \tag{3}$$

holds, unless both the integrals involved are non-existent.

Next let $f(x, y)$ be any summable function, and write

$$f(x, y) = f_1(x, y) - f_2(x, y),$$

where $f_1(x, y) = 0$, when $f(x, y) < 0$, and $f_2(x, y) = 0$, when $f(x, y) \geqslant 0$.

Then the preceding case covers $f_1(x, y)$ and $f_2(x, y)$. Therefore if these functions can be integrated over the curve, so can $f(x, y)$, and we again get the formula of transformation (3).

Conversely, if the existence of the integral of $f(x, y)$ over the area of C be only recognised, if the integral continues to exist when we change at any points we please the value of $f(x, y)$ to zero, the existence of this integral will necessitate the existence of the integrals of $f_1(x, y)$ and $f_2(x, y)$, and therefore the validity of the transformation formula.

On the other hand, the existence of the contour integrals for f_1 and f_2 necessitates that of the contour integral of f, and this will be then an absolutely convergent integral.

Conversely, the existence of this latter absolutely convergent integral ensures the existence of the two former contour integrals, and the consequent validity of the formula of transformation.

Summing up, we may say that, if $f(x, y)$ is a summable function, *the*

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* This involves the existence of the integral with respect to the total variation of $y(u)$.

transformation formula

$$\iint_C f(x, y)\,dx\,dy = \int_{u_0}^{U} F(x, y)\,dy(u) \tag{3}$$

holds, unless both these integrals fail to exist, the second as an absolutely convergent integral, and the first as an integral over the area of the curve, this being defined as in the present paper, with the stipulation that $f(x, y)$ *is such that any function got from* $f(x, y)$ *by maintaining its value at some points and at the others changing the value to zero, can be integrated over the area of the curve.*

18. It is now easily proved that the properties of integrals which hold in the case of bounded integrands, still hold for absolutely convergent integrals. In particular *a function which is zero except at a set of content zero, has an absolutely convergent integral over the area of our curve, and this integral is zero, and two functions which agree except at a set of content zero either have equal absolutely convergent integrals over the area of the curve or else neither function possesses such an integral.*

PART IV.—*Transformation of the Variables in a Two-Dimensional Integral.*

19. Now let us consider a continuous correspondence

$$x = x(u, v), \quad y = y(u, v),$$

and let the curve C be the image of a rectangle R in the (u, v)-plane, the sides of R being parallel to the axes of u and v.

The assumptions we make as to the correspondence are as follows :—

(i) *The formula for an area holds, viz.,*

$$C = \iint_C dx\,dy = \iint_R \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv,$$

not only for the particular rectangle R, *but for every such rectangle ; and*

(ii) *when the fundamental rectangle* R *is divided up into sub-rectangles* S, *by means of parallels to the axes of* u *and* v, *and these sub-rectangles are halved by means of their diagonals sloping down from left to right, the triangles* Δ' *in the* (x, y)*-plane, which I have called the "related"* *triangles of the semi-rectangles** Δ, *should be such that* $\Sigma\,|\Delta'|$ *is less than a fixed quantity, however the semi-rectangles be constructed.*

We shall now show that, *under these two conditions, the integral of a function $f(x, y)$ over the area of C, if it exists, may be expressed by the usual rule for change of the variables in a multiple integral, namely*

$$\iint_C f(x, y)\,dx\,dy = \iint_R f\{x(u, v),\, y(u, v)\}\,\frac{\partial(x, y)}{\partial(u, v)}\,du\,dv.$$

20. *First let $f(x, y)$ be a continuous function of (x, y).*

To prove the theorem we perform the division of the (u, v)-rectangle in two stages :—

Stage 1.—Divide up the fundamental rectangle R into sub-rectangles S, so small that each of the continuous functions $x(u, v)$, $y(u, v)$ and $f\{x(u, v), y(u, v)\}$ has an oscillation less than ϵ in each sub-rectangle S. Let there be M such rectangles.

Stage 2.—Determine the gauge necessary in order that, if we subdivide the S's by any network of parallels to the axes with this or any finer gauge, the area of each curve-image Γ of a sub-rectangle S, and the integral of $f(x, y)$ over Γ, differ by less than η/M from the area of the polygon Π, and the integral over Π respectively, where Π is the polygon inscribed in Γ, and determined by the points marked on the periphery of the corresponding sub-rectangle S.

The sub-sub-rectangles thus obtained are then to be divided into semi-rectangles Δ by the diagonals sloping down from left to right. Let Δ' denote the "related" triangles in the (x, y)-plane. Then the triangles Δ', positive and negative, group themselves into polygons Π, and these further, without being themselves simple polygons, constitute the polygon P inscribed in the curve C, the vertices of P being the images of all the points on the perimeter of R marked at both stages by our sub-divisions.

Now, by (i),

$$\Gamma = \iint_\Gamma dx\,dy = \iint_S \frac{d(x, y)}{d(u, v)}\,du\,dv.$$

Therefore, denoting by θ with or without subscript or dashes a quantity numerically $\leqslant 1$,

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* Namely, those whose vertices are the three-point images of the vertices of the semi-rectangles.

$$\Pi = \iint_{\Pi} dx\,dy = \iint_S \frac{d(x,\,y)}{d(u,\,v)}\,du\,dv = \frac{\theta_s\,\eta}{M}.$$

But, since the oscillation of $f\{x(u,\,v),\,y(u,\,v)\}$ in S is $\leqslant \epsilon$,

$$\iint_S f(x,\,y)\frac{d(x,\,y)}{d(u,\,v)}\,du\,dv = f(\xi,\,\eta)\iint_S \frac{d(x,\,y)}{d(u,\,v)}\,du\,dv + \theta'_s\epsilon \iint_S \left|\frac{d(x,\,y)}{d(u,\,v)}\right|\,du\,dv,$$

where $(\xi,\,\eta)$ is any point inside S or on its perimeter. Therefore, by the last two equations, denoting by B the upper bound of $|f(x,\,y)|$,

$$\iint_S f(x,\,y)\frac{d(x,\,y)}{d(u,\,v)}\,du\,dv = f(\xi,\,\eta)\iint_{\Pi} dx\,dy + \frac{\theta''_S B\eta}{M} + \theta'_S\epsilon \iint_S \left|\frac{d(x,\,y)}{d(u,\,v)}\right| du\,dv.$$

Now, if Δ' be any one of the related triangles in Π, and $(x,\,y)$ one of its vertices, $|x-\xi|$ and $|y-\eta|$ are, by our construction, each less than ϵ, and therefore the same is true when $(x,\,y)$ denotes any point inside or on the perimeter of Δ', since the x, or the y, of such a point lies between the greatest and the least of the x's, or the y's, of the vertices. We may therefore replace the term

$$f(\xi,\,\eta)\iint_{\Pi} dx\,dy = f(\xi,\,\eta)\,\Sigma \iint_{\Delta'} dx\,dy$$

by $\Sigma \iint_{\Delta'} f(x,\,y)\,dx\,dy + \theta\epsilon\,\Sigma \iint_{\Delta'} |dx\,dy| = \iint_{\Pi} f(x,\,y)\,dx\,dy + \theta_{\Pi}\epsilon\,\Sigma\,|\Delta'|$,

the summation being over all the Δ''s constituting Π.

Summing for all the M sub-rectangles S, we get

$$\iint_R f(x,\,y)\frac{d(x,\,y)}{d(u,\,v)}\,du\,dv$$

$$= \iint_P f(x,\,y)\,dx\,dy + \theta\epsilon\Sigma\,|\Delta'| + \theta'' B\eta + \theta'\epsilon \iint_R \left|\frac{d(x,\,y)}{d(u,\,v)}\right| du\,dv.$$

But, by (ii), $\Sigma\,|\Delta'|$ is bounded, the summation being all over the fundamental rectangle, however the division and sub-division be performed, and by (i) the Jacobian is absolutely integrable. Thus the three last terms in our last equation are as small as we please, say less than e.

Proceeding to the limit, by reducing the gauge of the sub-divisions indefinitely, we see that any limit of $\iint_P f(x,\,y)\,dx\,dy$ differs from the integral on the left by less than e. Thus all the limits must coincide with that integral, since e is as small as we please.

But, by definition, this unique limit is $\iint_C f(x, y)\,dx\,dy$, if this integral exists. Thus we have the usual formula for the transformation of the variables in a multiple integral

$$\iint_C f(x, y)\,dx\,dy = \iint_R f(x, y)\frac{d(x, y)}{d(u, v)}\,du\,dv,$$

with our extended meaning of integration, the integral on the left being the integral of the continuous function $f(x, y)$ *over the curve* C, *which need not be simple* and is the image of the perimeter of the fundamental rectangle* R. *The integral on the right is a Lebesgue double integral. The only conditions imposed on the correspondence*

$$x = x(u, v), \quad y = y(u, v)$$

are the conditions (i) *and* (ii) *of the preceding article.*

21. We have dealt in § 20 only with continuous integrands. Next let $f(x, y)$ denote any bounded function which is the limit of a monotone sequence of continuous functions, or other functions for which the theorem has been proved true,

$$f_1(x, y) \leqslant f_2(x, y) \leqslant \dots \leqslant f_n(x, y) \leqslant \dots \to f(x, y).$$

Then, by definition,

$$\iint_C f(x, y)\,dx\,dy = \operatorname*{Lt}_{n\to\infty} \iint_C f_n(x, y)\,dx\,dy.$$

Again our monotone sequence remains a monotone sequence when we substitute $x(u, v)$ and $y(u, v)$ for x and y. We may therefore multiply by the summable function $\dfrac{d(x, y)}{d(u, v)}$, and integrate term-by-term over the fundamental (u, v)-rectangle.

But, by hypothesis,

$$\iint_C f_n(x, y)\,dx\,dy = \int_a^c\int_b^d f_n\{x(u, v), y(u, v)\}\,\frac{d(x, y)}{d(u, v)}\,du\,dv.$$

Therefore, proceeding to the limit, $n \to \infty$,

$$\iint_C f(x, y)\,dx\,dy = \int_a^c\int_b^d f\{x(u, v), y(u, v)\}\,\frac{d(x, y)}{d(u, v)}\,du\,dv.$$

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* When C is simple, the integral on the left is an ordinary integral.

This proves the result of § 18 for any bounded l- or u-function, and hence for any bounded lu- or ul-function and so on, in fact for any bounded function of monotone type. We shall, however, only employ this argument for functions of the lowest types, as will be seen by the next following articles.

22. We now require the following theorem, given in my Royal Society paper, above quoted, under the conditions there laid down, but valid also under the conditions of the present paper.

THEOREM.—*If the conditions of* § 19 *hold for the correspondence between the* (x, y)- *and* (u, v)*-planes, then every set of zero content in the* (x, y)*-plane is either the image of a set of zero content in the* (u, v)*-plane, or else is such that* $\frac{d(x, y)}{d(u, v)} = 0$ *almost everywhere in the set.*

Every set of positive content is the sum of a set of zero content and of an o-set,* and the content of an o-set is the limit of the content of the closed sets defining it. Moreover in a continuous correspondence, a closed set in the (u, v)-plane has for image a closed set in the (x, y)-plane. It is therefore sufficient to prove that, if a closed set $S_{x,y}$ of zero content in the (x, y)-plane is the image of a closed set $S_{u,v}$ in the (u, v)-plane, then

$$\frac{d(x, y)}{d(u, v)} = 0,$$

almost everywhere in this closed set.

Let Q be the sub-set of this closed set at which

$$\frac{d(x, y)}{d(u, v)} > 0.$$

Now $\frac{d(x, y)}{d(u, v)}$ is the limit of a continuous function, and therefore is both a lu- and an ul-function. Therefore the points at which it is > 0 form an o-set. The set Q is therefore also an o-set, and the set Q' which is the image of Q is therefore also an o-set, since the closed sets generating Q correspond to closed sets generating Q'. Therefore the function, say

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* An o-set, or ordinary outer limiting set, is a set consisting of all the points of a succession G_1, G_2, ... of closed sets, each G_r contained in the set G_{r+1} following it. W. H. Young, "On Functions and their Associated Sets of Points," *Proc. London Math. Soc.*, Ser. 2, Vol. 12 (1912), pp. 260-287.

$\phi(x, y)$, which is unity at every point of Q' and zero elsewhere, and the transformed function, which is unity at every point of Q and zero elsewhere, are both *lu*-functions. We may therefore apply the formula for transformation of variables, which gives us

$$\iint_C \phi(x, y)\, dx\, dy = \int_a^c \int_b^d \phi\{x(u, v), y(u, v)\} \frac{d(x, y)}{d(u, v)}\, du\, dv.$$

But the left-hand side is the areal content of the set Q' with respect to the curve C, and is therefore zero, since Q' has content zero, (§ 16). Thus the integral on the right is also zero. But $\phi\{x(u, v), y(u, v)\}$ is zero, except at the points of the set Q where it is unity and the Jacobian is positive (> 0), this set Q must therefore have content zero.

Similarly the points of our closed set at which the Jacobian is negative (< 0) form a set of content zero. The closed set consists therefore of two sub-sets of content zero, and possibly other points at which the Jacobian is zero.

This proves the theorem.

23. *Now let $f(x, y)$ be any bounded function possessing a Lebesgue integral.* Then we know it can be enclosed between an *ul*-function $g(x, y)$ and a *lu*-function $h(x, y)$, differing only at a set of plane content zero. Therefore $f\{x(u, v), y(u, v)\}$, $g\{x(u, v), y(u, v)\}$, and $h\{x(u, v), y(u, v)\}$ only differ at a set of points in the (u, v)-plane, such that almost everywhere in it the Jacobian is zero, by § 21. Hence

$$\iint f\{x(u, v), y(u, v)\} \frac{d(x, y)}{d(u, v)}\, du\, dv = \iint g\{x(u, v), y(u, v)\} \frac{d(x, y)}{d(u, v)}\, du\, dv,$$

and by § 15,
$$\iint_C f(x, y)\, dx\, dy = \iint_C g(x, y)\, dx\, dy.$$

Since therefore the transformation formula holds for g it holds for f, that is

$$\iint_C f(x, y)\, dx\, dy = \int_a^c \int_b^d f\{x(u, v), y(u, v)\} \frac{d(x, y)}{d(u, v)}\, du\, dv,$$

whatever bounded summable function $f(x, y)$ may be.

24. Finally, *let $f(x, y)$ be any unbounded measurable function which is finite almost everywhere, then, either neither of the two integrals involved in the formula of transformation exists, or both exist and the formula holds.*

Here the existence of the integral over the curve implies that it exists in the sense of § 17, and the existence of the double integral over the fundamental (u, v)-rectangle implies that the integral is absolutely convergent.

To prove this, *consider first the case where*

$$f(x, y) \geqslant 0,$$

and $\dfrac{\partial(x, y)}{\partial(u, v)}$ *is of one sign, or is zero, at all the points where* $f(x, y)$ *is not zero.*

Let us take any monotone ascending sequence of bounded functions with $f(x, y)$ as limit,

$$f_1(x, y) \leqslant f_2(x, y) \leqslant \ldots \rightarrow f(x, y).$$

Then, by § 23,

$$\iint_C f_n(x, y)\,dx\,dy = \int_a^c \int_b^d f_n\{x(u, v),\ y(u, v)\} \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv. \tag{1}$$

Again, since, except where $f(x, y) = 0$, and therefore $f_n(x, y) \leqslant 0$, the Jacobian is always $\geqslant 0$, or always $\leqslant 0$, the integrand of the integral on the right-hand side traces out a monotone succession, whose limit is

$$f\{x(u, v),\ y(u, v)\} \frac{\partial(x, y)}{\partial(u, v)},$$

wherever this is definite. The points where it is indefinite, or infinite, consist of the points where the Jacobian is infinite, forming a set of content zero, and the points where $f\{x(u, v),\ y(u, v)\}$ is infinite, which form such a set as was discussed in § 22, since it has for image the set of content zero at which the function $f(x, y)$ is infinite. Thus the Jacobian is zero almost everywhere in this set. If therefore we agree to interpret the product as meaning zero at all the points where the Jacobian is zero,

$$\operatorname*{Lt}_{n\rightarrow\infty} f_n\{x(u, v),\ y(u, v)\} \frac{\partial(x, y)}{\partial(u, v)} = f\{x(u, v),\ y(u, v)\} \frac{\partial(x, y)}{\partial(u, v)},$$

except at a set of content zero, where the values of the functions may be taken to be any we please.

Thus if, with this understanding, $f\{x(u, v),\ y(u, v)\} \dfrac{\partial(x, y)}{\partial(u, v)}$ is a summable function of (u, v), we may integrate our monotone sequence term-by-term, and write

$$\operatorname*{Lt}_{n\to\infty} \int_a^c \int_b^d f_n \{(u, v),\, y(u, v)\} \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv$$

$$= \int_a^c \int_b^d f\{x(u, v),\, y(u, v)\} \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv. \tag{2}$$

In this case, therefore, the left-hand side of (1) has the right-hand side of (2) for unique limit. Since this is independent of the particular sequence $f_1(x, y)$, $f_2(x, y)$, ... chosen, our integral exists and we have the transformation formula

$$\iint_C f(x, y)\, dx\, dy = \int_a^c \int_b^d f\{x(u, v),\, y(u, v)\} \frac{\partial(x, y)}{\partial(u, v)}\, du\, dv. \tag{3}$$

Again, if $\iint_C f(x, y)\, dx\, dy$ exists, the left-hand side of (1) has the left-hand side of (3) for unique limit, and therefore, by (1) and (2), (3) again holds.

Thus (3) holds, or else neither of the integrals in (3) exists. This proves our statement in this first case.

25. The remaining case is when the signs of $f(x, y)$ and $\dfrac{\partial(x, y)}{\partial(u, v)}$ have no particular connection with one another. We then write

$$f(x, y) = f_1(x, y) + f_2(x, y) - f_3(x, y) - f_4(x, y),$$

where $f_1(x, y) = f(x, y)$ wherever $f(x, y) \geqslant 0$, and $\dfrac{\partial(x, y)}{\partial(u, v)} \geqslant 0$,

and $f_1(x, y) = 0$, elsewhere,

$$f_3(x, y) = f(x, y) \text{ wherever } f(x, y) \geqslant 0, \text{ and } \frac{\partial(x, y)}{\partial(u, v)} < 0,$$

and f_2 and f_4 are got from these by interchanging the signs $>$ and $<$.

The functions f_1, $-f_2$, f_3, $-f_4$ all fall under Class 1. Thus our required result is true for each of these functions separately, if the resulting formula has a sense.

But if each of these functions has an integral over the curve, so has $f(x, y)$, in the sense of § 17, and conversely, if $f(x, y)$ has such an integral, so have f_1, f_2, f_3, and f_4. Therefore, if $\iint_C f(x, y)\, dx\, dy$ exists in the sense of § 17, the transformation formula holds.

On the other hand, if for $i = 1, 2, 3$, or 4,

$$f_i\{x(u, v), y(u, v)\}\frac{\partial(x, y)}{\partial(u, v)},$$

is a summable function of (u, v), so is

$$f\{x(u, v), y(u, v)\}\frac{\partial(x, y)}{\partial(u, v)}.$$

Thus we see that unless both the integrals involved in the transformation formula are non-existent, the one in the sense of § 17, and the other as an absolutely convergent double integral, the formula transformation holds.

This proves the required result.

26. Basing ourselves on Theorem VI of my paper "On a Formula for an Area," we have now the following theorem.

THEOREM.—*If* (i) $x(u, v)$ *and* $y(u, v)$ *are integrals with respect to* u *for each fixed value of* v, *and integrals with respect to* v *for each fixed value of* a;

(ii) $$\left|\frac{\partial y(u, V)}{\partial V}\right| \leqslant \mu(V), \qquad \left|\frac{\partial x(u, V)}{\partial V}\right| \leqslant M(V),$$

where $\mu(V)$ *and* $M(V)$ *are summable functions of* V, *independent of* u;*

(iii) *The total variations of* $x(U, V)$ *and* $y(U, V)$ *with respect to* U, *for constant* V, *are bounded functions of* V, *and therefore of* (U, V); *then*

$$\iint_C f(x, y)\,dx\,dy = \int_a^c\int_b^d f(x, y)\frac{d(x, y)}{d(u, v)}\,du\,dv,$$

where C *is the image of the rectangle* $(a, b; c, d)$ *in the correspondence* $x = x(u, v)$, $y = y(u, v)$.

Indeed the conditions (i), (ii), and (iii) ensure not only the validity of the Formula for an Area, by the theorem above quoted, but also the rectifiability of the curve C, since x and y on the curve are functions of one of the variables only at a time, and these functions are integrals. Thus the integral of $f(x, y)$ over the area exists, by the results of the present paper.

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* This condition need only hold for a countable everywhere dense set of values of y.

Also

$$\left| \Delta' \right| = 1 \{x(u+h,\ v)-x(u,\ v)\}\,\{y(u,\ v+k)-y(u,\ v)\}$$

$$\leqslant \int_u^{u+h} \left| \frac{\partial x(U,\ v)}{\partial U} \right| dU \int_v^{v+k} \left| \frac{\partial y(u,\ V)}{\partial V} \right| dV$$

$$\leqslant \int_u^{u+h} \left| \frac{\partial x(U,\ v)}{\partial U} \right| dU \int_v^{v+k} \mu(V)\,dV.$$

Hence, summing all over the fundamental rectangle, we have, since the first of the two integrals, just written down, is the total variation of $x(U, v)$ in $(u, u+h)$, say $X(u+h, v)-X(u, v)$,

$$\Sigma\Sigma \left| \Delta' \right| \leqslant \Sigma \left[\{X(c,\ v)-X(a,\ v)\} \int_v^{v+k} \mu(V)\,dV \right] \leqslant B \int_b^d \mu(V)\,dV,$$

since, by (iii), there is a constant $B > X(c, v)-X(a, v)$, for all values of v. Thus $\Sigma\Sigma \left| \Delta' \right|$ is bounded, and the conditions of § 19 are satisfied, which proves the theorem.

CURVATURE AND TORSION IN ELLIPTIC SPACE

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[Received May 21st, 1921.—Read June 9th, 1921.]

Summary.

(i) Analogues to Frenet's formulæ are deduced in the form of eighteen equations between the line coordinates of the tangent, principal normal and binormal and their derivatives with respect to the arc.

(ii) By drawing from a given point quadrant lengths right parallel and left parallel to the tangents to a curve, we obtain two curves Γ_1, Γ_2 which may be termed the Right and Left Indicatrices of tangents.

It is proved that all curves having two given plane curves Γ_1 and Γ_2 as Right and Left Indicatrices are geodesics on a certain surface, and that this surface is a parabolic surface of constant curvature,* so that its geodesic geometry is Euclidean.

(iii) Extensions to elliptic space of theorems due to Bertrand and Mannheim.

(iv) Theorems on the curvature and torsion of curves whose tangents belong to a complex of any order. For example, it is proved that for all curves touching a singular line at the same point P (other than the singular point) the ratio of curvature to torsion at P is the same.

1. The fundamental equations are

$$x_0^2+x_1^2+x_2^2+x_3^2 = 1,$$

and

$$ds^2 = dx_0^2+dx_1^2+dx_2^2+dx_3^2,$$

where x_0, x_1, x_2, x_3 are the sines of the perpendiculars from the point x to the faces of the quadrantal tetrahedron of reference, and the unit of length is so chosen that the length of the whole line is π units in the "polar" form or 2π units in the "antipodal" form of elliptic space.

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* Cf. Whitehead, "Geodesic Geometry of Surfaces in Non-Euclidean Space," *Proc. London Math. Soc.*, Ser. 1, Vol. XXIX (1898), p. 275.

Denoting the vertices of this tetrahedron by V_0, V_1, V_2, V_3, it is convenient to regard V_0 as the origin and V_0V_1, V_0V_2, V_0V_3 as the positive senses of the edges through the origin. The positive senses of the remaining edges will be V_2V_3, V_3V_1, and V_1V_2. It is also supposed that the vertices are so taken that the positive senses of translation and rotation follow the right-hand screw convention.

We have to deal throughout with directed lines in space. If x and y are two points at a quadrant distance there is one direction in which, starting from x and proceeding a quadrant distance, we arrive at the point (y_0, y_1, y_2, y_3). If we proceed in the opposite direction a quadrant distance from x we arrive at the point $(-y_0, -y_1, -y_2, -y_3)$. The first mentioned direction will be termed the sense x to y of the directed line xy. The coordinates of the line in this sense may be taken to be the six quantities $p_{01}, p_{02}, p_{03}, p_{23}, p_{31}, p_{12}$, where

$$p_{mn} = x_m y_n - x_n y_m.$$

We have then $$p_{01}p_{23}+p_{02}p_{31}+p_{03}p_{12} = 0,$$

and $$\Sigma p_{mn}^2 = 1.$$

It is useful to note that the coordinates of the reciprocal line are $p_{23}, p_{31}, p_{12}, p_{01}, p_{02}, p_{03}$, where the positive senses of the two lines are so related that they have a positive mutual moment.

We have also to deal with the mutual angular or distance relations of two directed lines. If p and q are two lines and δ and δ_1 their common perpendiculars, we have the known formulæ

$$p_{01}q_{23}+p_{02}q_{31}+p_{03}q_{12}+p_{23}q_{01}+p_{31}q_{02}+p_{12}q_{03} = \sin\delta\sin\delta_1,$$

and $$p_{01}q_{01}+p_{02}q_{02}+p_{03}q_{03}+p_{23}q_{23}+p_{31}q_{31}+p_{12}q_{12} = \cos\delta\cos\delta_1. \qquad (1)$$

With regard to the signs of δ and δ_1 it will be observed that δ_1 is the angle between the planes (p, δ) and (q, δ), and may be regarded as the angle of divergence of the lines in the vicinity of their common perpendicular δ. We may then, in the first instance, suppose that δ is positive and less than π; δ_1 will then be positive or negative according as the lines p and q have a positive or negative mutual moment. If $\delta_1 = \delta$, the lines are right-parallel; and if $\delta_1 = -\delta$, the lines are left-parallel. With this convention equations (1) will be universally true.

2. A more useful system of line coordinates for our purpose is to take

the six quantities $p_1, p_2, p_3, p_4, p_5, p_6$ defined by the equations

$$p_1 = p_{01}+p_{23}, \quad p_2 = p_{02}+p_{31}, \quad p_3 = p_{03}+p_{12},$$

$$p_4 = p_{01}-p_{23}, \quad p_5 = p_{02}-p_{31}, \quad p_6 = p_{03}-p_{12},$$

and satisfying the relations

$$p_1^2+p_2^2+p_3^2 = 1, \quad p_4^2+p_5^2+p_6^2 = 1.$$

The coordinates of the reciprocal line will be

$$p_1, \; p_2, \; p_3, \; -p_4, \; -p_5, \; -p_6.$$

Equations (1) now become

$$p_1 q_1+p_2 q_2+p_3 q_3 = \cos(\delta-\delta_1) \quad \text{and} \quad p_4 q_4+p_5 q_5+p_6 q_6 = \cos(\delta+\delta_1),$$

or shortly $\overline{pq} = \cos(\delta-\delta_1)$ and $\underline{\underline{pq}} = \cos(\delta+\delta_1)$. (2)

The lines joining the origin to the points $(0, p_1, p_2, p_3)$ and $(0, p_4, p_5, p_6)$ are respectively right and left parallel to the line p. Calling these the right and left representative points, and denoting them by $R(p)$ and $L(p)$, we have the geometry of lines in space associated with that of point pairs* in a plane and a class of theorems in plane geometry can be immediately translated into theorems on lines in space.

If lines are right parallel their R points coincide and for left parallelism their L points coincide. If two lines intersect, the distance between the R points is equal to the distance between the L points and equals the angle between the lines. Other general relations are easily established, but these are sufficient for our purpose.

3. If a line is a tangent to a curve in space, the R and L points describe curves which may be called the Right and Left Indicatrices of tangents. There are also Indicatrices of principal normals and of binormals. Denoting the tangent, principal normal and binormal by the letters t, n, b the points $R(t)$, $R(n)$, $R(b)$ are the vertices of a quadrantal triangle.

Since the tangent is at right angles to two consecutive binormals and the binormal to two consecutive tangents, the Indicatrices of tangents and binormals are reciprocal curves, and the line joining $R(t)$ and $R(b)$ is a common normal to the two Indicatrices. The tangents at $R(t)$ and $R(b)$

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* Cf. Coolidge, *Non-Euclidean Geometry*, Oxford, 1909, pp. 124 and 227.

consequently pass through $R(n)$, and we have for a small displacement the relations

$$dt_1/n_1 = dt_2/n_2 = dt_3/n_3,$$

and

$$db_1/n_1 = db_2/n_2 = db_3/n_3.$$

Similarly from the consideration of the Left Indicatrices, we have

$$dt_4/n_4 = dt_5/n_5 = dt_6/n_6,$$

and

$$db_4/n_4 = db_5/n_5 = db_6/n_6.$$

4. We now proceed to an exact determination of these ratios in terms of the curvature and torsion at the point. Let the coordinates of a point P on a curve be supposed functions of the arc s, and let PT, PN, PB be quadrant lengths along the tangent, principal normal and binormal. We suppose PT in the direction of s increasing. There are two directions from P to the centre of curvature. For one of these the radius of curvature is less than a quadrant. We suppose PN drawn in this direction and that the curvature is then positive (this convention is only used for convenience in the demonstration and may be dispensed with in the subsequent use of the equations). The binormal PB is then taken in such a sense that the tetrahedron $(PTNB)$ regarded as a rigid body can be moved into coincidence with the fundamental tetrahedron $(V_0 V_1 V_2 V_3)$ so that the points P, T, N, B coincide respectively with V_0, V_1, V_2, V_3. The torsion will be supposed positive when the tetrahedron $(PTNB)$ has a right-hand screw motion along PT. With these conventions we have the known formulæ

$$\cot r = \kappa = (x_0''^2 + x_1''^2 + x_2''^2 + x_3''^2 - 1)^{\frac{1}{2}}, \tag{3}$$

and

$$\tau\kappa^2 = -(xx'x''x'''), \tag{4*}$$

where r is the radius of curvature, κ the curvature, τ the torsion, and the accents indicate differentiation with respect to the arc.

Taking the tetrahedron $(PTNB)$ as the system of reference and using equations (3) and (4) to determine the constants, the coordinates of a point on the curve in the vicinity of P have the values

$$\left.\begin{aligned} x_0 &= 1 - \tfrac{1}{2}s^2 + \tfrac{1}{24}(1+\kappa^2)s^4 + \dots \\ x_1 &= s - \tfrac{1}{6}(1+\kappa^2)s^3 + \dots \\ x_2 &= \tfrac{1}{2}\kappa s^2 + \dots \\ x_3 &= \tfrac{1}{6}\tau\kappa s^3 + \dots \end{aligned}\right\}. \tag{5}$$

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* Cf. Whitehead, *Universal Algebra*, p. 480.

The coordinates of the tangent and principal normal are

$$t_{01} = (x_0x_1'-x_1x_0'), \qquad t_{23} = (x_2x_3'-x_3x_2'), \quad \&c.,$$

$$n_{01} = \kappa^{-1}(x_0x_1''-x_1x_0''), \qquad n_{23} = \kappa^{-1}(x_2x_3''-x_3x_2''), \quad \&c.$$

The binormal is the reciprocal of the line joining the points x' and $\kappa^{-1}(x+x'')$. Its coordinates are therefore

$$b_{01} = \kappa^{-1}\{x_2'(x_3+x_3'')-x_3'(x_2+x_2'')\}, \qquad b_{23} = \kappa^{-1}\{x_0'(x_1+x_1'')-x_1'(x_0+x_0'')\}, \ \&c.$$

We have immediately $$\frac{dt_r}{ds} = \kappa n_r$$

for $$r = 1,\ 2,\ 3,\ 4,\ 5,\ 6.$$

Substituting the values in formula (5), we find

$$n_{01} = -\kappa s+\ldots, \qquad n_{23} = -\tfrac{1}{3}\tau\kappa s^3+\ldots,$$

and $$b_{01} = \tfrac{1}{2}\tau\kappa s^2+\ldots, \qquad b_{23} = \tfrac{1}{2}\kappa s^2+\ldots .$$

Hence $$\frac{db_1}{ds} = -(\tau+1)\,n_1 \quad \text{and} \quad \frac{db_4}{ds} = -(\tau-1)\,n_4.$$

Also $$t_1^2+n_1^2+b_1^2 = 1.$$

Therefore $$\frac{dn_1}{ds} = (\tau+1)\,b_1-\kappa t_1.$$

Similarly $$\frac{dn_4}{ds} = (\tau-1)\,b_4-\kappa t_4.$$

Summing up these results we get the following eighteen equations analogous to Frenet's formulæ

$$\left.\begin{aligned}
&\left.\begin{aligned}
\frac{dt_r}{ds} &= \kappa n_r\\
\frac{dn_r}{ds} &= (\tau+1)\,b_r-\kappa t_r\\
\frac{db_r}{ds} &= -(\tau+1)n_r
\end{aligned}\right\}\quad (r = 1,\ 2,\ 3)\\
\text{and}\quad
&\left.\begin{aligned}
\frac{dt_r}{ds} &= \kappa n_r\\
\frac{dn_r}{ds} &= (\tau-1)\,b_r-\kappa t_r\\
\frac{db_r}{ds} &= -(\tau-1)n_r
\end{aligned}\right\}\quad (r = 4,\ 5,\ 6)
\end{aligned}\right\}\qquad (6)$$

It is useful to note that these equations remain unaltered if the signs of κ, n and b are simultaneously changed.

5. Equations (6) of the preceding paragraph may also be obtained by considering the motion of the trihedron (t, n, b) as a solid body with one degree of freedom.

Let X be a point referred to a fixed quadrantal tetrahedron Δ, and x the same point referred to a moving tetrahedron Δ', the coordinates of whose vertices a, b, c, d are functions of t. Then

$$X_m = a_m x_0 + b_m x_1 + c_m x_2 + d_m x_3 \quad (m = 0, 1, 2, 3).$$

For a small displacement we have

$$dX_m = \Sigma a_m dx_0 + (\Sigma a'_m x_0) dt.$$

Hence

$$\left.\begin{array}{ll} a_0 dX_0 + a_1 dX_1 + a_2 dX_2 + a_3 dX_3 & = dx_0 - (\omega_{23} x_1 + \omega_{31} x_2 + \omega_{12} x_3) dt \\ b_0 dX_0 + \qquad\qquad ,, & = dx_1 - (-\omega_{23} x_0 + \omega_{03} x_2 - \omega_{02} x_3) dt \\ c_0 dX_0 + \qquad\qquad ,, & = dx_2 - (-\omega_{31} x_0 + \omega_{01} x_3 - \omega_{03} x_1) dt \\ d_0 dX_0 + \qquad\qquad ,, & = dx_3 - (-\omega_{12} x_0 + \omega_{02} x_1 - \omega_{01} x_1) dt \end{array}\right\}, \quad (7)$$

where

$$\begin{aligned} \omega_{01} &= c'_0 d_0 + c'_1 d_1 + c'_2 d_2 + c'_3 d_3, \\ \omega_{02} &= d'_0 b_0 + d'_1 b_1 + d'_2 b_2 + d'_3 b_3, \\ \omega_{03} &= b'_0 c_0 + b'_1 c_1 + b'_2 c_2 + b'_3 c_3, \\ \omega_{23} &= a'_0 b_0 + a'_1 b_1 + a'_2 b_2 + a'_3 b_3, \\ \omega_{31} &= a'_0 c_0 + a'_1 c_1 + a'_2 c_2 + a'_3 c_3, \\ \omega_{12} &= a'_0 d_0 + a'_1 d_1 + a'_2 d_2 + a'_3 d_3. \end{aligned}$$

Hence the displacements of the point x referred to the instantaneous position of Δ' are given by the right-hand sides of equations (7). That is, if $(\xi_0 \xi_1 \xi_2 \xi_3)$ are the coordinates of the point referred to the undisturbed position of Δ', we have

$$\begin{aligned} d\xi_0 &= dx_0 - (\omega_{23} x_1 + \omega_{31} x_2 + \omega_{12} x_3) dt, \\ d\xi_1 &= dx_1 - (-\omega_{23} x_0 + \omega_{03} x_2 - \omega_{02} x_3) dt, \\ d\xi_2 &= dx_2 - (-\omega_{31} x_0 + \omega_{01} x_3 - \omega_{03} x_1) dt, \\ d\xi_3 &= dx_3 - (-\omega_{12} x_0 + \omega_{02} x_1 - \omega_{01} x_2) dt. \end{aligned}$$

It is clear that the quantities ω_{01}, ω_{23}, &c., are the angular velocities round the edges ab, cd, &c.

The point (a_m, b_m, c_m, d_m), being a vertex of Δ, is fixed. Therefore

$$\left.\begin{aligned} \frac{da_m}{dt} &= \omega_{23} b_m + \omega_{31} c_m + \omega_{12} d_m \\ \frac{db_m}{dt} &= -\omega_{23} a_m + \omega_{03} c_m - \omega_{02} d_m \\ \frac{dc_m}{dt} &= -\omega_{31} a_m + \omega_{01} d_m - \omega_{03} b_m \\ \frac{dd_m}{dt} &= -\omega_{12} a_m + \omega_{02} b_m - \omega_{01} c_m \end{aligned}\right\} \quad (m = 0, 1, 2, 3). \qquad (8)$$

Let the coordinates of the edges ab, ac, ad be respectively

$$a_r,\ \beta_r,\ \gamma_r \quad (r = 1, 2, 3, 4, 5, 6).$$

We can readily find da_r/dt, &c., by means of equations (8).

The resulting equations are

$$\left.\begin{aligned} &\left.\begin{aligned} \frac{da_r}{dt} &= \omega_3 \beta_r - \omega_2 \gamma_r \\ \frac{d\beta_r}{dt} &= \omega_1 \gamma_r - \omega_3 a_r \\ \frac{d\gamma_r}{dt} &= \omega_2 a_r - \omega_1 \beta_r \end{aligned}\right\} \quad (r = 1, 2, 3) \\ &\left.\begin{aligned} \frac{da_r}{dt} &= \omega_6 \beta_r - \omega_5 \gamma_r \\ \frac{d\beta_r}{dt} &= \omega_4 \gamma_r - \omega_6 a_r \\ \frac{d\gamma_r}{dt} &= \omega_5 a_r - \omega_4 \beta_r \end{aligned}\right\} \quad (r = 4, 5, 6) \end{aligned}\right\}, \qquad (9)$$

where
$$\omega_1 = \omega_{01} + \omega_{23}, \quad \omega_2 = \omega_{02} + \omega_{31}, \quad \omega_3 = \omega_{03} + \omega_{12},$$
$$\omega_4 = \omega_{01} - \omega_{23}, \quad \omega_5 = \omega_{02} - \omega_{31}, \quad \omega_6 = \omega_{03} - \omega_{12}.$$

When the vertex a moves with unit speed along a curve and a, β, γ are respectively the tangent, normal and binormal the values of the angular velocities are

$$\omega_{23} = 1, \quad \omega_{31} = 0, \quad \omega_{12} = 0,$$

$$\omega_{01} = \tau, \quad \omega_{02} = 0, \quad \omega_{03} = \kappa,$$

and equations (9) reduce to the anologues given in the preceding paragraph.

The Indicatrices Γ_1, Γ_2.

6. Denoting by κ_1, s_1, r_1 the curvature, arc, and radius of curvature of the Right Indicatrix of tangents Γ_1, we have

$$ds_1^2 = dt_1^2 + dt_2^2 + dt_3^2 = \kappa^2 ds^2$$

from equations (6) or

$$ds_1 = \kappa ds. \tag{10}$$

The tangent to Γ_1 is reciprocal to the line joining the origin to the point $R(b)$. Its coordinates are therefore $(b_1, b_2, b_3, -b_1, -b_2, -b_3)$.

The normal to Γ_1 is reciprocal to the line joining $R(n)$ to the origin, and its coordinates are

$$(-n_1, \ -n_2, \ -n_3, \ n_1, \ n_2, \ n_3).$$

Therefore, from equations (6),

$$\frac{db_1}{ds_1} = -\kappa_1 n_1,$$

or

$$-\frac{\tau+1}{\kappa} n_1 = -\kappa_1 n_1.$$

Therefore

$$\kappa_1 = \cot r_1 = \frac{\tau+1}{\kappa}. \tag{11}$$

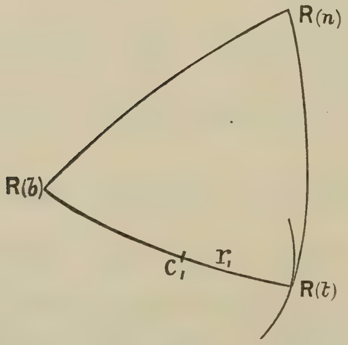

Fig. 1.

Fig. 1 shows the relative positions of the points $R(t)$, $R(n)$, $R(b)$ and C_1 the centre of curvature of the Right Indicatrix when κ and κ_1 are both positive.

Denoting by κ_2, s_2, r_2 the corresponding elements of the Left Indicatrix Γ_2, we have

$$ds_2 = \kappa ds = ds_1,$$

and

$$\kappa_2 = \cot r_2 = (\tau - 1)/\kappa. \tag{12}$$

7. Conversely two arbitrary plane curves Γ_1 and Γ_2 may be regarded as the Right and Left Indicatrices of a system of curves, and if we put $s_1 - s_2 = \alpha$ or $s_1 + s_2 = \alpha$, to each value of α corresponds a pair of curves and as α varies two surfaces are generated.

The properties of these surfaces may be investigated as follows.

We have a congruence of lines given by the coordinates

$$t_1(s_1), \quad t_2(s_1), \quad t_3(s_1), \quad t_4(s_2), \quad t_5(s_2), \quad t_6(s_2),$$

where

$$t_1^2 + t_2^2 + t_3^2 = 1, \qquad \left(\frac{dt_1}{ds_1}\right)^2 + \left(\frac{dt_2}{ds_1}\right)^2 + \left(\frac{dt_3}{ds_1}\right)^2 = 1,$$

$$t_4^2 + t_5^2 + t_6^2 = 1, \qquad \left(\frac{dt_4}{ds_2}\right)^2 + \left(\frac{dt_5}{ds_2}\right)^2 + \left(\frac{dt_6}{ds_2}\right)^2 = 1.$$

The line is a tangent to a curve, if

$$ds_1^2 = ds_2^2.$$

Consider, in the first instance, the case $s_1 - s_2 = \text{const}$. The principal normal is given by

$$t_1'(s_1), \quad t_2'(s_1), \quad t_3'(s_1), \quad t_4'(s_2), \quad t_5'(s_2), \quad t_6'(s_2),$$

and the binormal by

$$t_2 t_3' - t_3 t_2', \ \ldots, \ \ldots, \ t_5 t_6' - t_6 t_5', \ \ldots, \ \ldots.$$

The coordinates of the point on the curve may be expressed in terms of s_1 and s_2, but the expressions are unsymmetrical and unwieldy and are not required. It is however apparent that we may regard s_1 and s_2 as curvilinear coordinates for the surface. The curves $s_1 - s_2 = \text{constant}$ are generating curves having Γ_1 and Γ_2 as Indicatrices. The curve $s_1 = \text{constant}$ is clearly the locus of points on different generating curves, the tangents at which are right parallels and $s_2 = \text{constant}$ gives the locus for left parallelism.

It will now be proved that the surface has the following properties :—

(*a*) The generating curves $s_1 - s_2 =$ constant are geodesics on the surface.

(*b*) The curves $s_1 =$ constant and $s_2 =$ constant are the asymptotic lines.

(*c*) The Indicatrices of any geodesic on the surface are parallel curves to Γ_1, Γ_2.

(*d*) The geodesic geometry of the surface is Euclidean.

8. One method of investigation is as follows.

For given values of s_1 and s_2 the values of t, n, b are determined by means of the formulæ in the preceding paragraph. Consider then the curve $s_1 =$ constant, and let T, N, B denote the values of t, n, b for a small increase in s_2. Also let δ_{pq} and δ'_{pq} denote the common perpendiculars of two lines p and q. Then

$$\cos(\delta_{pq} - \delta'_{pq}) = \overline{pq},$$

$$\cos(\delta_{pq} + \delta'_{pq}) = \underline{pq},$$

and
$$4\sin^2 \tfrac{1}{2}(\delta_{pq} - \delta'_{pq}) = (p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2,$$

$$4\sin^2 \tfrac{1}{2}(\delta_{pq} + \delta'_{pq}) = (p_4 - q_4)^2 + (p_5 - q_5)^2 + (p_6 - q_6)^2.$$

We have therefore
$$(\delta_{tT} - \delta'_{tT})^2 = 0$$

and
$$(\delta_{tT} + \delta'_{tT})^2 = (t_4'^2 + t_5'^2 + t_6'^2)\, ds_2^2,$$

or
$$\delta_{tT} = \tfrac{1}{2} ds_2. \qquad (13)$$

Similarly
$$4\delta_{bB}^2 = \Sigma(b'_4 ds_2 + \ldots)^2$$

and
$$\frac{db_4}{ds_2} = \kappa^{-1}\frac{db_4}{ds} = -\frac{\tau - 1}{\kappa}\, n_4,$$

that is
$$\delta_{bB} = \tfrac{1}{2}\cot r_2\, ds_2. \qquad (14)$$

Also
$$4\delta_{nN}^2 = \Sigma\kappa^{-2}\left\{(\tau - 1)\, b_4 - \kappa t_4\right\}^2 ds_2^2,$$

that is
$$\delta_{nN} = \tfrac{1}{2}\operatorname{cosec} r_2\, ds_2. \qquad (15)$$

Again
$$\cos(\delta_{bT} - \delta'_{bT}) = 0,$$

since
$$T_r = t_r \quad (r = 1, 2, 3),$$

and
$$\cos(\delta_{bT}+\delta'_{bT}) = \Sigma b_4 (t_4+t'_4 ds_2+\tfrac{1}{2}t''_4 ds_2^2 \ldots)$$
$$= \tfrac{1}{2}\Sigma b_4 t''_4 ds_2^2.$$

Hence δ_{bT} is of the second order and the line b touches the surface. This proves that the generating curves are geodesics on the surface.

Now
$$\cos(\delta_{Nt}-\delta'_{Nt}) = 0$$
and
$$\cos(\delta_{Nt}+\delta'_{Nt}) = \Sigma t_4(n_4+n'_4 ds_2+\ldots) = -ds_2,$$
that is
$$\sin\delta_{Nt}\sin\delta'_{Nt} = \tfrac{1}{2}ds_2. \qquad (16)$$

Hence N and t have a positive mutual moment when ds_2 is positive.

Finally,
$$\cos(\delta_{Nb}-\delta'_{Nb}) = 0$$
and
$$\cos(\delta_{Nb}+\delta'_{Nb}) = \Sigma b_4(n_4+n'_4 ds_2 \ldots) = \cot r_2 ds_2.$$
Hence
$$\sin\delta_{Nb}\sin\delta'_{Nb} = -\tfrac{1}{2}\cot r_2 ds_2, \qquad (17)$$
that is N and b have a negative mutual moment if κ is positive and $\tau > 1$.

Equations (13), (14), (15), (16), (17) prove that when κ is positive and $\tau > 1$, consecutive points P and P' on the curve $s_1 =$ constant are situated as shown in Fig. 2, and that the distance $PP' = \tfrac{1}{2}\operatorname{cosec} r_2 ds_2$. (The principal normal is in the direction from P towards the observer.)

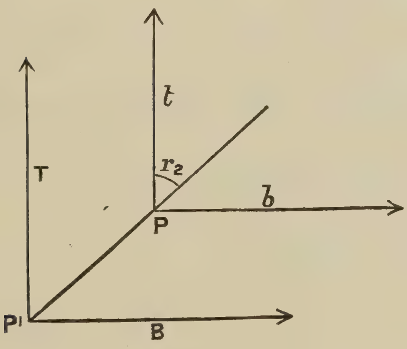

FIG. 2.

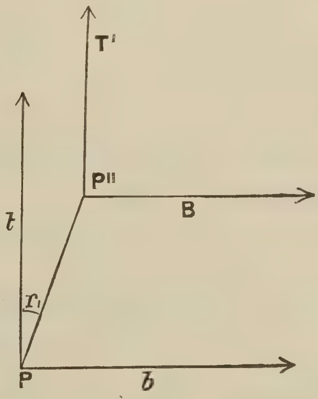

FIG. 3.

It is found in a similar manner that consecutive points P and P'' on the curve $s_2 =$ constant are situated as shown in Fig. 3, and that
$$PP'' = \tfrac{1}{2}\operatorname{cosec} r_1 ds_1.$$

If Q is any point $(s_1+ds_1,\ s_2+ds_2)$ on the surface near $P(s_1,\ s_2)$ and du, dv the projections of PQ on t and b, we have

$$du = \tfrac{1}{2}\cot r_1\,ds_1-\tfrac{1}{2}\cot r_2\,ds_2,$$

$$dv = \tfrac{1}{2}ds_1-\tfrac{1}{2}ds_2.$$

These are perfect differentials, since r_1 is a function of s_1 only and r_2 a function of s_2 only.

The element of length is then

$$dS^2 = du^2+dv^2,$$

and the geodesic geometry of the surface is Euclidean.

The simplest example of such a surface is a circular cylinder.

9. Another way of obtaining an expression for the element of length on the surface is as follows:—

Lemma.—"If t, n, b are three mutually rectangular lines intersecting in a point P, and T, N, B three mutually rectangular lines intersecting in a point Q, then

$$\begin{aligned}3-4\sin^2 PQ = {} & \overline{tT}.\underline{tT}+\overline{tN}.\underline{tN}+\overline{tB}.\underline{tB}\\ & +\overline{nT}.\underline{nT}+\overline{nN}.\underline{nN}+\overline{nB}.\underline{nB}\\ & +\overline{bT}.\underline{bT}+\overline{bN}.\underline{bN}+\overline{bB}.\underline{bB},\end{aligned}$$

where $$\overline{tT} = t_1T_1+t_2T_2+t_3T_3,$$

and $$\underline{tT} = t_4T_4+t_5T_5+t_6T_6, \text{ \&c.''}$$

Let d_1, d_2, d_3 be the perpendicular distance of Q from the lines t, n, b respectively. Then

$$2\sin^2 PQ = \sin^2 d_1+\sin^2 d_2+\sin^2 d_3.$$

Now a line through Q is given by the coordinates

$$\lambda T_r+\mu N_r+\nu B_r \quad \text{for} \quad r = 1,\ 2,\ 3,\ 4,\ 5,\ 6.$$

This line will be right-parallel to t, if

$$\lambda T_1+\mu N_1+\nu B_1 = t_1,\quad \lambda T_2+\mu N_2+\nu B_2 = t_2,\quad \lambda T_3+\mu N_3+\nu B_3 = t_3.$$

Hence $$\lambda = \overline{tT},\quad \mu = \overline{tN},\quad \nu = \overline{tB},$$

and $$\cos 2d_1 = \overline{tT}.\underline{tT}+\overline{tN}.\underline{tN}+\overline{tB}.\underline{tB}.$$

Similarly $$\cos 2d_2 = \overline{nT}.\underline{nT}+\overline{nN}.\underline{nN}+\overline{nB}.\underline{nB},$$

and $$\cos 2d_3 = \overline{bT}.\underline{bT}+\overline{bN}.\underline{bN}+\overline{bB}.\underline{bB},$$

and the Lemma follows immediately.

In the present instance, using equations (6), (10), (11), and (12) to effect the reductions, we have

$$\overline{tT} = \Sigma t_1 (t_1+t_1' ds_1+\tfrac{1}{2}t_1'' ds_1^2+\ldots) \qquad \underline{tT} = 1-\tfrac{1}{2}ds_2^2+\ldots,$$
$$= 1-\tfrac{1}{2}ds_1^2+\ldots.$$

Since $$\Sigma t_1 t_1'' = -\Sigma t_1'^2;$$

$$\overline{tN} = \Sigma t_1 (n_1+n_1' ds_1+\ldots) \qquad \underline{tN} = -ds_2+\ldots,$$
$$= -ds_1+\ldots,$$

$$\overline{tB} = \Sigma t_1 (b_1+b_1' ds_1+\tfrac{1}{2}b_1'' ds_1^2+\ldots) \qquad \underline{tB} = \tfrac{1}{2}\cot r_2\, ds_2^2+\ldots,$$
$$= \tfrac{1}{2}\cot r_1\, ds_1^2+\ldots.$$

Since $$\Sigma t_1 b_1'' = -\Sigma t_1' b_1' = (\tau+1)/\kappa;$$

$$\overline{nT} = \Sigma n_1 (t_1+t_1' ds_1+\ldots) \qquad \underline{nT} = ds_2+\ldots,$$
$$= ds_1+\ldots,$$

$$\overline{nN} = \Sigma n_1 (n_1+n_1' ds_1+\tfrac{1}{2}n_1'' ds_1^2+\ldots) \qquad \underline{nN} = 1-\tfrac{1}{2}\operatorname{cosec}^2 r_2\, ds_2^2+\ldots,$$
$$= 1-\tfrac{1}{2}\operatorname{cosec}^2 r_1\, ds_1^2+\ldots.$$

Since $$\Sigma n_1 n_1'' = -\kappa^{-2}\Sigma\left\{(\tau+1)b_1-\kappa t_1\right\}^2;$$

$$\overline{nB} = \Sigma n_1 (b_1+b_1' ds_1+\tfrac{1}{2}b_1'' ds_1^2+\ldots) \qquad \underline{nB} = -\cot r_2\, ds_2+\ldots,$$
$$= -\cot r_1\, ds_1+\ldots;$$

$$\overline{bT} = \Sigma b_1 (t_1+t_1' ds_1+\tfrac{1}{2}t_1'' ds_1^2+\ldots) \qquad \underline{bT} = \tfrac{1}{2}\cot r_2\, ds_2^2+\ldots,$$
$$= \tfrac{1}{2}\cot r_1\, ds_1^2+\ldots;$$

$$\overline{bN} = \Sigma b_1 (n_1+n_1' ds_1+\ldots) \qquad \underline{bN} = \cot r_2\, ds_2+\ldots,$$
$$= \cot r_1\, ds_1+\ldots;$$

$$\overline{bB} = \Sigma b_1 (b_1+b_1' ds_1+\tfrac{1}{2}b_1'' ds_1^2+\ldots) \qquad \underline{bB} = 1-\tfrac{1}{2}\cot^2 r_2\, ds_2^2+\ldots,$$
$$= 1-\tfrac{1}{2}\cot^2 r_1\, ds_1^2+\ldots.$$

Substituting the above values in the formula of the Lemma, we get

$$4\sin^2 PQ = (ds_1-ds_2)^2+(\cot r_1 ds_1-\cot r_2 ds_2)^2$$
$$+\text{terms of the third and higher orders.}$$

Hence $$dS^2 = du^2+dv^2,$$

where $$du = \tfrac{1}{2}\cot r_1\, ds_1-\tfrac{1}{2}\cot r_2 ds_2,$$

and $$dv = \tfrac{1}{2}ds_1-\tfrac{1}{2}ds_2,$$

as in § 8.

10. We have now to examine the nature of the curve $s_1 =$ constant.

Let the tangent, normal and binormal to this curve be denoted by T, N, and B.

We have $$T = t\cos r_2+b\sin r_2,$$

and $$dS = \tfrac{1}{2}\operatorname{cosec} r_2\, ds_2.$$

Now $$\frac{dT_1}{dS} = 2\sin r_2(-t_1\sin r_2+b_1\cos r_2)\frac{dr_2}{ds_2},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{dT_4}{dS} = 2\sin r_2(-t_4\sin r_2+b_4\cos r_2)\frac{dr_2}{ds_2}+2\sin r_2\left(\frac{dt_4}{ds_2}\cos r_2+\frac{db_4}{ds_2}\sin r_2\right)$$

$$= 2\sin r_2(-t_4\sin r_2+b_4\cos r_2)\frac{dr_2}{ds_2},$$

since $$\cot r_2 = (\tau-1)/\kappa = -\frac{db_4}{dt_4}.$$

Hence N is the line $(-t\sin r_2+b\cos r_2)$ and lies in the tangent plane. The curves $s_1=$ constant are consequently one system of asymptotic lines. The curvature is $2\sin r_2 \,.\, dr_2/ds_2$, and can easily be expressed in terms of κ, τ and their derivatives. The torsion is equal to -1, since B is the normal to the surface and moves right-parallel to itself along the curve $s_1 =$ constant.

The curves $s_2 =$ constant are the other system of asymptotic lines and have constant torsion equal to 1.

11. The tangent to the geodesic making a constant angle θ with the geodesics $v=$ constant is $t\cos\theta+b\sin\theta$. The Right and Left Indicatrices

of this geodesic are therefore curves parallel to Γ_1 and Γ_2 and at a normal distance θ from these curves.

There is a second surface corresponding to the relation $s_1+s_2=$ constant, and the two surfaces constitute the focal surface of the congruence of lines whose R and L points move on Γ_1 and Γ_2 respectively.

12. Equations (6) have the advantage over the Euclidean analogues in so far as they concern all six coordinates of a line and determine the line completely. It is not surprising then to find that they are particularly powerful in all investigations concerning curvature and torsion.

A general method of procedure is as follows. The lines t, n, b and their reciprocals t', n', b' form a quadrantal tetrahedron, and the coordinates of any other line can be expressed linearly in terms of t, n, b, t', n', b'. Let the elements of an associated curve be T, N, B and its curvature, torsion and arc K, τ, S respectively. If T be expressed in terms of t, n, b, &c., the coordinates of N may be obtained by differentiation. B can then be found and further differentiations enable us to determine K and τ.

13. Take, for example, the associated curve to be the edge of regression of the polar developable surface.

Here $$T = b\cos r + t'\sin r,$$

that is, $$T_1 = b_1\cos r + t_1 \sin r, \text{ \&c.},$$

and $$T_4 = b_4\cos r - t_4\sin r.$$

Therefore

$$KN_1\,dS/ds = \{\kappa \sin r - (\tau+1)\cos r\}\, n_1 + t_1\cos r\,\frac{dr}{ds} - b_1 \sin r\,\frac{dr}{ds}$$

$$= t_1\cos r\,\frac{dr}{ds} - b_1\sin r\,\frac{dr}{ds} - n_1\tau\cos r. \qquad (18)$$

Similarly $$KN_4\,dS/ds = -t_4\cos r\,\frac{dr}{ds} - b_4\sin r\,\frac{dr}{ds} - n_4\tau\cos r. \qquad (19)$$

Consequently N is the line

$$\left\{(t'\cos r - b\sin r)\,\frac{dr}{ds} - n\tau\cos r\right\}.$$

The line B' meets T and N at right angles and is given by

$$n\frac{dr}{ds}+t'\tau\cos^2 r-b\tau\cos r\sin r.$$

Therefore B is the line

$$n'\frac{dr}{ds}+t\tau\cos^2 r-b'\tau\cos r\sin r.$$

The angle between B and t is R, the radius of spherical curvature.

Hence $$\cos R=\tau\cos^2 r\Big/\left\{\left(\frac{dr}{ds}\right)^2+\tau^2\cos^2 r\right\}^{\frac{1}{2}}, \tag{20}$$

and $$\sin h=\frac{dr}{ds}\Big/\left\{\left(\frac{dr}{ds}\right)^2+\tau^2\cos^2 r\right\}^{\frac{1}{2}},$$

where h is the distance of the centre of spherical curvature from the principal normal. The line B can now be expressed in the form

$$B=t\cos R-b'\cos h\sin r+n'\sin h.$$

Differentiating we find

$$-(\tau+1)\,N_1\,dS/ds=t_1\{-\sin R\,dR/ds-\kappa\sin h\}+\&c.,$$

and $$-(\tau-1)\,N_4\,dS/ds=t_4\{-\sin R\,dR/ds+\kappa\sin h\}+\&c.$$

Comparing with (18) and (19), we obtain the relations

$$(\tau+1)/K=(\kappa\sin h+\sin R\,dR/ds)/\cos r\,\frac{dr}{ds},$$

$$(\tau-1)/K=(\kappa\sin h-\sin R\,dR/ds)/\cos r\,\frac{dr}{ds}.$$

Therefore $$K^{-1}=\sin R\sec r\,dR/dr,$$

and $$\tau K^{-1}=\sin h\,\mathrm{cosec}\,r\Big/\frac{dr}{ds}.$$

Equation (20) may be reduced to

$$\tan^2 R=\tan^2 r+\left(\frac{1}{\tau}\,\frac{d}{ds}\tan r\right)^2. \tag{21}$$

The Reciprocal Curve.

14. In this case we may write

$$T \equiv t'.$$

Therefore $\quad KN_1\,dS = \kappa n_1\,ds, \quad$ and $\quad KN_4\,dS = -\kappa n_4\,ds.$

Hence we may write $\quad N \equiv n'.$

Consequently $\quad B \equiv b.$

From which $\quad (\mathcal{T}+1)\,N_1\,dS = (\tau+1)\,n_1\,ds$

and $\quad (\mathcal{T}-1)\,N_4\,dS = (\tau-1)\,n_4\,ds.$

Hence $\quad (\mathcal{T}+1)/K = (\tau+1)/\kappa,$

and $\quad (\mathcal{T}-1)/K = -(\tau-1)/\kappa.$

So that $\quad K = \kappa/\tau \quad$ and $\quad \mathcal{T} = 1/\tau.$

These values of K and $\mathcal{T}$ are otherwise immediately evident, except possibly as regards sign.

The rectifying developable of a curve is the polar developable of the reciprocal curve. Consequently the edge of regression of the rectifying developable is the centre of a sphere which touches four consecutive osculating planes. Suitably modifying equation (21) we find R_1 the radius of this sphere by the formula

$$\cot^2 R_1 = \tau^2/\kappa^2 + \left\{ \frac{d}{ds}(\tau/\kappa) \right\}^2.$$

Curves for which κ and τ are Constant.

15. Putting $\quad (\tau+1)/\kappa = \cot r_1 \quad$ and $\quad (\tau-1)/\kappa = \cot r_2,$

we infer from equations (6) that

$$t_m \cos r_1 + b_m \sin r_1 = c_m \quad (m = 1,\ 2,\ 3),$$

and

$$t_m \cos r_2 + b_m \sin r_2 = c_m \quad (m = 4,\ 5,\ 6).$$

Evidently $\quad c_1^2+c_2^2+c_3^2 = 1 = c_4^2+c_5^2+c_6^2.$

Therefore $(c_1,\ c_2,\ c_3,\ c_4,\ c_5,\ c_6)$ are the coordinates of a fixed line c.

Also $\overline{cn} = 0 = \underline{cn}$, and therefore n meets c at right angles.

Moreover $$\overline{c(t\cos r_1 + b\sin r_1)} = 1,$$

and $$\underline{c(t\cos r_2 + b\sin r_2)} = 1.$$

Hence the lines $(t\cos r_1 + b\sin r_1)$ and $(t\cos r_2 + b\sin r_2)$ are respectively right and left parallel to c, and the length of the normal is equal to $\frac{1}{2}(r_1 \sim r_2)$.

The curve is consequently a geodesic on a circular cylinder of which $(t\cos r_1 + b\sin r_1)$ and $(t\cos r_2 + b\sin r_2)$ are generators.

Curves for which $(\tau \pm 1)/\kappa$ is constant.

16. If $(\tau + 1)/\kappa$ is constant, we have

$$t_m \cos r_1 + b_m \sin r_1 = c_m \quad (m = 1, 2, 3).$$

Hence the line $(t\cos r_1 + b\sin r_1)$ is right-parallel to a fixed line, and the curve is a geodesic on a ruled surface whose generators are right-parallels.

The interpretation of $(\tau - 1)/\kappa =$ constant is analogous. These surfaces may be generated by the screw motion of a curve about a given axis, the pitch of the screw being ± 1.

Bertrand Curves.

16. In this case $$n \equiv N.$$

Now $$\frac{d}{ds}\,\overline{tT} = \kappa\,\overline{nT} + S'K\,\overline{tN} = 0,$$

and $$\frac{d}{ds}\,\underline{tT} = \kappa\,\underline{nT} + S'K\,\underline{tN} = 0,$$

where $$S' = dS/ds.$$

Therefore $\cos(\delta - \delta_1)$ and $\cos(\delta + \delta_1)$ are constant, where δ and δ_1 are the perpendicular distances of t and T.

Hence corresponding points on the two curves are at a constant distance apart and the osculating planes are inclined at a constant angle.

Supposing δ to be the distance and δ_1 the angle, we find on differentiating $\overline{tN}$, $\underline{tN}$, $\overline{bN}$, $\underline{bN}$, the relations

$$\kappa = S' \left\{(\mathcal{T}+1)\sin(\delta_1-\delta)+K\cos(\delta_1-\delta)\right\},$$

$$\kappa = S' \left\{(\mathcal{T}-1)\sin(\delta_1+\delta)+K\cos(\delta_1+\delta)\right\},$$

$$\tau+1 = S' \left\{(\mathcal{T}+1)\cos(\delta_1-\delta)-K\sin(\delta_1-\delta)\right\},$$

$$\tau-1 = S' \left\{(\mathcal{T}-1)\cos(\delta_1+\delta)-K\sin(\delta_1+\delta)\right\}.$$

The following results are easily deduced:

$$\tau\mathcal{T} = \sin^2\delta_1 \operatorname{cosec}^2\delta,$$

showing that the torsions have the same sign:

$$(\cot\delta-\kappa)(\cot\delta+K) = \cos^2\delta_1 \operatorname{cosec}^2\delta,$$

giving an anharmonic property for the two centres of curvature and the two points on the curves:

$$\kappa+\tau\cot\delta_1 = \cot\delta,$$

which is the necessary relation between κ and τ:

$$R_1 = r_1+\delta-\delta_1, \quad R_2 = r_2-\delta-\delta_1,$$

where R_1, r_1 are the radii of curvatures of the Right Indicatrices of the two curves, and R_2, r_2 those of the Left Indicatrices. Hence the Right Indicatrices are parallel curves at a normal distance $\delta_1-\delta$, and the Left Indicatrices are parallel and at a normal distance $\delta_1+\delta$.

Mannheim Curves.

17. If the principal normal of the first curve is the binormal of the second curve, we have

$$n \equiv B.$$

Let δ be the distance between the points on the curves, and δ_1 the second common perpendicular to t and T.

We have $$\frac{d}{ds}\overline{bT} = -(\tau+1)\overline{nT}+S'K\overline{bN},$$

and $$\frac{d}{ds}\underline{bT} = -(\tau-1)\underline{nT}+S'K\underline{bN}.$$

Now $\overline{bT} = \cos\left(\delta_1-\frac{\pi}{2}-\delta\right)$ and $\overline{bN} = \cos(\delta_1-\pi-\delta)$.

Therefore $$\frac{d}{ds}(\delta_1-\delta) = -S'K,$$

and similarly $$\frac{d}{ds}(\delta_1+\delta) = -S'K.$$

Accordingly δ is constant, and corresponding points on the two curves are at a constant distance apart. Also

$$\frac{d}{ds}\overline{tB} = 0 = \kappa\,\overline{nB} - S'(\tau+1)\overline{tN} = \kappa - S'(\tau+1)\cos\left(\delta_1-\frac{\pi}{2}-\delta\right).$$

Therefore $$\kappa = S'(\tau+1)\sin(\delta_1-\delta).$$

Similarly $$\kappa = S'(\tau-1)\sin(\delta_1+\delta).$$

Differentiating $\overline{bB}$ and $\underline{bB}$, we find

$$\tau+1 = S'(\tau+1)\cos(\delta_1-\delta),$$

and $$\tau-1 = S'(\tau-1)\cos(\delta_1+\delta).$$

Hence $$\cot r_1 = (\tau+1)/\kappa = \cot(\delta_1-\delta),$$

and $$\cot r_2 = (\tau-1)/\kappa = \cot(\delta_1+\delta).$$

Therefore $$r_2-r_1 = 2\delta \pmod{\pi},$$

so that for a Mannheim curve the difference of the radii of curvature of the Right and Left Indicatrices is constant.

The relation between κ and τ is

$$(\kappa-\cot 2\delta)^2+\tau^2 = \operatorname{cosec}^2 2\delta.$$

If this relation is given, δ is determined to a multiple of $\pi/2$. This is explained by the fact that a curve and its reciprocal have the same binormal.

Curves of a Linear Complex.

18. A linear complex may be defined by the equation

$$a_1p_1+a_2p_2+a_3p_3+a_4p_4+a_5p_5+a_6p_6 = 0,$$

or shortly $$\overline{ap}+\underline{ap} = 0.$$

If t is a tangent at a point P to a curve of the complex, we have

$$\overline{at}+\underline{at} = 0.$$

Differentiating with respect to the arc, we get

$$\overline{an}+\underline{an} = 0.$$

Therefore n belongs to the complex, and the polar plane of P is the osculating plane.

A second differentiation gives

$$(\tau+1)\,\overline{ab}+(\tau-1)\,\underline{ab} = 0.$$

Hence the torsion depends only on b and is the same for all curves through P.

Differentiating again, we get

$$d\tau/ds\,(\overline{ab}+ab) = 2\tau(\overline{an}-\underline{an}).$$

Hence $d\tau/ds$ is the same for all curves with the same tangent at P. We note also that for a maximum or minimum value of τ the principal normal meets the axes of the complex at right angles. [It is evident that the lines $(\theta a_1,\ \theta a_2,\ \theta a_3,\ \pm\phi a_4,\ \pm\phi a_5,\ \pm\phi a_6)$ where $\theta^{-2} = a_1^2+a_2^2+a_3^2$ and $\phi^{-2} = a_4^2+a_5^2+a_6^2$ are the axes of the complex.]

Two further differentiations enable us to eliminate $\overline{at}$, $\underline{at}$, &c., and we obtain the following differential equation* connecting the torsion and curvature and their derivatives with respect to the arc of a curve of any linear complex

$$4\tau^2\kappa\tau_3-2\tau\tau_2(2\tau\kappa_1+9\tau_1\kappa)+15\kappa\tau_1^3+6\tau\kappa_1\tau_1^2$$
$$+4\tau^2\tau_1\kappa\,(1+3\tau^2+\kappa^2)-8\tau^3\kappa_1(\tau^2-1) = 0,$$

the suffixes denoting differentiation with respect to s.

Curves of a Line Complex of any Order.

19. Let $$F(p) \equiv F(p_1,\, p_2,\, p_3,\, p_4,\, p_5,\, p_6) = 0,$$

be the equation of a general line complex.

For a curve of the complex through an arbitrary point P, we have

$$F(t) = 0,$$

and differentiating, $$\Sigma n_r F_r = 0,$$

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* Cf. Egan, "The Linear Complex and a certain Class of Twisted Curves," *Proc. Royal Irish Acad.*, 1911, p. 61.

where
$$F_r = \frac{\partial F}{\partial t_r}.$$

Substituting $t+\lambda n$ for t, we find that

$$F(t+\lambda n) = 0$$

is divisible by λ^2. Hence the osculating plane touches the cone corresponding to the point P.

A second differentiation gives

$$\kappa \Sigma n_r n_s F_{rs} + (\tau+1)(b_1 F_1 + b_2 F_2 + b_3 F_3) + (\tau-1)(b_4 F_4 + b_5 F_5 + b_6 F_6) = 0. \quad (22)$$

There is consequently a linear relation connecting the curvature and torsion at a point of all curves of a complex which touch one another at the point.

If P is a singular point and t a singular line of the complex, we have the relation
$$F_1^2 + F_2^2 + F_3^2 = F_4^2 + F_5^2 + F_6^2,$$

and the coordinates of the conjugate line are

$$F_1,\ F_2,\ F_3,\ -F_4,\ -F_5,\ -F_6.$$

Denoting this line by q, equation (22) becomes

$$\kappa \Sigma n_r n_s F_{rs} + \overline{bq} + \underline{bq} = 0.$$

In the general case, when the singular line is a double line on the cone, we have the following theorems :—

For all curves of a complex which touch a singular line at the singular point, those which have the same osculating plane have the same curvature at the point.

For curves which have the singular plane as osculating plane, the singular point is a point of inflexion.

For curves which have for osculating plane either tangent plane through the double line, the singular point is a cusp.

By considering the reciprocal curve, we find the following theorems for curves of the complex which touch the singular line at points other than the singular point :—

Curves which touch the singular line at points other than the singular *point have the singular plane as osculating plane, and the ratio* κ/τ *is the same for all curves touching at the same point.*

There are two points on a singular line for which $\tau = 0$, *viz. the points of contact with the complex curve in the singular plane.*

For a plane inclined at an angle θ to the singular plane the curvature is given by a relation of the form

$$\kappa(A \cos^2 \theta + 2B \cos \theta \sin \theta + C \sin^2 \theta) = D \sin \theta. \tag{23}$$

In the particular case when the complex consists of the tangent lines to a surface, we find $A = C = 0$ and equation (23) reduces to Meusnier's theorem.

A METHOD OF SOLVING CERTAIN LINEAR PARTIAL DIFFERENTIAL EQUATIONS

By T. W. Chaundy.

[Received March 3rd, 1921.—Read March 10th, 1921.]

1. The method enables us, for certain types of linear partial differential equation (of various orders and numbers of independent variables) to obtain a solution which has as many arbitrary elements as we ought to expect in the general solution, and which may be made to coincide with the general solution in those cases in which such solution has already been established. The method, however, is not powerful enough to determine whether such solutions are actually general solutions.

The method arises out of the method of Frobenius for solving linear ordinary differential equations in series in the special case in which the differential equation can be thrown into the form

$$\phi(\delta)y = x\psi(\delta)y,$$

where δ is the differential operator xd/dx and ϕ, ψ are any polynomial functions of their arguments. Such an equation has, we know, in general, a solution

$$y = x^a + \frac{\psi(a)}{\phi(a+1)}x^{a+1} + \frac{\psi(a)\,\psi(a+1)}{\phi(a+1)\,\phi(a+2)}x^{a+2} + \dots,$$

where the initial exponent a is a root of the "indicial" equation

$$\phi(a) = 0.$$

In application to partial differential equations we write similarly

$$\delta = x\frac{\partial}{\partial x}, \quad \delta' = y\frac{\partial}{\partial y}, \quad \text{or} \quad \delta_1 = x_1\frac{\partial}{\partial x_1}, \quad \delta_2 = x_2\frac{\partial}{\partial x_2}, \quad \delta_3 = x_3\frac{\partial}{\partial x_3}, \quad \dots.$$

The equations to which the method is appropriate are of the type

$$\phi(\delta_1, \delta_2, \dots)z = x_1^\alpha x_2^\beta \dots \psi(\delta_1, \delta_2, \dots)z,$$

and in some cases

$$\phi(\delta_1, \delta_2, \ldots)z = \Sigma x_1^\alpha x_2^\beta \ldots \psi(\delta_1, \delta_2, \ldots)z.$$

The method is best expounded by applying it to one or two well-known equations. I take (i) the hyperbolic form $z_{xy} = z$, and (ii) the parabolic form $z_{xx} = z_y$. A full discussion of these equations is to be found, for instance, in Goursat, *Cours d'Analyse*, tome 3.

The Equation $z_{xy} = z$.

2. Just as the exponential function is fundamental in the theory of the ordinary equation $z_x = z$, so in discussion of the equation $z_{xy} = z$, we need the Bessel's function

$$E_2(w) \equiv \sum_0^\infty \frac{w^n}{(n!)^2}. \tag{1}$$

I adopt this notation, for presently it is convenient to write

$$E_k(w) \equiv \sum_0^\infty \frac{w^n}{(n!)^k} \tag{2}$$

for any positive integer k.

Write the equation $z_{xy} = z$ in terms of δ's as

$$\delta\delta' z = xy . z.$$

The "indicial" equation for the initial indices is $\delta\delta' = 0$. Hence we start either with x^a, free of y, or with y^b, free of x, where in either case the index is arbitrary. Moreover every solution in series of the type considered will ascend by powers of xy.

The solution led by x^a, or better by x^a/a, is

$$\frac{x^a}{a} + \frac{x^{a+1}y}{a(a+1).1} + \frac{x^{a+2}y^2}{a(a+1)(a+2).1.2} + \ldots \text{ to } \infty,$$

i.e. $$x^a \sum_0^\infty \frac{(xy)^n \,\Pi(a-1)}{n!\,\Pi(a+n)},$$

where $\Pi(w)$ is the Gaussian equivalent of $\Gamma(w+1)$.

Now with certain limitations as to a, b, we have

$$\int_0^1 t^a (1-t)^b \, dt = \Pi(a)\,\Pi(b)/\Pi(a+b+1). \tag{3}$$

Hence the solution is

$$x^a \sum_0^\infty \frac{(xy)^n \Pi(a-1)\Pi(n)}{(n!)^2 \Pi(a+n)} = x^a \sum_0^\infty \frac{(xy)^n}{(n!)^2} \int_0^1 t^{a-1}(1-t)^n\,dt$$

$$= \int_0^1 (xt)^a \sum_0^\infty \frac{\{xy(1-t)\}^n}{(n!)^2} \frac{dt}{t}$$

(with necessary assumptions as to convergence)

$$= \int_0^1 (xt)^a E_2\{xy(1-t)\} \frac{dt}{t},$$

with the notation of (1).

But a remains arbitrary. Thus, if we multiply $(xt)^a$ by an arbitrary constant as coefficient and sum for every possible a, we have the most general function of xt expansible in powers of its argument. Write in its place $\phi(xt)$, where ϕ is an arbitrary functional symbol.* Then the initial term x^a has given rise to the solution

$$\int_0^1 \phi(xt) E_2\{xy(1-t)\} \frac{dt}{t}.$$

Similarly the initial term y^b will give rise to the solution

$$\int_0^1 \psi(yt) E_2\{xy(1-t)\} \frac{dt}{t}.$$

Hence we have as solution

$$z = \int_0^1 \phi(xt) E_2\{xy(1-t)\} \frac{dt}{t} + \int_0^1 \psi(yt) E_2\{xy(1-t)\} \frac{dt}{t}.$$

Take xt instead of t as new variable in the first integral, yt instead of t in the second integral, and write $\phi(t)$, $\psi(t)$ respectively in place of $\phi(t)/t$, $\psi(t)/t$, and we have

$$z = \int_0^x \phi(t) E_2\{(x-t)y\}\,dt + \int_0^y \psi(t) E_2\{x(y-t)\}\,dt.$$

But this solution reduces to 0 for $x = 0 = y$. We must therefore add a complementary term representing the part of z which is led by a term independent of x and of y. It is found to be $CE_2(xy)$, where C is some constant. Thus the complete solution is now

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* But ϕ must satisfy the restrictions necessary to secure the convergence of the integral in which it occurs. A like assumption is to be made in respect of all the arbitrary functions subsequently introduced.

$$z = CE_2(xy) + \int_0^x \phi(t) E_2\{(x-t)y\}\, dt + \int_0^y \psi(t) E_2\{x(y-t)\}\, dt. \qquad (4)$$

But how complete is it?

From the manner in which it was obtained it is clear that, if $\phi(t)$, $\psi(t)$ represent all functions expansible in positive powers of t, then the expression (4) represents all solutions of $z_{xy} = z$ expansible in positive powers of x, y. By a natural leap of the imagination it would therefore seem reasonable to suppose that, if $\phi(t)$, $\psi(t)$ represent *all* functions of t, then the expression (4) represents *all* solutions of the differential equation.

Actually the expression (4) is the solution obtained by Riemann's method as the general solution.* Hence in this example our belief as to the generality of the solution is justified. This form of solution is certainly not unique, since if z is a solution, so also is $\frac{\partial^{m+n} z}{\partial x^m \partial y^n}$. For greater variety introduce the more general Bessel's function†

$$F(k, w) \equiv \sum_0^\infty \frac{w^n}{n!\, \Pi(k+n)}, \qquad (5)$$

for any k not a negative integer.

The solution of $z_{xy} = z$ which was led by x^a/a, namely,

$$x^a \sum_0^\infty \frac{(xy)^n\, \Pi(a-1)}{n!\, \Pi(a+n)},$$

may be written, save for a constant factor, as

$$x^a \sum_0^\infty \frac{(xy)^n}{n!\, \Pi(k+n)} \; \frac{\Pi(k+n)\, \Pi(a-k-1)}{\Pi(a+n)}$$

$$= \int_0^1 x^a (1-t)^{k+n} t^{a-k-1} \sum_0^\infty \frac{(xy)^n}{n!\, \Pi(k+n)}\, dt$$

$$= \int_0^1 (xt)^a F\{k, xy(1-t)\} \left(\frac{1-t}{t}\right)^k \frac{dt}{t}.$$

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* Cf. Goursat, *Cours d'Analyse* (edition 1913), t. 3, Ch. xxvi, § 499, pp. 152 *et seq.*

† The notation is Schläfli's, *Annali di Mat.* (2), Vol. 1 (1868).

This leads to the solution

$$z = CE_2(xy) + \int_0^x \phi(t)\, F\{k,\, (x-t)y\} \left(\frac{x-t}{t}\right)^k \frac{dt}{t}$$
$$+ \int_0^y \psi(t) F\{k,\, x(y-t)\} \left(\frac{y-t}{t}\right)^k \frac{dt}{t}.$$

(The complementary term $CE_2(xy)$ remains, since $F(k, xy)$ itself is not, in general, a solution.)

If now we write $\frac{1}{2}(x \pm iy)$ instead of x, y, we must write also $\partial_x^2 + \partial_y^2$ instead of $\partial_x \partial_y$ and x^2+y^2 instead of $4xy$. This gives a solution of

$$z_{xx} + z_{yy} = z,$$

namely,

$$z = CE_2\{\tfrac{1}{4}(x^2+y^2)\} + \int_0^1 \phi(xt \pm iyt) F\{k,\, \tfrac{1}{4}(x^2+y^2)(1-t)\} \left(\frac{1-t}{t}\right)^k \frac{dt}{t}.$$

In particular it is known that

$$\Pi(\tfrac{1}{2})\, F(\tfrac{1}{2},\, w) = \frac{\sinh 2\sqrt{w}}{2\sqrt{w}} \quad \text{and} \quad \Pi(-\tfrac{1}{2})\, F(-\tfrac{1}{2},\, w) = \cosh 2\sqrt{w}.$$

Hence two particular solutions of $z_{xx} + z_{yy} = z$ are

$$z = CE_2\{\tfrac{1}{4}(x^2+y^2)\} + \int_0^1 \phi(xt \pm iyt) \cosh \sqrt{\{(x^2+y^2)(1-t)\}}\, \frac{dt}{\sqrt{\{t(1-t)\}}}, \tag{6}$$

and

$$z = CE_2\{\tfrac{1}{4}(x^2+y^2)\} + \int_0^1 \phi(xt \pm iyt) \sinh \sqrt{\{(x^2+y^2)(1-t)\}}\, \frac{dt}{\sqrt{\{(x^2+y^2)\, t^3\}}}. \tag{7}$$

There is a solution of the equation $z_{xy} = z$ of the same general form

$$z = \int \phi(t) f(x,\, y,\, t)\, dt,$$

which is well-known, namely,

$$z = \int_\gamma \phi(t)\, e^{xt + y/t}\, dt, \tag{8}$$

where γ is a simple contour surrounding the origin.

It can be obtained by adding particular solutions $e^{xt+y/t}$, each with its own coefficient $\phi(t)dt$. Only a single arbitrary function so arises, and this second type of solution appears therefore to be of a character quite distinct from the type (4). Both types occur in connection with the various equations to be discussed here, and it will be convenient to distinguish them by terming the type represented by (8) the "restricted" solution, without prejudice to the question of its comparative generality. It is the type of solution obtained by Whittaker,* and more generally by Bateman,† for a large class of equations. I shall state the corresponding solution of this type for all the equations which I discuss, not as novel, therefore, but for purposes of comparison.

The equations $\dfrac{\partial^3 z}{\partial x_1 \partial x_2 \partial x_3} = z$, *&c.*

3. We may extend the methods of the foregoing paragraph to a similar equation in k variables. It will be enough to give the analysis for the case $k = 3$. In terms of δ's the equation is

$$\delta_1 \delta_2 \delta_3 z = x_1 x_2 x_3 z.$$

The indicial equation, $\delta_1 \delta_2 \delta_3 = 0$, gives δ_1 or δ_2 or $\delta_3 = 0$, so that the initial term must lack one variable. Suppose it $x_2^a x_3^b$.

The solution it introduces is

$$x_2^a x_3^b \sum_0^\infty \frac{(x_1 x_2 x_3)^n\, \Pi(a-1)\, \Pi(b-1)}{n!\, \Pi(a+n)\, \Pi(b+n)}$$

$$= x_2^a x_3^b \sum_0^\infty \frac{(x_1 x_2 x_3)^n}{(n!)^3} \frac{\Pi(a-1)\, \Pi(n)}{\Pi(a+n)} \frac{\Pi(b-1)\, \Pi(n)}{\Pi(b+n)}$$

$$= x_2^a x_3^b \sum_0^\infty \frac{(x_1 x_2 x_3)^n}{(n!)^3} \int_0^1 (1-t)^n t^{a-1}\, dt \int_0^1 (1-u)^n u^{b-1}\, du$$

$$= \int_0^1 \int_0^1 (x_2 t)^a (x_3 u)^b E_3 \{x_1 x_2 x_3 (1-t)(1-u)\} \frac{dt}{t} \frac{du}{u},$$

where

$$E_3(w) \equiv \sum_0^\infty \frac{w^n}{(n!)^3}.$$

We may now multiply $(x_2 t)^a (x_3 u)^b$ by any constant as coefficient and sum for every a and b. We are led to the arbitrary function of two arguments

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* *Math. Ann.*, Vol. 57 (1903), p. 333.

† *Proceedings*, Ser. 2, Vol. 1, p. 451.

$\phi(x_2 t, x_3 u)$, and we have ultimately the solution

$$\int_0^{x_2}\int_0^{x_3}\phi(t, u)\, E_3\{x_1(x_2-t)(x_3-u)\}\, dt\, du$$

$$+\int_0^{x_3}\int_0^{x_1}\psi(t, u)\, E_3\{(x_1-u)\, x_2(x_3-t)\}\, dt\, du$$

$$+\int_0^{x_1}\int_0^{x_2}\chi(t, u)\, E_3\{(x_1-t)(x_2-u)\, x_3\}\, dt\, du. \tag{9}$$

But if we put $x_2 = 0 = x_3$, the expression (9) vanishes, and thus the solution led by x_1^a has not been included, nor, consequently, those led by x_2^a, x_3^a and by a term independent of the variables. For the complete solution, then, we must add to (9) the expression

$$\int_0^{x_1}\lambda(t)\, E_3\{(x_1-t)x_2 x_3\}\, dt+\int_0^{x_2}\mu(t)\, E_3\{x_1(x_2-t)x_3\}\, dt$$

$$+\int_0^{x_3}\nu(t)\, E_3\{x_1 x_2(x_3-t)\}\, dt+CE_3(x_1 x_2 x_3), \tag{10}$$

where λ, μ, ν are arbitrary functions.

The form of solution for the general equation in k variables $\partial_1\partial_2 \dots \partial_k z = z$ is now sufficiently evident. The principal terms, corresponding to (9), consist of k arbitrary functions, each of $k-1$ arguments, and each integrated $k-1$ times. To this we add the secondary terms composed of $k(k-1)/2!$ arbitrary functions, each of $k-2$ arguments and each integrated $k-2$ times, and so forth, ending with the single term $CE_k(x_1 x_2 \dots x_k)$.

It is not difficult to see that for any linear partial equation of order p and in k independent variables the general solution, or at any rate its principal terms, should consist of p arbitrary functions each of $k-1$ arguments. These will in general be represented by p distinct solutions in series, corresponding to the p roots of the indicial equation which is of order p.

When $k = 1$, these p arbitrary functions reduce to the p arbitrary constants that characterize the general solution of an ordinary differential equation of order p. Each of these arbitrary functions will be subject, at most, to $k-1$ integrations. For we may take the $k-1$ arguments of the arbitrary function as $k-1$ variables of integration; and further quadratures do not affect the arbitrary function and serve therefore only to define the specific* function under the integral sign.

The equation $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial z}{\partial y}$.

4. Writing this as $\delta(\delta-1)z = x^2y^{-1}\delta' z$, we see that the solution proceeds by ascending† powers of x^2/y. The two leading terms are y^a, xy^b; they lead the two series

$$\sum_0^\infty \frac{(-x^2)^n\, y^{a-n}\, \Pi(n-a-1)}{(2n)!\; \Pi(-a-1)} \quad \text{and} \quad \sum_0^\infty \frac{(-)^n\, x^{2n+1}\, y^{b-n}\, \Pi(n-b-1)}{(2n+1)!\; \Pi(-b-1)}.$$

If we differentiate the first series for x and write $a \equiv b+1$, we obtain the second. The first series gives the general solution

$$\int_0^1 \phi\left\{\frac{y(1-t)}{t}\right\} F\left(-\frac{x^2t}{y}\right)\frac{dt}{t}, \tag{11}$$

where

$$F(w) \equiv \sum_0^\infty \frac{w^n\, n!}{(2n)!}.$$

If we differentiate for x, and write

$$\phi(w) \equiv -w\psi(w),$$

we have for the other solution

$$x\int_0^1 \psi\left\{\frac{y(1-t)}{t}\right\} 2F'\left(-\frac{x^2t}{y}\right)\frac{1-t}{t}\,dt, \tag{12}$$

where

$$2F'(w) \equiv \sum_1^\infty \frac{w^{n-1}\, n!}{(2n-1)}.$$

We may also write the series led by y^a, save for a constant multiplier, in the form

$$\sum_0^\infty \left(-\frac{x^2}{4y}\right)^n y^a\, \frac{\Pi(n-\tfrac{1}{2})\, 2^{2n}\, \Pi(n-a-1)\, \Pi(a-\tfrac{1}{2})}{(2n)!\; \Pi(n-\tfrac{1}{2})}.$$

This leads to the solution

$$\int_0^1 \phi\left\{\frac{y(1-t)}{t}\right\} e^{-x^2t/4y}\, \frac{dt}{t\sqrt{(1-t)}}. \tag{13}$$

If we take the argument of ϕ as the new variable of integration, this solu-

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* I mean such a function as $E_2\{x\,(y-t)\}$, which may of course be defined by quadratures.

† This is necessary for convergence.

tion becomes
$$\int_0^{\pm\infty} \psi(t)\, e^{-x^2/4(y+t)} \frac{dt}{\sqrt{(y+t)}}, \tag{14}$$

where $\pm\infty$ has the sign of y.

We may obtain the better known solution involving the exponential function, if we let the definite integral enter by way of the Γ-function, rather than of the B-function. In the series led by y^a I write, then,

$$\Pi(a) = \int_0^\infty e^{-t} t^a \, dt,$$

and obtain the expression

$$\sum_0^\infty \frac{(-x^2/y)^n}{(2n)!} y^a \int_0^\infty e^{-t} t^{n-a-1} dt.$$

If the infinite summation and the infinite integration may cross, we derive the general solution

$$\int_0^\infty \phi\left(\frac{y}{t}\right) e^{-t} \cos\left(x\sqrt{\frac{t}{y}}\right) \frac{dt}{t}.$$

By differentiation we obtain the other solution

$$\int_0^\infty \psi\left(\frac{y}{t}\right) e^{-t} \sin\left(x\sqrt{\frac{t}{y}}\right) \frac{dt}{t}.$$

More conveniently we may combine the two solutions as

$$\int_0^{\pm\infty} \phi(t)\, e^{-yt \pm ix\sqrt{t}}\, dt, \tag{15}$$

where the sign of $\pm\infty$ is the sign of y.

The "restricted" solution is

$$\int_\gamma e^{xt+yt^2} \phi(t)\, dt, \tag{16}$$

where γ is a simple contour in the t-plane surrounding the origin. It will be noticed that this is of a form virtually equivalent to (15).

We may extend the method to any "binomial" equation in variables $x_1 \dots x_k$, $x_l \dots x_m$ of the form

$$\partial_1^{a_1} \dots \partial_k^{a_k} z = \partial_l^{a_l} \dots \partial_m^{a_m} z.$$

The solutions will all proceed by powers of

$$x_1^{a_1} \dots x_k^{a_k} / x_l^{a_l} \dots x_m^{a_m},$$

and those will converge which proceed by positive powers, if

$$a_1+\ldots+a_k > a_l+\ldots+a_m.$$

I shall suppose this inequality to hold (excluding for the present the possibility $a_1+\ldots+a_k = a_l+\ldots+a_m$).

But I shall include the hyperbolic case

$$\partial_1^{a_1}\ldots\partial_k^{a_k} z = z,$$

which can be derived from the more general case by omitting variables $x_l, \ldots, x_m$, and supposing $a_l, \ldots, a_m$ all zero.

Now the "indicial" equation gives

$$\delta_1 = 0, 1, \ldots, a_1-1, \quad \delta_2 = 0, 1, \ldots, a_2-1, \quad \&c.$$

There will thus be a set of a_1 solutions led respectively by

$$x_2^{a_2}\ldots x_m^{a_m}, \quad x_1 x_2^{a_2}\ldots x_m^{a_m}, \quad \ldots, \quad x_1^{a_1-1}x_2^{a_2}\ldots x_m^{a_m},$$

a set of a_2 solutions led by

$$x_1^{a_1}x_3^{a_3}\ldots x_m^{a_m}, \quad x_1^{a_1}x_2x_3^{a_3}\ldots x_m^{a_m}, \quad \ldots, \quad x_1^{a_1}x_2^{a_2-1}x_3^{a_3}\ldots x_m^{a_m},$$

and so forth.

It will be sufficient to give the form of the general solution represented by the series led by $x_2^{a_2}\ldots x_m^{a_m}$; the other solutions may be derived from it by differentiation or by considerations of symmetry. This general solution is

$$\int_0^1 \frac{dt_2}{t_2}\ldots\int_0^1 \frac{dt_m}{t_m}\,\phi\left\{x_2t_2, \ldots, x_kt_k, \frac{x_l(1-t_l)}{t_l}, \ldots, \frac{x_m(1-t_m)}{t_m}\right\}$$
$$\times F\left\{x_1^{a_1}x_2^{a_2}(1-t_2)^{a_2}\ldots x_k^{a_k}(1-t_k)^{a_k}\left(-\frac{x_l}{t_l}\right)^{-a_l}\ldots\left(-\frac{x_m}{t_m}\right)^{-a_m}\right\}, \quad (17)$$

where
$$F(w) \equiv \sum_0^\infty \frac{(a_ln)!\ldots(a_mn)!}{(a_1n)!\ldots(a_kn)!}\,w^n.$$

With appropriate change of variable we may write this as

$$\int_0^{x_2} dt_2\ldots\int_0^{x^k} dt_k\int_0^{\pm\infty}\frac{dt_l}{x_l+t_l}\ldots\int_0^{\pm\infty}\frac{dt_m}{x_m+t_m}\,\phi\{t_2, \ldots, t_k, t_l, \ldots, t_m\}$$
$$\times F\{x_1^{a_1}(x_2-t_2)^{a_2}\ldots(x_k-t_k)^{a_k}(-x_l-t_l)^{a_l}\ldots(-x_m-t_m)^{a_m}\}. \quad (18)$$

The "restricted" solution is

$$\int_y \phi(t_2, \ldots, t_m) \exp(x_1 T + x_2 t_2^{\alpha_1} + \ldots + x_k t_k^{\alpha_1} + x_l t_l^{-\alpha_1} + \ldots + x_m t_m^{-\alpha_1})\, dt_2 \ldots dt_m, \tag{19}$$

where T is written for the product

$$t_2^{-\alpha_2} \ldots t_k^{-\alpha_k} t_l^{\alpha_l} \ldots t_m^{\alpha_m}.$$

The equations $\dfrac{\partial^2 z}{\partial x_1 \partial x_2} = \dfrac{\partial^2 z}{\partial x_3^2}$ *and* $\dfrac{\partial^2 z}{\partial x_1 \partial x_2} = \dfrac{\partial^2 z}{\partial x_3 \partial x_4}$.

5. The case in which each term of the binomial is of the same order merits special consideration. The simplest example not amenable to elementary methods is the equation

$$\partial_1 \partial_2 z = \partial_3^2 z,$$

which we may write

$$\delta_1 \delta_2 z = (x_1 x_2 / x_3^2)\, \delta_3(\delta_3 - 1)\, z.$$

The leading terms are $x_2^a x_3^b$ and $x_1^a x_3^b$.

The first leads the series

$$x_2^a x_3^b \sum_0^\infty \frac{(x_1 x_2/x_3^2)^n\, \Pi(a)\, \Pi(2n-1-b)}{n!\, \Pi(a+n)\, \Pi(-1-b)}$$

$$= x_2^a x_3^b \sum_0^\infty \frac{(4x_1 x_2/x_3^2)^n\, \Pi(n-1-\tfrac{1}{2}b)}{n!}\, \frac{\Pi(n-\tfrac{1}{2}-\tfrac{1}{2}b)\, \Pi(a-\tfrac{1}{2}+\tfrac{1}{2}b)}{\Pi(a+n)},$$

save for a constant multiplier. This becomes

$$\int_0^1 dt (x_2 t)^a \left(\frac{x_3^2 t}{1-t}\right)^{\frac{1}{2}b} \left\{1 - \frac{4x_1 x_2(1-t)}{x_3^2}\right\}^{\frac{1}{2}b} \frac{dt}{\sqrt{\{t(1-t)\}}},$$

indicating the solution

$$\int_0^1 \phi\left\{x_2 t,\ \frac{x_3^2 t}{1-t} - 4x_1 x_2 t\right\} \frac{dt}{\sqrt{\{t(1-t)\}}}. \tag{20}$$

The other solution comes, of course, by interchange of x_1 and x_2.

Since the terms of the binomial are of equal order, we may also write the equation as

$$\delta_3(\delta_3 - 1)\, z = (x_3^2 / x_1 x_2)\, \delta_1 \delta_2 z.$$

The leading terms are $x_1^a x_2^b$ and $x_3 x_1^a x_2^b$. The second series will be the x_3-derivative of the first.

The series led by $x_1^a x_2^b$ may be written in either of the three forms (save for constant multipliers)

$$\text{(A)} \qquad x_1^a x_2^b \sum_0^\infty \frac{(x_3^2/4x_1x_2)^n\,\Pi(n-a-1)}{n!\,\Pi(-a-1)}\,\frac{\Pi(n-b-1)\,\Pi(b-\tfrac{1}{2})}{\Pi(n-\tfrac{1}{2})},$$

$$\text{(B)} \qquad x_1^a x_2^b \sum_0^\infty \frac{(x_3^2/4x_1x_2)^n\,\Pi(2n-2a-1)}{(2n)!\,\Pi(-2a-1)}\,\frac{\Pi(n-b-1)\,\Pi(b-a-\tfrac{1}{2})}{\Pi(n-a-\tfrac{1}{2})},$$

$$\text{(C)} \qquad x_1^a x_2^b \sum_0^\infty \frac{(x_3^2/x_1x_2)^n\,\Pi(2n-a-b-1)}{(2n)!}\,\frac{\Pi(n-a-1)\,\Pi(n-b-1)}{\Pi(2n-a-b-1)}.$$

Of these (A) gives the general solution

$$\int_0^1 \phi\left\{\frac{x_2 t}{1-t},\ x_1 - \frac{x_3^2(1-t)}{4x_2}\right\}\frac{dt}{(1-t)\sqrt{t}}.$$

If we change the arguments of ϕ and take $-t/(1-t)$ as new variable of integration, this becomes

$$\int_{-\infty}^0 \phi\left\{x_2 t,\ \frac{x_3^2 t}{1-t} - 4x_1x_2t\right\}\frac{dt}{\sqrt{\{t(1-t)\}}}. \tag{21}$$

The other solution is the x_3-derivative of this.

It is to be remarked that (20) and (21) differ only as to the limits of integration.

The form (B) represents

$$\int_0^1 \left(\frac{x_1}{t}\right)^a \left(\frac{x_2 t}{1-t}\right)^b \left\{1 \pm x_3\sqrt{\left[\frac{(1-t)}{4x_1x_2}\right]}\right\}^{2a}\frac{dt}{(1-t)\sqrt{t}},$$

where we retain only even terms in the binomial expansion of $\{\ \}^{2a}$.

The odd terms are supplied by the other solution, led by $x_1^a x_2^b x_3$. We have thus the pair of solutions

$$\int_0^1 \phi\left\{\frac{x_2 t}{1-t},\ x_3 \pm 2\sqrt{\left(\frac{x_1x_2}{1-t}\right)}\right\}\frac{dt}{(1-t)\sqrt{t}}. \tag{22}$$

With $-t(1-t)$ a new variable of integration this becomes

$$\int_{-\infty}^0 \phi\,\{x_2 t,\ x_3 \pm 2\sqrt{[x_1x_2(1-t)]}\}\,\frac{dt}{\sqrt{\{t(1-t)\}}}. \tag{23}$$

So from the form (C) we derive the pair of solutions

$$\int_0^1 \phi\left\{\frac{x_1 t}{x_2(1-t)},\ x_3 \pm \sqrt{\left[\frac{x_1x_2}{(1-t)t}\right]}\right\}\frac{dt}{t(1-t)}. \tag{24}$$

The restricted solution is

$$\int_\gamma \phi\left(t,\ x_1 t+\frac{x_2}{t}+x_3\right) dt. \tag{25}$$

For the equation $\partial_1\partial_2 z=\partial_3\partial_4 z$, which is written as

$$\delta_1\delta_2 z=(x_1x_2/x_3x_4)\,\delta_3\delta_4 z,$$

we have a leading term $x_2^a x_3^b x_4^c$ and its corresponding series

$$x_2^a x_3^b x_4^c \sum_0^\infty \left(\frac{x_1x_2}{x_3x_4}\right)^n \frac{\Pi(n-c-1)}{n!\,\Pi(-c-1)}\,\frac{\Pi(n-b-1)\,\Pi(a+b)}{\Pi(n+a)}.$$

This indicates the general solution

$$\int_0^1 \phi\left\{x_2(1-t),\ \frac{x_3(1-t)}{t},\ x_4-\frac{x_1x_2t}{x_3}\right\}\frac{dt}{t},$$

or, adjusting the arguments of ϕ,

$$\int_0^1 \phi\left\{x_2(1-t),\ x_3\left(1-\frac{1}{t}\right),\ x_1x_2(1-t)+x_3x_4\left(1-\frac{1}{t}\right)\right\}\frac{dt}{t}. \tag{26}$$

The other part of the solution comes by interchange of x_1 and x_2. Interchange of x_1, x_2 with x_3, x_4 gives a series ascending by powers x_3x_4/x_1x_2. It merely changes the limits of integration in (26), namely from 0, 1 to 1, ∞.

The restricted solution is

$$\int_\gamma \phi\left(t,\ x_1t+x_3,\ \frac{x_2}{t}+x_4\right) dt.$$

For the more general equation

$$\partial_1\partial_2\ldots\partial_k z=\partial_{k+1}\partial_{k+2}\ldots\partial_{2k} z,$$

we have similarly as a typical term of the general solution

$$\int_0^1\cdots\int_0^1 \phi\left\{x_2(1-t_2),\ \ldots,\ x_k(1-t_k),\ x_{k+2}\left(1-\frac{1}{t_2}\right),\ \ldots,\ x_{2k}\left(1-\frac{1}{t_k}\right),\right.$$
$$\left. x_1x_2\ldots x_k(1-t_2)\ldots(1-t_k)+x_{k+1}x_{k+2}\ldots x_{2k}\left(1-\frac{1}{t_2}\right)\ldots\left(1-\frac{1}{t_k}\right)\right\}\frac{dt_2}{t_2}\cdots\frac{dt_k}{t_k}. \tag{27}$$

The corresponding "restricted" solution is

$$\int_\gamma \phi\left\{t_2,\ \ldots,\ t_k,\ x_1t_2\ldots t_k+x_{k+1},\ \frac{x_2}{t_2}+x_{k+2},\ \ldots,\ \frac{x_k}{t_k}+x_{2k}\right\} dt_2\ldots dt_k. \tag{28}$$

The foregoing methods can be applied to the general homogeneous binomial equation. It is, perhaps, worthy of notice that, unlike the non-homogeneous equation discussed in § 4, it needs for its solution no specific functional type more advanced than the surd.

I may mention that the symmetrical equation

$$\partial_1^\alpha \partial_2^\beta z = \partial_3^\alpha \partial_4^\beta z$$

has $\alpha+\beta$ terms in its general solution of which β are of the form

$$\int_0^1 \phi \left\{ x_1(1-t),\ x_3\left(1-\frac{1}{t}\right),\ x_2[x_1(1-t)]^{\alpha/\beta}+\omega x_4\left[x_3\left(1-\frac{1}{t}\right)\right]^{\alpha/\beta} \right\} \frac{dt}{t}, \tag{29}$$

where $$\omega^\beta = 1.$$

Laplace's Equation, &c.

6. It is, however, perhaps of greater interest to make a somewhat different generalisation of the results of the preceding section.

If in the equation $\partial_1 \partial_2 z = \partial_3^2 z$ we make the substitutions

$$2x_1 = x_1'+ix_2', \quad 2x_2 = -x_1'+ix_2',$$

and drop accents, we have the following solutions of Laplace's equation

$$(\partial_1^2+\partial_2^2+\partial_3^2)\,z = 0\,;$$

from (20), $$\int_0^1 \phi \left\{ (x_1 \pm ix_2)t,\ (x_1^2+x_2^2)t+\frac{x_3^2 t}{1-t} \right\} \frac{dt}{\sqrt{\{t(1-t)\}}}\,; \tag{30}$$

from (22), $$\int_0^1 \phi \left\{ \frac{(x_1+ix_2)t}{1-t},\ x_3 \pm i\sqrt{\left(\frac{x_1^2+x_2^2}{1-t}\right)} \right\} \frac{dt}{(1-t)\sqrt{t}}; \tag{31}$$

from (24), $$\int_0^1 \phi \left\{ \frac{(x_1+ix_2)t}{(x_1-ix_2)(1-t)},\ 2x_3 \pm i\sqrt{\left[\frac{x_1^2+x_2^2}{t(1-t)}\right]} \right\} \frac{dt}{t(1-t)}. \tag{32}$$

The "restricted" solution (25) now becomes

$$\int_\gamma \phi \left\{ t,\ x_1\left(t-\frac{1}{t}\right)+ix_2\left(t+\frac{1}{t}\right)+2x_3 \right\} \frac{dt}{t}.$$

If we take γ to be unit circle with centre at $t = 0$, then $t = e^{2i\theta}$, and the solution is

$$\int_{-\pi}^{\pi} \phi \{\theta,\ ix_1 \sin\theta + ix_2 \cos\theta + x_3\}\, d\theta, \tag{33}$$

which is Whittaker's well known form.

In like fashion, if in the equation $\partial_1\partial_2 z = \partial_3\partial_4 z$, we write

$$2x_1 = x_1'+ix_2', \qquad 2x_3 = x_3'+ix_4',$$

$$2x_2 = x_1'+ix_2', \qquad 2x_4 = -x_3'+ix_4',$$

and drop accents, we obtain Laplace's equation in four variables (or the "wave" equation)

$$(\partial_1^2+\partial_2^2+\partial_3^2+\partial_4^2)z = 0,$$

and a general solution

$$\int_0^1 \phi\left\{(x_1 \pm ix_2)(1-t),\ (x_3+ix_4)\left(1-\frac{1}{t}\right),\right.$$
$$\left.(x_1^2+x_2^2)(1-t)-(x_3^2+x_4^2)\left(1-\frac{1}{t}\right)\right\}\frac{dt}{t}. \quad (34)$$

We may extend these results to the general Laplacian equation of the second order in k variables. We need as lemma the well-known result that, if U is homogeneous of degree n in these k variables and is annihilated by $\partial_1^2+\partial_2^2+\ldots+\partial_k^2$, then

$$(\partial_1^2+\ldots+\partial_k^2)(x_1^2+\ldots+x_k^2)^a\, U = 2a(2a+2n+k-2)(x_1^2+\ldots+x_k^2)^{a-1}\, U.$$

Consider then the equation in five variables of the form

$$\partial_4\partial_5 z = (\partial_1^2+\partial_2^2+\partial_3^2)z,$$

i.e.

$$\delta_4\delta_5 z = x_4 x_5(\partial_1^2+\partial_2^2+\partial_3^2)z.$$

A leading term is $x_1^a x_2^b x_3^c x_4^d$. But I am looking for a solution proceeding by ascending powers of $x_4x_5/(x_1^2+x_2^2+x_3^2)$, and I shall take as leading term $(x_1^2+x_2^2+x_3^2)^a\, x_4^d\, V$, where

$$V \equiv \int_0^1 \{(x_2+ix_3)(1-t)\}^{2b}\left\{(x_2^2+x_3^2)(1-t)-x_1^2\left(1-\frac{1}{t}\right)\right\}^c \frac{dt}{\sqrt{\{t(1-t)\}}}.$$

This contains as many arbitrary exponents as does $x_1^a x_2^b x_3^c x_4^d$, and is, therefore, probably of equal generality. With this as first term, the second term is obtained in the usual way as

$$\frac{x_4^{d+1}\,x_5}{(d+1)\,1}(\partial_1^2+\partial_2^2+\partial_3^2)(x_1^2+x_2^2+x_3^2)^a\, V.$$

But V, in virtue of (30), is a solution of the Laplacian equation in x_1, x_2, x_3, and it is homogeneous of degree $2(b+c)$ in the x's.

Therefore, by the lemma, this second term is

$$\frac{x_4^{d+1}x_5}{(d+1)\,1}(x_1^2+x_2^2+x_3^2)^{a-1}\,V\,.\,2a(2a+4b+4c+1).$$

Proceeding in this fashion we develop (save for a constant factor) the series

$$(x_1^2+x_2^2+x_3^2)^a\,x_4^d\,V\sum_0^\infty \frac{\{4x_4x_5/(x_1^2+x_2^2+x_3^2)\}^n\,\Pi(n-a-1)}{n!\,\Pi(-a-1)}$$

$$\times\frac{\Pi(a+2b+2c+d+\frac{1}{2})\,\Pi(n-a-2b-2c-\frac{3}{2})}{\Pi(d+n)}.$$

If we substitute for V and proceed as in earlier examples, we have the expression

$$\int_0^1\!\!\int_0^1\left\{(x_1^2+x_2^2+x_3^2)\left(1-\frac{1}{s}\right)+4x_4x_5(1-s)\right\}^a\left\{(x_2+ix_3)(1-t)\left(1-\frac{1}{s}\right)\right\}^{2b}$$

$$\times\left\{(x_2^2+x_3^2)(1-t)-x_1^2\left(1-\frac{1}{t}\right)\right\}^c\left(1-\frac{1}{s}\right)^{2c}$$

$$\times\{x_4(1-s)\}^d\sqrt{(1-s)}\,\frac{ds}{s\sqrt{s}}\,\frac{dt}{\sqrt{\{t(1-t)\}}}.$$

Write $2x_4=x_4'+ix_5'$, $2x_5=-x_4'+ix_5'$, and drop accents.

We have as a general solution of

$$(\partial_1^2+\partial_2^2+\partial_3^2+\partial_4^2+\partial_5^2)z=0,$$

the expression

$$\int_0^1\!\!\int_0^1\phi\left\{\left[x_1^2\left(1-\frac{1}{t}\right)-(x_2^2+x_3^2)(1-t)\right]\left(1-\frac{1}{s}\right)^2,\ (x_2+ix_3)(1-t)\left(1-\frac{1}{s}\right),\right.$$

$$\left.(x_1^2+x_2^2+x_3^2)\left(1-\frac{1}{s}\right)-(x_4^2+x_5^2)(1-s),\quad (x_4\pm ix_5)(1-s)\right\}$$

$$\times\left(\frac{1}{t}-1\right)^{-\frac{1}{2}}\left(\frac{1}{s}-1\right)^{\frac{1}{2}}\frac{dt}{t}\,\frac{ds}{s}. \qquad (35)$$

We may proceed in this way for any *odd* number of variables.

Thus for the equation

$$(\partial_1^2+\partial_2^2+\partial_3^2+\partial_4^2+\partial_5^2+\partial_6^2+\partial_7^2)\,z=0,$$

there is the solution

$$\int_0^1\int_0^1\int_0^1 \{\phi\} \left(\frac{1}{t}-1\right)^{-\frac{1}{2}} \left(\frac{1}{s}-1\right)^{\frac{1}{2}} \left(\frac{1}{r}-1\right)^{\frac{3}{2}} \frac{dt}{t}\,\frac{ds}{s}\,\frac{dr}{r},$$

where $\{\phi\}$ is an arbitrary function of the six arguments

$$\left.\begin{aligned}
&\left\{x_1^2\left(1-\frac{1}{t}\right)-(x_2^2+x_3^2)(1-t)\right\}\left(1-\frac{1}{s}\right)^2\left(1-\frac{1}{r}\right)^2,\\
&(x_2+ix_3)(1-t)\left(1-\frac{1}{s}\right)\left(1-\frac{1}{r}\right),\\
&\left\{(x_1^2+x_2^2+x_3^2)\left(1-\frac{1}{s}\right)-(x_4^2+x_5^2)(1-s)\right\}\left(1-\frac{1}{r}\right)^2,\\
&(x_4+ix_5)(1-s)\left(1-\frac{1}{r}\right),\\
&(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2)\left(1-\frac{1}{r}\right)-(x_6^2+x_7^2)(1-r),\\
&(x_6\pm ix_7)(1-r).
\end{aligned}\right\}$$

These results are the generalisations of (30); I have not found analogous generalisations of (31) and (32).

For an *even* number of independent variables we commence with the equation

$$(\partial_1^2+\partial_2^2+\partial_3^2+\partial_4^2)\,z = \partial_5\partial_6 z,$$

and take as our leading term, in view of (34),

$$(x_1^2+x_2^2+x_3^2+x_4^2)^a\, x_5^e \int_0^1 \left\{(x_1+ix_2)\left(1-\frac{1}{t}\right)\right\}^{2b}$$

$$\times\left\{(x_1^2+x_2^2)\left(1-\frac{1}{t}\right)-(x_3^2+x_4^2)(1-t)\right\}^c \{(x_3+ix_4)(1-t)\}^{2d}\frac{dt}{t}.$$

Proceeding much as for an even number of variables we are led to a solution of the equation

$$(\partial_1^2+\partial_2^2+\partial_3^2+\partial_4^2+\partial_5^2+\partial_6^2)\,z = 0,$$

in the form

$$\int_0^1\int_0^1 \{\phi\} \left(\frac{1}{s}-1\right)\frac{dt}{t}\,\frac{ds}{s},$$

where $\{\phi\}$ is any function of the five arguments

$$\left.\begin{array}{l} (x_1+ix_2)\left(1-\dfrac{1}{t}\right)\left(1-\dfrac{1}{s}\right) \\ \left\{(x_1^2+x_2^2)\left(1-\dfrac{1}{t}\right)-(x_3^2+x_4^2)(1-t)\right\}\left(1-\dfrac{1}{s}\right)^2 \\ (x_3+ix_4)(1-t)\left(1-\dfrac{1}{s}\right) \\ (x_1^2+x_2^2+x_3^2+x_4^2)\left(1-\dfrac{1}{s}\right)-(x_5^2+x_6^2)(1-s) \\ (x_5 \pm ix_6)(1-s) \end{array}\right\}. \tag{36}$$

If two more variables x_7, x_8 enter, we multiply the foregoing arguments of $\{\phi\}$ by $1-1/r$ and $(1-1/r)^2$ alternately, suppress the double sign in $x_5 \pm ix_6$, and add two new arguments

$$(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2)\left(1-\frac{1}{r}\right)-(x_7^2+x_8^2)(1-r), \quad (x_7 \pm ix_8)(1-r).$$

The integrand is multiplied by $(1/r-1)^2\, dr/r$.

In like manner we may generalise the "telegraph" equation

$$z_{xx}+z_{yy} = z,$$

so that, for example,

$$(\partial_1^2+\partial_2^2+\partial_3^2)\, z = z$$

has a general solution

$$\int_0^1\int_0^1 \cosh\sqrt{\{(x_1^2+x_2^2+x_3^2)(1-t')\}}$$
$$\times\,\phi\left\{(x_1+ix_2)t',\ t'^2\left[(x_1^2+x_2^2)(1-t)-x_3^2\left(1-\frac{1}{t}\right)\right]\right\}\frac{dt}{\sqrt{\{t(1-t)\}}}\;\frac{dt'}{\sqrt{(1-t')}}. \tag{37}$$

Again, the parabolic form, *e.g.*

$$(\partial_1^2+\partial_2^2+\partial_3^2)\, z = \partial_4 z,$$

is solved similarly. This particular equation has the solution

$$\int_0^1\int_0^1 \exp\left\{-\frac{(1-t')(x_1^2+x_2^2+x_3^2)}{4x_4}\right\}$$
$$\times\,\phi\left\{(x_1+ix_2)(1-t)\,t',\ \left[(x_1^2+x_2^2)(1-t)-x_3^2\left(1-\frac{1}{t}\right)\right]t'^2,\ \frac{x_4t'}{1-t'}\right\}$$
$$\times\,\frac{dt}{\sqrt{\{t(1-t)\}}}\;\frac{dt'}{(1-t')\sqrt{t'}}. \tag{38}$$

There is another method, somewhat different in character, of deriving a "general" solution of the generalised Laplacian equation

$$(\partial_1^2+\ldots+\partial_k^2)\,z=0,$$

which is a little suggestive of Green's theorem.

I commence with the equation in three variables and write it as

$$\delta_1(\delta_1-1)=-x_1^2x_2^{-2}\delta_2(\delta_2-1)-x_1^2x_3^{-2}\delta_3(\delta_3-1).$$

A leading term is $x_2^a x_3^b$. It is followed in its series by the two terms

$$x_2^a x_3^b\left\{-\frac{x_1^2x_2^{-2}}{2!}a(a-1)-\frac{x_2^2x_3^{-2}}{2!}b(b-1)\right\}.$$

Each of these in turn is followed by two terms: together the four terms constitute

$$x_2^a x_3^b\left\{\frac{x_1^4x_2^{-4}\,\Pi(3-a)}{4!\,\Pi(-1-a)}+2\,\frac{x_1^4x_2^{-2}x_3^{-2}\,\Pi(1-a)\,\Pi(1-b)}{4!\,\Pi(-1-a)\,\Pi(-1-b)}+\frac{x_1^4x_3^{-4}\,\Pi(3-b)}{4!\,\Pi(-1-b)}\right\}.$$

Proceeding in this way we develop the double series

$$x_2^a x_3^b\sum_0^\infty\sum_0^\infty\frac{(m+n)!}{m!\,n!}\,\frac{(-)^{m+n}x_1^{2m+2n}x_2^{-2m}x_3^{-2n}}{(2m+2n)!}\,\frac{\Pi(2m-a-1)\,\Pi(2n-b-1)}{\Pi(-a-1)\,\Pi(-b-1)}.$$

Now, by Dirichlet's formula,

$$\Pi(p)\,\Pi(q)/\Pi(p+q+2)=\iint t^p s^q\,dt\,ds,$$

where the double integral is taken over that part of the positive quadrant in the $(s,\,t)$-plane which is cut off by the line $s+t=1$.

Hence we may write our solution as

$$x_2^a x_3^b\sum_0^\infty\sum_0^\infty\frac{(m+n)!}{m!\,n!}\,\frac{(-)^{m+n}x_1^{2m+2n}x_2^{-2m}x_3^{-2n}}{(2m+2n)!}\,\frac{\Pi(2m+2n-a-b)}{\Pi(-a-1)\,\Pi(-b-1)}$$
$$\times\iint t_2^{2m-a-1}\,t_3^{2n-b-1}\,dt_2\,dt_3$$
$$=\iint\left(\frac{x_2}{t_2}\right)^a\left(\frac{x_3}{t_3}\right)^b\sum_{m+n=0}^\infty\frac{(ix_1)^{2m+2n}\,\Pi(2m+2n-a-b)}{(2m+2n)!\,\Pi(-a-1)\,\Pi(-b-1)}$$
$$\times(t_2^2x_2^{-2}+t_3^2x_3^{-2})^{m+n}\,\frac{dt_2}{t_2}\,\frac{dt_3}{t_3}.$$

Summing now for $m+n$, we obtain the even terms of the binomial expansion in

$$\iint\left(\frac{x_2}{t_2}\right)^a\left(\frac{x_3}{t_3}\right)^b\left\{1-ix_1\sqrt{(t_2^2x_2^{-2}+t_3^2x_3^{-2})}\right\}^{a+b+1}\frac{dt_2}{t_2}\,\frac{dt_3}{t_3}.$$

The odd terms in the expansion are supplied by the double series led by $x_1 x_2^{a'} x_3^{b'}$. Hence the general solution is

$$\iint \phi \left\{ \frac{x_2}{t_2} \pm i x_1 \sqrt{\left(1+\frac{x_2^2 t_3^2}{x_3^2 t_2^2}\right)},\ \frac{x_3}{t_3} \pm i x_1 \sqrt{\left(1+\frac{x_3^2 t_2^2}{x_2^2 t_3^2}\right)} \right\} \sqrt{\left(\frac{t_2 t_3}{x_2 x_3}\right)} \frac{dt_2}{t_2} \frac{dt_3}{t_3}. \tag{39}$$

The integration is effected for every positive t_2 and t_3 for which $t_2+t_3 < 1$. We may more conveniently change the variables of integration and write

$$\iint \phi \left\{ \frac{1 \pm i x_1 \sqrt{(t_2^2+t_3^2)}}{t_2},\ \frac{1 \pm i x_1 \sqrt{(t_2^2+t_3^2)}}{t_3} \right\} \frac{dt_2 dt_3}{\sqrt{(t_2 t_3)}}, \tag{40}$$

the integration extending over every t_2, t_3 for which $x_2 t_2$, $x_3 t_3$ are positive and their sum less than unity.

Similarly with k variables the solution of

$$(\partial_1^2+\ldots+\partial_k^2)\, z = 0$$

is

$$\iint \ldots \int \phi \left\{ \ldots,\ \frac{1 \pm i x_1 \sqrt{(t_2^2+\ldots+t_k^2)}}{t_r},\ \ldots \right\} \Pi \left\{ \frac{dt_r}{\sqrt[k-1]{t_r}} \right\}, \tag{41}$$

where $r = 2, 3, \ldots, k$, and there is a corresponding definition of the region of integration.

We may also extend our method to the Laplacian equation of the p-th order

$$(\partial_1^p+\partial_2^p+\ldots+\partial_k^p)\, z = 0.$$

Its solution is composed of p terms of the type

$$\int \ldots \int \phi \left\{ \ldots,\ \frac{1-\omega x_1 \sqrt[p]{(t_2^p+\ldots+t_k^p)}}{t_r},\ \ldots \right\} \Pi \left\{ \frac{dt_r}{\sqrt[k-1]{t_r}} \right\}, \tag{42}$$

where $r = 2, 3, \ldots, k$, and ω is any root of $\omega^p = 1$.

Conclusion.

7. I here conclude this somewhat summary catalogue of results obtainable by the method. I have restricted myself to equations in which the independent variables do not enter explicitly, as likely to be of greater interest. The method covers a variety of equations with explicit independent variables, but is evidently limited to equations having solutions of a generalised hypergeometric type.

The limits of integration, perpetually inserted, are conventional. The procedure has consisted essentially in finding, in respect of a given equation, an appropriate integrand, say $\phi(t) f(x, y, t)$, involving an arbitrary function, and such that after the operation which represents the differential equation it becomes a perfect t-derivative $\partial/\partial t\{F(x, y, t)\}$. The path of integration may therefore be any path joining two zeroes of F or a contour enclosing poles of F. Presumably, in general, the path will depend on the form of ϕ.

Certain questions are raised by these results which seem of interest, but which I cannot answer. Are the various solutions actually "general," *i.e.* can they be shown to include every solution, or every solution of some definite type? Or at least can the several "general" solutions obtained for the same equation be reconciled? In particular, can the "restricted" solution, which is derived by a different process and possesses ordinarily but one arbitrary function, embrace the "general" solutions which involve several arbitrary functions? If the several "general" solutions are equivalent, can formulæ of transformation be determined for the variables of integration and for the arguments of the arbitrary function which will actually transform one "general" solution into the other? Finally, is there a "general" "general solution," *i.e.* a form of general solution of which all the others are particular cases?

PROOFS OF SOME FORMULÆ ENUNCIATED BY RAMANUJAN

By B. M. WILSON.

[Received October, 1921.—Read November 17th, 1921.]

IN a short note* Ramanujan enunciated, as found "incidentally in the course of other investigations," a number of curious and interesting formulæ in the analytic theory of numbers. The following paper contains proofs either of Ramanujan's formulæ or of the improved formulæ to which the recent work of Hardy and Littlewood† on the Riemann ζ-function naturally leads. Neither the ideas involved nor the methods used are new, and, for this reason, in several places little more than an outline of the proof is given.

For convenience we refer to the numbered formulæ of Ramanujan's note by prefixing the letter R.; thus (R. 1) denotes the first formula of Ramanujan's paper.

1. *Proofs of the formulæ*

$$\text{(R. 15)} \qquad \frac{\zeta(s)\,\zeta(s-a)\,\zeta(s-b)\,\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty} n^{-s}\sigma_a(n)\,\sigma_b(n)$$

$$[s=\sigma+it;\ \sigma>1,\ \sigma>1+\mathfrak{R}(a),\ \sigma>1+\mathfrak{R}(b),\ \sigma>1+\mathfrak{R}(a+b)],$$

$$\text{(R. 1)} \qquad \frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} n^{-s} d^2(n) \quad (\sigma>1),$$

$$\text{(R. 3)} \qquad \sum_{n\leqslant x} d^2(n) = Ax(\log x)^3 + Bx(\log x)^2 + Cx\log x + Dx + O(x^{\frac{3}{5}+\epsilon}).$$

We write

$$\text{(1.01)} \qquad Z(s) = \frac{\zeta(s)\,\zeta(s-a)\,\zeta(s-b)\,\zeta(s-a-b)}{\zeta(2s-a-b)},$$

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* *Messenger of Mathematics*, Vol. 45 (1915), pp. 81–84.

† I take this opportunity of expressing my grateful thanks to Prof. Hardy first for calling my attention to Ramanujan's note, and then for putting at my disposal his advice and communications from as yet unpublished researches of himself and Mr. Littlewood.

and assume, in the first place, that

$$a \neq b, \quad a \neq 0, \quad b \neq 0. \tag{1.02}$$

Then, if

$$\sigma > 1, \quad \sigma > 1+\Re(a), \quad \sigma > 1+\Re(b), \quad \sigma > 1+\Re(a+b), \tag{1.03}$$

we have

$$Z(s) = \prod_p \left\{ \frac{(1-p^{a+b}\,p^{-2s})}{(1-p^{-s})(1-p^a\,p^{-s})(1-p^b\,p^{-s})(1-p^{a+b}\,p^{-s})} \right\}$$

$$= \prod_p \left\{ \frac{1}{(1-p^a)(1-p^b)} \sum_{n=0}^{\infty} p^{-ns}(1-p^{(n+1)a})(1-p^{(n+1)b}) \right\},$$

by expressing in partial fractions and expanding each fraction in powers of p^{-s}; *i.e.*

$$Z(s) = \sum_{n=1}^{\infty} \rho_n n^{-s}, \tag{1.10}$$

where, if n when expressed as the product of distinct prime factors has the form

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r},$$

$$\rho_n = \prod_{\nu=1}^{r} \frac{(1-p_\nu^{(\alpha_\nu+1)a})(1-p_\nu^{(\alpha_\nu+1)b})}{(1-p_\nu^a)(1-p_\nu^b)} = \sigma_a(n)\,\sigma_b(n), \tag{1.15}$$

and, as usual,*

$$\sigma_a(n) = \sum_{d|n} d^a.$$

The extension of this result to the cases excluded under (1 . 02) follows at once by considerations of continuity; and formula (R. 15) is thus established for all values of a and b.

Formula (R. 1) is, of course, the particular case of (R. 15) obtained when $a = b = 0$.

Formula (R. 3) was *probably* deduced by Ramanujan somewhat as follows. If we write (R. 1) in the form

$$\sum_{n=1}^{\infty} n^{-s} d^2(n) = \sum_{n=1}^{\infty} n^{-s} d_4(n) \sum_{n=1}^{\infty} n^{-2s} \mu(n), \tag{1.20}$$

where $d_4(n)$ denotes the number of decompositions of n into four factors, we see, by the rule of Dirichlet multiplication, that

$$d^2(n) = \sum_{d^2|n} \mu(\delta)\, d_4(n/\delta^2),* \tag{1.21}$$

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* Following Landau, we write $d|n$ for "d is a divisor of n." Thus $\sum_{\delta^2|n}$ is a sum extended over all square divisors of n.

so that

$$(1.22) \qquad \sum_{n \leqslant x} d^2(n) = \sum_{\alpha^2\beta \leqslant x} \mu(\alpha)\, d_4(\beta) = \sum_{\alpha \leqslant \sqrt{x}} \mu(\alpha) \sum_{\beta \leqslant x/\alpha^2} d_4(\beta).$$

The sum with regard to β may be approximated to by using the formula, due to Landau,*

$$(1.24) \qquad \sum_{n \leqslant y} d_4(n) = a_0 y(\log y)^3 + a_1 y(\log y)^2 + a_2 y \log y + a_3 y + O(y^{\frac{3}{5}+\epsilon})$$
$$= \Phi(y) + O(y^{\frac{3}{5}+\epsilon}),$$

where $\Phi(y)$ denotes the residue of the function $y^s s^{-1}\{\zeta(s)\}^4$ at its pole $s = 1$. In this manner one obtains, after simple calculations, a formula of the type

$$(1.25) \qquad \sum_{n \leqslant x} d^2(n) = x\{A(\log x)^3 + B(\log x)^2 + C \log x + D\}$$
$$+ \sum_{\alpha \leqslant \sqrt{x}} \mu(\alpha)\, O\{(x/\alpha^2)^{\frac{3}{5}+\epsilon}\} + O\{\sqrt{x}(\log x)^3\}$$
$$= x\{A(\log x)^3 + B(\log x)^2 + C \log x + D\} + O(x^{\frac{3}{5}+\epsilon}),$$

since $\Sigma\mu(\alpha)\,\alpha^{-\frac{6}{5}}$ is convergent. The values of the constants A, B, C, D would, of course, be obtained in terms of the known constants a_0, a_1, a_2, a_3. It is all the more unnecessary to follow out the details in this attempted reconstruction of Ramanujan's proof because, as will be shown immediately, it is now possible, by using concepts introduced and results obtained by Hardy and Littlewood in their more recent joint researches, to improve upon formula (R. 3) to the extent of replacing therein the error-term $O(x^{\frac{3}{5}+\epsilon})$, by the lesser error $O(x^{\frac{1}{2}+\epsilon})$, and thus to prove, without unproved assumptions, what, without Hardy and Littlewood's researches, could only be proved on the assumption of the Riemann hypothesis.†

Order on the average.

Let $f(s)$ be a function of the complex variable s, defined on the line $\sigma = \alpha$, and $\phi(t)$ a positive, even function of the real variable t, monotone for positive values of t. We shall say that $f(s)$ is, on the line $\sigma = \alpha$, of order $\phi(t)$ *on the average*, if a constant K exists such that

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* "Über die Anzahl der Gitterpunkte in gewissen Bereichen," *Göttinger Nachrichten*, 1912, p. 723.

† See the first footnote of Ramanujan's paper.

$$\int_{-T}^{T} |f(a+it)|\,dt < KT\phi(T) \tag{1.30}$$

for all (or for all sufficiently large) values of T. And, to represent this fact symbolically, we introduce the notation

$$f(s) = O_1\{\phi(t)\}. \tag{1.35}$$

It is at once obvious that, in all ordinary cases, the idea of order "on the average" is wider than and includes the ordinary idea of "order." If, for example, $f(s) = O\{\phi(t)\}$, where $\phi(t)$ is, for positive values of t, a steadily increasing function, so that $|f(s)| < K\phi(t)$ for $t \geqslant t_0$ or $t \leqslant -t_0$, then

$$\begin{aligned}\int_{-T}^{T} |f(a+it)|\,dt &< \int_{-t_0}^{t_0} |f(a+it)|\,dt + 2K\int_{t_0}^{T}\phi(t)\,dt \\ &< 2K\int_{0}^{T}\phi(t)\,dt + \int_{-t_0}^{t_0}|f(a+it)|\,dt \\ &< K'T\phi(T) \quad (K' > 2K),\end{aligned}$$

for sufficiently large values of T, since the second integral is fixed and finite, as T increases. That is to say $f(s) = O_1\{\phi(t)\}$.

It follows at once from the definition of order, that if

$$f(s) = O\{\phi(t)\}, \qquad g(s) = O\{\psi(t)\},$$

then

$$f(s)\,g(s) = O\{\phi(t)\,\psi(t)\}.$$

The analogous theorem does not remain true when orders on the average are taken into account. The restricted result embodied in the following two Lemmas is, however, amply sufficient for our purposes.

Lemma 1.—*If* $f(s) = f(a+it) = O_1\{\phi(t)\}$ *and* $g(s) = O(1)$, *then*

$$f(s)\,g(s) = O_1\{\phi(t)\}.$$

The proof is obvious.

Lemma 2.—*With the same notation as in Lemma* 1,

$$f(s)\,s^{\eta} = O_1\{\phi(t)\,t^{\eta}\} \quad (\eta > -1); \qquad \frac{f(s)}{s} = O_1\left\{\phi(t)\,\frac{\log t}{t}\right\}.$$

For let

$$\int_{t_0}^{T} |f(a+it)|\,dt = F(T),$$

so that $F(t)$ vanishes when $t = t_0$, and

$$F(t) < Kt\phi(t)$$

for (as we may suppose) all values of t which exceed t_0. Then, by an integration by parts and by use of the Second Theorem of the Mean ($\phi(t)$ being monotone), we obtain

$$\int_{t_0}^{T} |f(a+it)\,t^{\eta}|\,dt < KT^{1+\eta}\,\phi(T)+K\int_{t_0}^{T} t^{\eta}\phi(t)\,dt < KT^{1+\eta}\,\phi(T)\text{[*]} \quad (\eta > -1);$$

$$< K\log T\phi(T) \quad (\eta = -1);$$

giving the result enunciated.

We should also state explicitly the theorems :—

If $a_1, a_2, a_3, \ldots$ are an unlimited sequence of numbers such that

(1.40) $$|a_n| = O(n^{\beta-1}) \quad (\beta > 0),$$

and if we write

(1.41) $$Z(s) = \sum_{n=1}^{\infty} a_n n^{-s},\text{[†]}$$

and

(1.42) $$f(x) = \sum_{n \leqslant x}{}' \; a_n,$$

where the dash over the sign of summation indicates that, if x is an integer,

(1.43) $$f(x) = a_1+a_2+\ldots+a_{x-1}+\tfrac{1}{2}a_x,$$

then

(1.45) $$\left| f(x)-\frac{1}{2\pi i}\int_{a-i\omega}^{a+i\omega} \frac{x^s}{s}Z(s)\,ds \right| < K\frac{x^a}{\omega} \quad (a > \beta,\ \omega > 0),$$

where K is independent both of x and of ω.[‡]

Finally, we make use of the result recently established by Hardy and Littlewood, that

(1.50) $$\{\zeta(\tfrac{1}{2}+it)\}^4 = O_1(t^{\epsilon}).$$

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[*] The K's which occur denote, of course, *different* constants.

[†] So that $Z(s)$ is absolutely convergent if $\sigma > \beta$.

[‡] This Lemma, although not explicitly enunciated in Landau's *Handbuch*, follows at once by combining equation (1) on p. 826 with Hilfssatz 1 of pp. 806–808; and is, of course, well known.

Proof of (*R.* 3).

From formulæ (R. 1) and (1 . 45) we have, taking $\omega = \sqrt{x}$,

$$(1.60)\qquad \sum_{n \leqslant x} d^2(n) = \frac{1}{2\pi i} \int_{1+\delta-i\sqrt{x}}^{1+\delta+i\sqrt{x}} \frac{x^s}{s} \frac{\zeta^4(s)}{\zeta(2s)}\, ds + O(x^{\frac{1}{2}+\epsilon}).$$

The integral on the right-hand side of (1 . 60) is evaluated by applying Cauchy's theorem to the rectangle of vertices $1+\delta \pm i\sqrt{x}$, $\frac{1}{2} \pm i\sqrt{x}$, allowing of course for the fourfold pole at $s = 1$. The integrals along the sides $t = \pm\sqrt{x}$ are easily seen to be both $O(x^{\frac{1}{2}+\epsilon})$, since

$$|\zeta(s)| = O(t^{\frac{1}{2}-\frac{1}{2}\sigma+\delta}) \qquad (\tfrac{1}{2} \leqslant \sigma \leqslant 1).$$

Along $\sigma = \frac{1}{2}$ we have, by (1 . 50) and Lemma 1,

$$(1.63)\qquad \frac{\zeta^4(s)}{\zeta(2s)} = O_1(t^{\epsilon}).$$

From (1 . 63) and Lemma 2 it follows that

$$(1.65)\qquad \int_{\frac{1}{2}-i\sqrt{x}}^{\frac{1}{2}+i\sqrt{x}} \frac{x^s}{s} \frac{\zeta^4(s)}{\zeta(2s)}\, ds = O\left(x^{\frac{1}{2}} \log x \int_1^{\sqrt{x}} \frac{dt}{t^{1-\epsilon}}\right) = O\{x^{\frac{1}{2}}(\sqrt{x})^{\epsilon}\} = O(x^{\frac{1}{2}+\epsilon}).$$

Calculating the residue of the integrand at $s = 1$, and collating the above results, we have then

$$(1.70)\qquad \sum_{n \leqslant x} d^2(n) = x\{A(\log x)^3 + B(\log x)^2 + C\log x + D\} + O(x^{\frac{1}{2}+\epsilon}),$$

where

$$(1.71)\qquad A = \frac{1}{\pi^2}, \qquad B = \frac{12\gamma-3}{\pi^2} - \frac{36}{\pi^4}\zeta'(2), \quad \ldots .$$

Equation (1 . 70) is the desired improved form of (R. 3).

2. (R. 5). *The Dirichlet Series* $\sum_{n=1}^{\infty} n^{-s} d^r(n)$.

We have

$$(2.01)\qquad \sum_{n=1}^{\infty} n^{-s} d^r(n) = \prod_p \left\{1 + \sum_{\nu=1}^{\infty} (\nu+1)^r p^{-\nu s}\right\},$$

where both members of this equation are absolutely convergent if $\sigma > 1$. Hence, if $\sigma > 1$,

$$(2.02)\qquad \left\{\sum_{n=1}^{\infty} n^{-s} d^r(n)\right\} \{\zeta(s)\}^{-2^r} = \prod_p \{(1-p^{-s})^{2^r}(1+2^r p^{-s} + 3^r p^{-2s} + \ldots\}$$

$$= \prod_p \{1 + a_2 p^{-2s} + a_3 p^{-3s} + \ldots\},$$

where $$|a_\nu| = \left|(\nu+1)^r - 2^r\nu^r + \frac{2^r(2^r-1)}{\underline{|2}}(\nu-1)^r - \ldots\right| < K\nu^r,$$

where K is independent of ν. Hence

$$(2.03) \qquad \sum_{\nu=2}^{\infty} |a_\nu| p^{-\nu s} < Kp^{-2s} \sum_{\nu=0}^{\infty} (\nu+2)^r p^{-\nu s} \leqslant Kp^{-2s} \sum_{\nu=0}^{\infty} (\nu+2)^r 2^{-\nu s}$$
$$< K'p^{-2s},$$

where K' is independent of p.

But if $\sigma > \frac{1}{2}$, $\sum_p p^{-2s}$ is absolutely convergent, and so therefore is also

$$(2.04) \qquad \phi(s) = \prod_p \left\{1 + \sum_{\nu=2}^{\infty} a_\nu p^{-\nu s}\right\}.$$

Hence we obtain equation (R. 5) that

$$(2.10) \qquad \sum_1^{\infty} n^{-s} d^r(n) = \{\zeta(s)\}^{2^r} \phi(s),$$

where $\phi(s)$ is absolutely convergent if $\sigma > \frac{1}{2}$. It follows at once that

$$(2.15) \qquad \sum_{n \leqslant x} d^r(n) = \frac{1}{2\pi i} \int_{i+\epsilon-i\omega}^{1+\epsilon+i\omega} \frac{x^s}{s} \{\zeta(s)\}^{2^r} \phi(s)\, ds + O\left(\frac{x^{1+\epsilon}}{\omega}\right).$$

The consideration of the integral gives rise to two distinct cases, according as 2^r is, or is not, an integer.

(R. 8). 2^r *is an integer.*

If 2^r is an integer we apply Cauchy's theorem to the rectangle whose vertices are $1+\epsilon \pm i\sqrt{x}$, $\frac{1}{2}+\epsilon \pm i\sqrt{x}$, within which the function

$$\frac{x^s}{s} \{\zeta(s)\}^{2^r} \phi(s)$$

is analytic, except for its pole at $s = 1$, where it has a residue of the form

$$(2.20) \qquad R(x) = x\left\{A_1(\log x)^{2^r-1} + A_2(\log x)^{2^r-2} + \ldots + A_{2^r}\right\},$$

where $$A_1 = \frac{\phi(1)}{\underline{|2^r-1}}, \ \ldots .$$

Thus

$$(2.21) \qquad \sum_{n \leqslant x} d^r(n) = R(x) + \frac{1}{2\pi i} \int_{1+\epsilon-i\sqrt{x}}^{\frac{1}{2}+\epsilon-i\sqrt{x}} + \int_{\frac{1}{2}+\epsilon-i\sqrt{x}}^{\frac{1}{2}+\epsilon+i\sqrt{x}}$$

$$+ \int_{\frac{1}{2}+\epsilon+i\sqrt{x}}^{1+\epsilon+i\sqrt{x}} \frac{x^s}{s} \{\zeta(s)\}^{2^r} \phi(s)\, ds + O(x^{\frac{1}{2}+\epsilon}).$$

On the assumption of the truth of the Riemann hypothesis

$$(2.22) \qquad |\zeta(s)| = O(t^\epsilon),$$

for $\sigma \geqslant \frac{1}{2}+\epsilon$, so that the first and third integrals on the right-hand side of (2.21) are obviously $O(x^{\frac{1}{2}+\epsilon})$. The middle integral also is

$$O \int_1^{\sqrt{x}} \frac{x^{\frac{1}{2}+\epsilon}}{t}\, dt = O(x^{\frac{1}{2}+\epsilon} \log \sqrt{x}) = O(x^{\frac{1}{2}+\epsilon}).$$

Hence we have the formula required, that

$$(2.25) \qquad \sum_{n \leqslant x} d^r(n) = R(x) + O(x^{\frac{1}{2}+\epsilon}),$$

where $R(x)$ is of the form given in (2.20). Formula (2.25) is proved on the assumption of the truth of the Riemann hypothesis.

The most that can be proved, *without the Riemann hypothesis*, is that

$$(2.28) \qquad \sum_{n \leqslant x} d^r(n) = R(x) + O(x^{\frac{2^r-1}{2^r+2}+\epsilon}),$$

for $r \geqslant 2$.

(R. 9). 2^r *is not an integer.*

If 2^r is not an integer, the above proof becomes invalid, since the integrand is not a uniform analytic function in the rectangle. The method to be used in this case was first used by Landau in a well known paper,* and will here (particularly as the calculations involved are a trifle long) be merely briefly described.

From the known equations

$$(2.30) \qquad \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{y^s}{s^2}\, ds = \log y \quad (y \geqslant 1)$$

$$= 0 \qquad (0 < y \leqslant 1),$$

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* "Über die Einteilung der positiven ganzen Zahlen u.s.w.," *Archiv der Math. und Physik* (*Grunert's Archiv*), Vol. 13 (1908), pp. 305–312.

we easily deduce by term-by-term integration that

$$(2.31) \qquad \sum_{n \leqslant x} d^r(n) \log \frac{x}{n} = \frac{1}{2\pi i} \int_{2-ix^2}^{2+ix^2} \frac{x^s}{s^2} f(s)\, ds + O(1),$$

where we have written

$$(2.32) \qquad f(s) = \sum_{n=1}^{\infty} n^{-s} d^r(n) = \{\zeta(s)\}^{2^r} \phi(s),$$

and $\phi(s)$ is absolutely convergent if $\sigma > \frac{1}{2}$.

It is known (independently of the Riemann hypothesis) that positive constants c and K exist such that

$$0 < |\zeta(s)| < K \log |t| \qquad \left(|t| \geqslant 3,\ \sigma \geqslant 1 - \frac{c}{\log |t|}\right).$$

The integral on the right-hand side of (2 . 32) is approximated to by applying Cauchy's theorem to the contour $ABCDEFEGHA$, where* B is the point $2+ix^2$, C is the point $1-\dfrac{c}{2\log x}+ix^2$, CD is the curve $\sigma = 1-\dfrac{c}{\log t}$, D is the point $1-\dfrac{c}{\log 3}+3i$, E is the point $1-\dfrac{c}{\log 3} = \vartheta$, say, where $\frac{1}{2} < \vartheta < 1$, F is the point $s = 1$, and the lower half of the contour is obtained from the upper part by reflection in the real axis.

The function $\dfrac{x^s}{s^2} f(s)$ is uniform and without singularity inside and on this contour; and the known upper bounds of $|\zeta(s)|$ on the various rectilinear and curvilinear parts of the contour lead us, after some calculation, to deduce from (2 . 31) the relation

$$(2.33) \qquad \sum_{n \leqslant x} d^r(n) \log \frac{x}{n} = \frac{1}{2\pi i} \int_{EFE} \frac{x^s}{s^2} f(s)\, ds + O\left(x e^{-\sqrt[3]{(\log x)}}\right),$$

where, in describing the path EFE, the point s goes round the point 1 in the positive sense.

In the integral along EFE we now replace $f(s)/s^2$ by its approximate expression there

$$(2.34) \quad \frac{f(s)}{s^2} = (s-1)^{-\rho} + a_1(s-1)^{-\rho+1} + \ldots + a_\nu(s-1)^{-\rho+\nu} + O\{(s-1)^{-\rho+\nu+1}\},$$

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* A figure is given in the paper of Landau quoted, p. 310.

where we have written ρ in place of 2^r; and so we deduce without difficulty that

$$(2.35)\quad \sum_{n\leqslant x} d^r(n)\log\frac{x}{n}$$
$$=\frac{1}{2\pi i}\sum_{\lambda=0}^{\nu} a_\lambda \int_{EFE} x^s(s-1)^{-\rho+\lambda}\,ds+O\left(\frac{x}{(\log x)^{(-\rho+\nu+2)}}\right)\quad (a_0=1),$$

provided that $\nu>\rho-2.$

Let now C be a contour of the form used by Hankel,* *i.e.* a contour starting at infinity on the positive real axis, encircling the origin in the positive direction, and then returning to the starting point; let γ denote that part of C which lies to the left of $\sigma=1-\vartheta$, and Γ the part to the left of $\sigma=(1-\vartheta)\log x$.

Then, for all values of μ,

$$\int_{EFE} x^s(s-1)^\mu\,ds$$
$$=-x\int_\gamma x^{-s}(-s)^\mu\,ds=-\frac{x}{(\log x)^{\mu+1}}\int_\Gamma e^{-z}(-z)^\mu\,dz$$
$$=-\frac{x}{(\log x)^{\mu+1}}\left\{\int_C e^{-z}(-z)^\mu\,dz+O\int_{(1-\vartheta)\log x}^{\infty} e^{-\sigma}\sigma^\mu\,d\sigma\right\}$$
$$=-\frac{x}{(\log x)^{\mu+1}}\left\{2i\sin(\mu\pi)\,\Gamma(\mu+1)+O\,e^{-(1-\vartheta)\log x}(\log x)^{\mu+2}\int_1^\infty\frac{d\sigma}{\sigma^2}\right\},$$

supposing that x is so large that $(1-\vartheta)\log x>\mu+2.$

Hence, *a fortiori*,

$$(2.36)\quad \frac{1}{2\pi i}\int_{EFE} x^s(s-1)^\mu\,ds=-\frac{\sin(\mu\pi)\,\Gamma(\mu+1)}{\pi}\,\frac{x}{(\log x)^{\mu+1}}+O(1).$$

From equations (2.35) and (2.36),

$$(2.37)\quad \sum_{n\leqslant x} d^r(n)\log\frac{x}{n}$$
$$=\frac{\sin(\rho\pi)}{\pi}\sum_{\lambda=0}^{\nu}(-1)_\lambda\,a_\lambda\Gamma(\lambda-\rho+1)\,\frac{x}{(\log x)^{\lambda-\rho+1}}+O\left(\frac{x}{(\log x)^{\nu-\rho+2}}\right)$$
$$=\sum_{\lambda=0}^{\nu} b_\lambda\,\frac{x}{(\log x)^{\lambda-\rho+1}}+O\left(\frac{x}{(\log x)^{\nu-\rho+2}}\right).$$

---

* Hankel's Γ contour. See, *e.g.* Whittaker and Watson's *Modern Analysis* (2nd edition), p. 239.

The desired asymptotic formula for $\sum\limits_{n \leqq x} d^r(n)$ is now deduced from (2.37) in the standard manner; *i.e.* take δ, a positive function of x, tending to zero as x tends to infinity, but otherwise quite arbitrary and to be determined later. By writing $x+\delta x$ in place of x in (2.37), subtracting the first equation from the second, and using the obvious relation

$$\sum_{n \leqslant x} d^r(n) \sim A_1 x (\log x)^{\rho-1},$$

we obtain, after division by δ,

$$(2.38) \quad \sum_{n \leqslant x} d^r(n)$$

$$= x \sum_{\lambda=0}^{\nu} \frac{A_\lambda}{(\log x)^{\lambda-\rho+1}} + O\{\delta x (\log x)^{\rho-1}\} + O\{x \delta^{-1} (\log x)^{\rho-\nu-2}\},$$

where $$A_0 = b_0, \quad A_\lambda = b_\lambda + (\rho-\lambda)\, b_{\lambda-1} \quad (\lambda \geqslant 1).$$

Finally, choosing

$$\delta = \delta(x) = (\log x)^{-\frac{1}{2}(\nu+1)},$$

we obtain Ramanujan's result that, if 2^r is not an integer,

$$(2.39) \quad \sum_{n \leqslant x} d^r(n) = x (\log x)^{2^r-1} \left\{ \sum_{\lambda=0}^{\nu} \frac{A_\lambda}{(\log x)^\lambda} + O\left(\frac{1}{(\log x)^{\nu+1}}\right) \right\},$$

where the A's are constants.

3. (R. 6) *and* (R. 7). *Case of* $r = -1$.

As a particular case of the above, we may take $r = -1$, we thus have

$$(3.01) \qquad f(s) = \sum_{n=1}^{\infty} \frac{1}{n^s d(n)} = \sqrt{\{\zeta(s)\}}\, \phi(s),$$

where $\phi(s)$ is absolutely convergent if $\sigma > \frac{1}{2}$. But, clearly,

$$(3.02) \quad \sum_{n=1}^{\infty} \frac{1}{n^s d(n)} = \prod_p \left(1 + \tfrac{1}{2} p^{-s} + \tfrac{1}{3} p^{-2s} + \dots + \frac{1}{\nu+1} p^{-\nu s} + \dots\right)$$

$$= \prod_p \left\{ p^s \log\left(\frac{1}{1-p^{-s}}\right) \right\},$$

and hence

$$(3.03) \qquad \phi(s) = \prod_p \left\{ p^s \sqrt{(1-p^{-s})} \log\left(\frac{1}{1-p^{-s}}\right) \right\}.$$

If now we write

$$(2.34\ bis)\qquad \frac{f(s)}{s^2}=\frac{1}{(s-1)^\rho}\sum_{\lambda=0}^{\infty}a_\lambda(s-1)^\lambda\quad(\text{about } s=1),$$

$$(2.37\ bis)\qquad \sum_{n\leqslant x}d^r(n)\log\frac{x}{n}=\sum_{\lambda=0}^{\nu}b_\lambda\frac{x}{(\log x)^{\lambda-\rho+1}}+O\left(\frac{x}{(\log x)^{\nu-\rho+2}}\right),$$

$$(3.06)\qquad \sum_{n\leqslant x}d^r(n)=\sum_{\lambda=0}^{\nu}c_\lambda\frac{x}{(\log x)^{\lambda-\rho+1}}+O\left(\frac{x}{(\log x)^{\nu-\rho+2}}\right),$$

we easily show, by algebraical transformation, that c_λ is derived from $a_\lambda+a_{\lambda-1}$ in the same way as b_λ is from a_λ.* Thus, as we should expect, when once the validity of formula (2.39) has been established, the coefficients on the right-hand side are most conveniently calculated from the residue at $s=1$ not of the function $x^s f(s)s^{-2}$ but of the function $x^s f(s)s^{-1}$. Further, if we write

$$f(s)=\frac{1}{(s-1)^\rho}\sum_{\lambda=0}^{\infty}a_\lambda(s-1)^\lambda,$$

we find at once, since

$$\frac{f(s)}{s}=\frac{f(s)}{1+(s-1)},$$

that in equation (3.06) c_λ is given by

$$c_\lambda=(-1)^\lambda\frac{\sin(\rho\pi)}{\pi}\Gamma(\lambda-\rho+1)\{a_\lambda-a_{\lambda-1}+\dots+(-1)^\lambda a_0\}.$$

Applying these considerations to the case $r=-1$, where

$$f(s)=\phi(s)\surd\zeta(s)=\frac{1}{\surd(s-1)}[\phi(1)+\{\phi'(1)+\tfrac{1}{2}\gamma\phi(1)\}(s-1)+\dots],$$

we find at once that in the formula

$$(3.10)\qquad \sum_{n\leqslant x}\frac{1}{d(n)}=\frac{x}{\surd(\log x)}\left\{c_0+\frac{c_1}{\log x}+\dots+\frac{c_\nu}{(\log x)^\nu}+O\left(\frac{1}{(\log x)^{\nu+1}}\right)\right\},$$

the coefficients are given by

$$c_0=\frac{1}{\surd\pi}\phi(1)=\frac{1}{\surd\pi}\prod_p\left\{\surd(p^2-p)\log\left(\frac{p}{p-1}\right)\right\},$$

$$c_1=-\frac{1}{2\surd\pi}\{\phi'(1)+(\tfrac{1}{2}\gamma-1)\phi(1)\},\ \dots,$$

---

* The relation between b_λ and a_λ is shown in (2.37).

where

$$\zeta(1+s) = \frac{1}{s} + \gamma - \gamma_1 s + \gamma_2 s^2 - \dots .$$

4. (R. 11), (R. 12), and (R. 13). On $D_v(x) = \sum_{n \leqslant x} d(nv)$.

By an induction from k to $k+1$, and by use of the identity

$$\alpha + \gamma + 1 = (\alpha+1)(\gamma+1) - \alpha\gamma,$$

we easily prove the identity

$$
\begin{aligned}
(4.01) \quad & (\alpha_1+\gamma_1+1)(\alpha_2+\gamma_2+1)\dots(\alpha_k+\gamma_k+1) \\
&= (\alpha_1+1)(\gamma_1+1)(\alpha_2+1)(\gamma_2+1)\dots(\alpha_k+1)(\gamma_k+1) \\
&\quad - \sum_1 \alpha_1\gamma_1(\alpha_2+1)(\gamma_2+1)(\alpha_3+1)(\gamma_3+1)\dots(\alpha_k+1)(\gamma_k+1) \\
&\quad + \sum_2 \alpha_1\gamma_1\alpha_2\gamma_2(\alpha_3+1)(\gamma_3+1)\dots(\alpha_k+1)(\gamma_k+1) \\
&\quad - \dots + (-1)^r \sum_r \alpha_1\gamma_1\alpha_2\gamma_2\dots\alpha_r\gamma_r(\alpha_{r+1})\dots(\gamma_k+1) \\
&\quad + \dots + (-1)^k \alpha_1\gamma_1\alpha_2\gamma_2\dots\alpha_k\gamma_k,
\end{aligned}
$$

where the symbol r under the sign of summation indicates that there are ${}_nC_r$ terms in the sum.

Let then the two positive integers u and v, when exhibited as the product of their prime factors, be

$$u = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k} q_1^{\beta_1} \dots q_l^{\beta_l}, \qquad v = p_1^{\gamma_1} p_2^{\gamma_2} \dots p_k^{\gamma_k} r_1^{\delta_1} r_2^{\delta_2} \dots r_m^{\delta_m},$$

the prime factors which are common to u and v all being represented by letters p.

Then $$d(uv) = \prod_{\kappa=1}^{k} (\alpha_\kappa+\gamma_\kappa+1) \prod_{\lambda=1}^{l} (\beta_\lambda+1) \prod_{\mu=1}^{m} (\delta_\mu+1).$$

On substituting for the first product from equation (4.01), we find

$$
\begin{aligned}
(4.10) \quad d(uv) &= d(u)\,d(v) - \sum_1 d\left(\frac{u}{p_1}\right) d\left(\frac{v}{p_1}\right) + \dots \\
&\qquad + (-1)^r \sum_r d\left(\frac{u}{p_1 p_2 \dots p_r}\right) d\left(\frac{v}{p_1 p_2 \dots p_r}\right) + \dots \\
&= \sum_{\delta|u,\, \delta|v} \mu(\delta)\, d\left(\frac{u}{\delta}\right) d\left(\frac{v}{\delta}\right).
\end{aligned}
$$

From equation (4 . 10) it follows that, if we write

$$D_v(x) = \sum_{n \leqslant x} d(nv), \tag{4.12}$$

where v is any positive integer, then

$$D_v(x) = \sum_{n \leqslant x,\ \delta|v,\ \delta|n} \sum \mu(\delta)\, d\left(\frac{v}{\delta}\right) d\left(\frac{n}{\delta}\right) = \sum_{\delta|v} \mu(\delta) d\left(\frac{v}{\delta}\right) D_1\left(\frac{x}{\delta}\right) \tag{4.15}$$

$$= \sum_{\delta|v} \mu(\delta)\, d\left(\frac{v}{\delta}\right) \left\{ \frac{x}{\delta} \log\left(\frac{x}{\delta}\right) + (2\gamma - 1)\frac{x}{\delta} + O(x^{\frac{1}{3}} \log x) \right\}^{*}$$

$$= \alpha(v)\, x(\log x + 2\gamma - 1) + \beta(v) x + O(x^{\frac{1}{3}} \log x),$$

where
$$\alpha(v) = \sum_{\delta|v} \frac{\mu(\delta)}{\delta} d\left(\frac{v}{\delta}\right),$$

so that
$$\sum_{n=1}^{\infty} \alpha(n) n^{-s} = \frac{\zeta^2(s)}{\zeta(s+1)},$$

and
$$\beta(v) = -\sum_{\delta|v} \frac{\mu(\delta)}{\delta} \log \delta . d\left(\frac{v}{\delta}\right),$$

so that
$$\sum_{n=1}^{\infty} \beta(n)\, n^{-s} = -\frac{\zeta^2(s)\, \zeta'(s+1)}{\zeta^2(s+1)}.$$

5. (R. 16).

We have

$$\eta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} = \prod_q \frac{1}{1-q^{-s}} \prod_r \frac{1}{1+r^{-s}} \quad (\sigma > 1), \tag{5.01}$$

where q takes all prime values of the form $4m+1$, and r all prime values of the form $4m+3$. Hence, as in the proof of formula (15) of Ramanujan's paper,

$$\frac{\eta(s)\, \eta(s-a)\, \eta(s-b)\, \eta(s-a-b)}{(1-2^{-2s+a+b})\, \zeta(2s-a-b)}$$

$$= \prod_q \frac{1-q^{-2s+a+b}}{(1-q^{-s})(1-q^a q^{-s})(1-q^b q^{-s})(1-q^{a+b} q^{-s})}$$

$$\times \prod_r \frac{1-r^{-2s+a+b}}{(1+r^{-s})(1+r^a r^{-s})(1+r^b r^{-s})(1+r^{a+b} r^{-s})}$$

---

* Using the best known approximation (Voronoï's, 1903) for $D_1(x) = \sum_{n \leqslant x} d(n)$.

$$= \prod_q \left[\frac{1}{(q^a-1)(q^b-1)} \left(\frac{1}{1-q^{-s}} - \frac{q^a}{1-q^{a-s}} - \frac{q^b}{1-q^{b-s}} + \frac{q^{a+b}}{1-q^{a+b-}}\right)\right]$$

$$\times \prod_r \left[\frac{1}{(r^a-1)(r^b-1)} \left(\frac{1}{1+r^{-s}} - \frac{r^a}{1+r^{a-s}} - \frac{r^b}{1+r^{b-s}} + \frac{r^{a+b}}{1+r^{a+b-s}}\right)\right]$$

$$= \Sigma \frac{a_n}{n^s},$$

where, if $n = q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_l^{\alpha_l} \,.\, r_1^{\beta_1} r_2^{\beta_2} \ldots r_m^{\beta_m}$ $[q_\lambda \equiv 1,\ r_\mu \equiv 3 \pmod 4]$,

$$a_n = \prod_{\lambda=1}^{l} \frac{(q_\lambda^{a(\alpha_\lambda+1)}-1)(q_\lambda^{b(\alpha_\lambda+1)}-1)}{(q_\lambda^a-1)(q_\lambda^b-1)} \prod_{\mu=1}^{m} (-1)^{\beta_\mu} \frac{(r_\mu^{a(\beta_\mu+1)}-1)(r_\mu^{b(\beta_\mu+1)}-1)}{(r_\mu^a-1)(r_\mu^b-1)}$$

$$= \epsilon_n = (-1)^{\beta_1+\beta_2+\ldots+\beta_m} \sigma_a(n)\, \sigma_b(n).$$

The index is even or odd according as n is congruent to 1 or to 3 (mod 4); thus we have Ramanujan's equation (16), that

$$\frac{\eta(s)\,\eta(s-a)\,\eta(s-b)\,\eta(s-a-b)}{(1-2^{-2s+a+b})\,\zeta(2s-a-b)} = \sum_{n=0}^{\infty} \frac{(-1)^n \sigma_a(n)\, \sigma_b(n)}{(2n+1)^s}.$$

6. *An approximate formula* (R. 10) *for* $\prod_{n \leq x} d(n)$.

We consider the Dirichlet series

$$(6.01) \qquad f(s) = \sum_{n=1}^{\infty} n^{-s} \log_2 d(n).$$

$f(s)$ is clearly absolutely convergent if $\sigma > 1$, for $d(n) = O(n^\epsilon)$. If $n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r}$, a product of r *distinct* prime factors, then

$$(6.05) \qquad \log_2 d(n) = \sum_{\nu=1}^{r} \log_2(a_\nu+1)$$

$$= \sum_{\nu=1}^{r} \left\{ 1 + \log_2\left(\frac{a_\nu+1}{a_\nu}\right) + \log_2\left(\frac{a_\nu}{a_\nu-1}\right) + \ldots + \log_2 \tfrac{3}{2} \right\}$$

$$= \sum_{\nu=1}^{r} \left\{ 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{a_\nu} + \log_2\left(1+\frac{1}{a_\nu}\right) - \frac{1}{a_\nu} \right.$$

$$\left. + \log_2\left(1+\frac{1}{a_\nu-1}\right) - \frac{1}{a_\nu-1} + \ldots + \log_2(1+\tfrac{1}{2}) - \tfrac{1}{2} \right\}.$$

Now, if $\sigma > 1$,

$$(6.10)\qquad \log \zeta(s) = -\sum_{p} \log(1-p^{-s}) = \sum_{\nu=1}^{\infty} \sum_{p} \frac{1}{\nu} p^{-\nu s},$$

that branch of the logarithm being taken which is real when s is real and greater than 1. Thus

$$(6.12)\qquad \zeta(s) \log \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \sum_{\nu,\, p} \frac{1}{\nu} p^{-\nu s} = \sum_{n=1}^{\infty} a_n n^{-s},$$

where

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r},$$

and

$$(6.13)\qquad a_n = \sum_{\nu=1}^{r} \left(1+\tfrac{1}{2}+\tfrac{1}{3}+\dots+\frac{1}{\alpha_\nu}\right).$$

Also if we write

$$(6.15)\qquad \phi(s) = \sum_{\nu=2}^{\infty} \sum_{p} \left\{ \log_2 \left(1+\frac{1}{\nu}\right) - \frac{1}{\nu} \right\} p^{-\nu s},$$

then

$$(6.16)\qquad \zeta(s)\, \phi(s) = \sum_{n=1}^{\infty} b_n n^{-s},$$

where

$$(6.17)\quad b_n = \sum_{\nu=1}^{r} \left\{ \log_2 \left(1+\frac{1}{\alpha_\nu}\right) - \frac{1}{\alpha_\nu} + \log_2 \left(1+\frac{1}{\alpha_\nu - 1}\right) - \frac{1}{\alpha_\nu - 1} + \dots \right.$$
$$\left. + \log_2 (1+\tfrac{1}{2}) - \tfrac{1}{2} \right\}.$$

From equations (6.01), (6.05), (6.12), (6.13), (6.16), and (6.17), it follows that

$$(6.20)\qquad f(s) = \zeta(s) \log \zeta(s) + \zeta(s)\, \phi(s) \quad (\sigma > 1).$$

But $\phi(s)$ is absolutely convergent, if $\sigma > \frac{1}{2}$; for

$$\left| \log_2 \left(1+\frac{1}{\nu}\right) - \frac{1}{\nu} \right| < K \quad (\nu \geqslant 2);$$

and therefore

$$\sum_{\nu=2}^{\infty} \left| \log_2 \left(1+\frac{1}{\nu}\right) - \frac{1}{\nu} \right| p^{-\nu s} < \frac{Kp^{-2s}}{1-p^{-s}} < Kp^{-2s}.$$

Hence

$$\sum_{p,\,\nu} \left| \log_2 \left(1+\frac{1}{\nu}\right) - \frac{1}{\nu} \right| p^{-\nu s} < K \sum_{p} p^{-2s},$$

which is convergent if $\sigma > \frac{1}{2}$.

Thus we have at once

$$(6.25) \quad \sum_{n \leqslant x} \log_2 d(n)$$

$$= \frac{1}{2\pi i} \int_{1+\epsilon-i\omega}^{1+\epsilon+i\omega} \frac{x^s}{s} \zeta(s) \log \zeta(s)\, ds + \frac{1}{2\pi i} \int_{1+\epsilon-i\omega}^{1+\epsilon+i\omega} \frac{x^s}{s} \zeta(s)\, \phi(s)\, ds + O\left(\frac{x^{1+\epsilon}}{\omega}\right).$$

The integrand of the second integral is analytic in the rectangle with vertices

$$1+\epsilon \pm i\omega, \quad \tfrac{1}{2}+\epsilon \pm i\omega,$$

and has inside this rectangle only a simple pole at $s = 1$, with residue

$$(6.26) \qquad x\phi(1) = x \sum_{\nu=2}^{\infty} \sum_{p} \left\{ \log_2 \left(1+\frac{1}{\nu}\right) - \frac{1}{\nu} \right\} p^{-\nu}.$$

An application of Cauchy's theorem to this rectangle therefore gives at once

$$(6.28) \quad \frac{1}{2\pi i} \int_{1+\epsilon-i\omega}^{1+\epsilon+i\omega} \frac{x^s}{s} \zeta(s)\, \phi(s)\, ds = x\phi(1) + O\left(\frac{x^{1+\epsilon}}{\omega}\right) + O(x^{\frac{1}{2}+\epsilon}\, \omega^{\frac{1}{2}}).$$

For the other part of the integral we must use the contour and method of 2.3.*

If we write

$$(6.29) \qquad \zeta(s) \log \zeta(s) = \sum_{n=1}^{\infty} \delta(n)\, n^{-s},$$

we thus obtain

$$(6.30) \quad \sum_{n \leqslant x} \delta(n) \log \frac{x}{n} = \frac{1}{2\pi i} \int_{EFE} \frac{x^s}{s^2} \zeta(s) \log \zeta(s)\, ds + O\left(\frac{x}{\log^N x}\right),$$

however large N may be, and hence, since in the neighbourhood of $s = 1$,

$$\log \zeta(s) = \log \left(\frac{1}{s-1}\right) + \wp_1(s-1),$$

where $\wp_1(s-1)$ denotes a power-series in $(s-1)$, vanishing when $s = 1$, we have

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* See pp. 242–245.

$$(6.31)\quad \frac{1}{2\pi i}\int_{EFE}\frac{x^s}{s^2}\zeta(s)\log\zeta(s)\,ds=\frac{1}{2\pi i}\int_{EFE}\frac{x^s}{s^2}\zeta(s)\log\left(\frac{1}{s-1}\right)ds$$

$$=\frac{1}{2\pi i}\sum_{\lambda=0}^{\nu}k_\lambda\int_{EFE}x^s(s-1)^{\lambda-1}\log\left(\frac{1}{s-1}\right)ds+O\int_{\vartheta}^{1}x^\sigma(1-\sigma)^\nu\log\left(\frac{1}{1-\sigma}\right)d\sigma,$$

where, near $s=1$,

$$(6.32)\qquad \frac{\zeta(s)}{s^2}=\frac{1}{s-1}\sum_{\lambda=0}^{\infty}k_\lambda(s-1)^\lambda.$$

Proceeding as in the former case, we have (with the same notation)

$$\frac{1}{2\pi i}\int_{EFE}x^s(s-1)^\mu\log\left(\frac{1}{s-1}\right)ds=-\frac{1}{2\pi i}\,x\int_\gamma x^{-s}(-s)^\mu\log\left(\frac{1}{-s}\right)ds$$

$$=-\frac{1}{2\pi i}\,\frac{x}{(\log x)^{\mu+1}}\int_\Gamma e^{-z}(-z)^\mu\log\left(\frac{\log x}{-z}\right)dz$$

$$=-\frac{1}{2\pi i}\,\frac{x}{(\log x)^{\mu+1}}\int_C e^{-z}(-z)^\mu\log\left(\frac{\log x}{-z}\right)dz+O\left(\frac{x}{\log^N x}\right).$$

And hence, by Hankel's formula

$$\frac{1}{\Gamma(y)}=\frac{i}{2\pi}\int_C(-z)^{-y}e^{-z}\,dz,$$

and the derived formula

$$\frac{\Gamma'(y)}{\{\Gamma(y)\}^2}=\frac{i}{2\pi}\int_C(-z)^{-y}e^{-z}\log(-z)dz,$$

we have

$$(6.34)\quad \frac{1}{2\pi i}\int_{EFE}x^s(s-1)^{-1}\log\left(\frac{1}{s-1}\right)ds$$

$$=x\log\log x+\gamma x+O\left(\frac{x}{\log^N x}\right)\quad\bigl(\Gamma'(1)=-\gamma\bigr),$$

and if $\mu\geqslant 0$,

$$(6.35)\quad \frac{1}{2\pi i}\int_{EFE}x^s(s-1)^\mu\log\left(\frac{1}{s-1}\right)ds$$

$$=-\frac{x}{(\log x)^{\mu+1}}\,\frac{\Gamma'(-\mu)}{\{\Gamma(-\mu)\}^2}+O\left(\frac{x}{\log^N x}\right)$$

$$=(-1)^\mu(\mu)!\,\frac{x}{(\log x)^{\mu+1}}+O\left(\frac{x}{\log^N x}\right).$$

Hence we have

$$(6.36)\quad \sum_{n\leqslant x} \delta(n)\log\frac{x}{n}$$

$$= k_0(x\log\log x+\gamma x)+\sum_{\lambda=0}^{\nu-1} k_{\lambda+1}(-1)^{\lambda}\,\lambda!\,\frac{x}{(\log x)^{\lambda+1}}+O\,\frac{x}{(\log x)^{\nu}}.$$

The transition from $\sum\limits_{n\leqslant x}\delta(n)\log\dfrac{x}{n}$ to $\sum\limits_{n\leqslant x}\delta(n)$ is now performed in just the same way as before, as also is the transformation of the coefficients in the expansion, so that we obtain

$$(6.37)\quad \sum_{n\leqslant x}\log_2 d(n)$$

$$= (x\log\log x+\gamma x)+(\gamma-1)\frac{x}{\log x}$$

$$+1!\,(\gamma_1+\gamma-1)\frac{x}{(\log x)^2}+2!\,(\gamma_2+\gamma_1+\gamma-1)\frac{x}{(\log x)^3}+\dots$$

$$+(r-1)!\,(\gamma_{r-1}+\gamma_{r-2}+\dots+\gamma-1)\frac{x}{(\log x)^r}+x\phi(1)$$

$$+O\left(\frac{x}{(\log x)^{r+1}}\right),$$

where $$\zeta(1+s)=\frac{1}{s}+\gamma-\gamma_1 s+\gamma_2 s^2-\dots,$$

and hence, we have the formula (R. 10),

$$(6.40)\quad \prod_{n\leqslant x} d(n)=2^{x(\log\log x+C)+\psi(x)},$$

where $$C=\gamma+\phi(1)=\gamma+\sum_{\nu=2}^{\infty}\left\{\log_2\left(1+\frac{1}{\nu}\right)-\frac{1}{\nu}\right\}\sum_p p^{-\nu},$$

and $$\psi(x)=x\sum_{\lambda=1}^{r}\frac{(\lambda-1)!}{(\log x)^{\lambda}}(\gamma_{\lambda-1}+\gamma_{\lambda-2}+\dots+\gamma_1+\gamma-1)+O\left(\frac{x}{(\log x)^{r+1}}\right).$$

7. *Other formulæ.*

Formula (R. 15) leads, of course, to an asymptotic formula for the general sum $\sum\limits_{n<x}\sigma_a(n)\,\sigma_b(n)$. Ramanujan gives the formulæ for the two cases $a=0$, $b=1$,[*] and $a=b=1$, ignoring the case $a=b=0$, given

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[*] The π^2 occurring in this formula should be corrected to read π^4.

in (R. 3). The discussion of the formulæ for general values of a and b falls naturally into four cases: $a = b = 0$; $a = 0$, $b > 0$; $a = b > 0$; $a > b > 0$.* Without attempting to give this exhaustive discussion we think it worth while just to state a few of the formulæ which are obtained in the second (and second simplest) case.

If $r > \frac{2}{3}$,

$$(7\,.\,10) \qquad \sum_{n \leqslant x} d(n)\,\sigma_r(n) = \frac{\zeta^2(r+1)}{(r+1)\,\zeta(r+2)}\,x^{r+1}\,\{\log x + 2c_r\} + O(x^{r+\frac{1}{3}}\log x),$$

where
$$c_r = \gamma - \frac{1}{2(r+1)} + \frac{\zeta'(r+1)}{\zeta(r+1)} - \frac{\zeta'(r+2)}{\zeta(r+2)}.$$

If $r < \frac{1}{2}$, the error-term on the right-hand side of (7 . 10) can be shown to be both

$$O(x^{r+\frac{2r+5}{8(r+1)}+\epsilon}) \quad \text{and} \quad O(x^{r+\frac{2r+3}{4r+5}}\log x),$$

the latter being a better approximation than the former if

$$r \leqslant \frac{\sqrt{33}-5}{8} = {\cdot}092\ldots .$$

Such formulæ are obviously not "best possible," so that there would be no interest in setting down more of them.

Ramanujan's formulæ (R. 21) and (R. 22) for $\sum n^{-s} r^2(n)$ and $\sum\limits_{n \leqslant x} r^2(n)$ are too similar to the analogous formulæ involving $d^2(n)$, and proved in too similar a way to be worthy of special discussion. We prove, in addition, without difficulty, the further equality, analogous to (2 . 10):

$$(7\,.\,20) \qquad \sum_{n=1}^{\infty} n^{-s} r^k(n) = 4^k (1-2^{-s})^{2^{(k-1)}-1}\,\{\zeta(s)\,\eta(s)\}^{2^{(k-1)}}\,\phi(s),$$

where
$$\eta(s) = 1^{-s} - 3^{-s} + 5^{-s} - 7^{-s} + \ldots,$$

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* The cases of negative values of a or b are easily reduced to be above by using the equation
$$\sigma_{-r}(n) = n^{-r}\sigma_r(n).$$
Thus, if we write
$$S_y = \sum_{n \leqslant y} \sigma_a(n)\,\sigma_b(n),$$
we obtain, by Abel's formula of partial summation,
$$\sum_{n \leqslant x} \sigma_{-a}(n)\,\sigma_b(n) = [x]^{-r} S_{[x]} - \sum_{n \leqslant x-1} S_n\,\{n^{-r} - (n-1)^{-r}\}.$$

and $\phi(s)$ is absolutely convergent for $\sigma > \frac{1}{2}$; k denotes a positive integer. Proof of the equation

$$\{\zeta(\tfrac{1}{2}+it)\,\eta(\tfrac{1}{2}+it)\}^2 = O_1(t^{\epsilon})$$

would, of course, enable us to replace

(R. 22) $$\sum_{n \leqslant x} r^2(n) = \frac{x}{4}(\log x + C) + O(x^{\frac{3}{5}+\epsilon})$$

by the better approximation with error-term $O(x^{\frac{1}{2}+\epsilon})$, with corresponding improvement in the asymptotic formula arising from (7 . 20) for values of k greater than 2.

ON LINES OF ELECTRIC INDUCTION AND THE CONFORMAL TRANSFORMATIONS OF A SPACE OF FOUR DIMENSIONS

By H. Bateman.

[Received November 1920.—Read November 11th, 1920.—
Received, in revised form, June 6th, 1921.]

1. It is generally agreed that the equations of an electromagnetic field may be expressed in the form

$$\left.\begin{aligned} \operatorname{curl} \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} &= \mathbf{s}/c, \quad & \operatorname{div} \mathbf{D} &= \rho \\ \operatorname{curl} \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= 0, \quad & \operatorname{div} \mathbf{B} &= 0 \end{aligned}\right\}, \tag{1}$$

where E is the electric field strength or intensity, H the magnetic field strength, B the magnetic induction, and D the electric displacement or induction. Since the time of Minkowski* there has been some difference of opinion as to the structure of s and the precise definitions of H and D. It is clear, in fact, that if a part of s is of an appropriate form it may be brought over to the left-hand side of the first equation and included in the expression on the left by altering the definitions of H and D. A part of ρ which is of an appropriate form may be treated in a similar way.

On account of this lack of definiteness we shall call s *the proper electric current density*, ρ *the proper density of electricity* for a scheme in which the quantities D, H, B, and E are supposed to be already defined. We shall make no attempt to give names to the different parts of s, for we shall be interested here in the case when $\mathbf{s} = 0$, $\rho = 0$. If, moreover, this condition can be secured by the artifice just mentioned and a change in the definition of the quantities D and H, we shall favour this change.

A solution of the above equations may be built up from elementary solutions in many ways, but we are interested here only in the case when the elementary solutions* are either of type

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* *Gött. Nach.* (1908), *Math. Ann.*, Vol. 68 (1910).

$$\mathrm{D} = \gamma(\nabla\alpha \times \nabla\beta), \qquad c\mathrm{H} = \gamma \left\{ \frac{\partial\alpha}{\partial t}\nabla\beta - \frac{\partial\beta}{\partial t}\nabla\alpha \right\}, \tag{2}$$

where α, β, γ are appropriate functions of x, y, z, t, or of type

$$\mathrm{B} = \nabla\sigma \times \nabla\tau, \qquad c\mathrm{E} = \frac{\partial\tau}{\partial t}\nabla\sigma - \frac{\partial\sigma}{\partial t}\nabla\tau. \tag{3}$$

In these expressions we use the vector notation of Gibbs; $\nabla\alpha$, as usual, denotes the vector with components $\frac{\partial\alpha}{\partial x}$, $\frac{\partial\alpha}{\partial y}$, $\frac{\partial\alpha}{\partial z}$. The quantity c represents the velocity of light in a vacuum, and is supposed to be constant.

When the solution is of type (2) the moving curves represented by the equations $\alpha =$ constant, $\beta =$ constant may be called *lines of electric induction*, and the analysis may be connected in a natural manner with Sir J. J. Thomson's beautiful researches on moving lines of electric induction.† When the solution is of type (3) the moving curves represented by the equations $\sigma =$ constant, $\tau =$ constant may be called *lines of magnetic induction*.

In some cases when the constitutive relations between the vectors B, D, H and E are of a particularly simple character, *e.g.* $\mathrm{B} = \mathrm{H}$, $\mathrm{D} = \mathrm{E}$, an elementary solution may be of both types, and we may speak of lines of electric force as synonymous with lines of electric induction or lines of magnetic force as synonymous with lines of magnetic induction.

In the simple case just mentioned the functions α, β, γ, σ, τ, are connected by a set of differential equations of type

$$\frac{\partial(\sigma, \tau)}{\partial(y, z)} = -\frac{\gamma}{c}\,\frac{\partial(\alpha, \beta)}{\partial(x, t)}, \tag{4}$$

which imply that there is a differential relation of form‡

$$dx^2 + dy^2 + dz^2 - c^2dt^2 = A\,d\alpha^2 + 2C\,d\alpha\,d\beta + B\,d\beta^2 + S\,d\sigma^2 + 2R\,d\sigma\,d\tau + T\,d\tau^2, \tag{5a}$$

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* These are related to the elementary functions defined by Volterra, "The Generalisation of Analytic Functions," *The Rice Institute Pamphlet*, Vol. 4 (Jan. 1917); *Rend. Lincei* (4), Vol. 5 (1889), pp. 599, 630.

† *Phil. Mag.*, (5), Vol. 31 (1891), p. 149; *Recent Researches* (1893), Ch. 1. See also H. Bateman, *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1919), p. 95.

‡ H. Bateman, *Messenger of Math.*, Vol. 46 (Jan. 1917).

where A, B, C, S, T, R are functions of α, β, σ and τ connected by the relation

$$AB-C^2+\gamma^2(ST-R^2)=0. \tag{5b}$$

The case in which γ is constant is of special interest, because then $\rho=0$, $\mathrm{s}=0$, and we have an elementary solution of the simple Maxwellian equations, which, prior to the advent of Einstein's ideas regarding gravitation, were regarded almost universally as the field equations for the æther.

A simple example in which the functions α, β, σ and τ can be found (except for the solution of a Riccatian equation) is furnished by the field of an electric pole moving with a velocity less than c in an otherwise arbitrary manner. The case in which the motion is rectilinear is particularly simple, because then the Riccatian equation can be solved in terms of known functions and the solution expressed as follows.

Let $x=\xi(t)$, $y=0$, $z=0$ be the coordinates of the moving pole at time t, then τ, σ, α and β are defined by the equations

$$\left.\begin{aligned}
&[x-\xi(\tau)]^2+y^2+z^2=c^2(t-\tau)^2, && \beta=\tan^{-1}\frac{y}{z} \\
&\sigma=\frac{k}{c}\,\frac{[x-\xi(\tau)]\,\xi''(\tau)+c^2-v^2}{[x-\xi(\tau)]\,v-c^2(t-\tau)}, && \alpha=k\,\frac{x-\xi(\tau)-(t-\tau)v}{[x-\xi(\tau)]\,v-c^2(t-\tau)}
\end{aligned}\right\}, \tag{6}$$

where $v=\xi'(\tau)$ and $k=ec/4\pi$ is a constant proportional to the electric charge $-e$ associated with the moving pole. As usual $v<c$ and the quantity τ is subject to the inequality $\tau\leqslant t$. We easily find that

$$c^2dt^2-dx^2-dy^2-dz^2$$

$$=\frac{(c^2-v^2)^3}{[c(c^2-v^2)\sigma-c^2\alpha\xi''+kv\xi'']^2}\left[\frac{c^2k^2d\alpha^2}{c^2\alpha^2-k^2}+(c^2\alpha^2-k^2)\,d\beta^2+2kcd\sigma\, d\tau+Kd\tau^2\right],$$

where
$$K=c^2-v^2+\frac{2c^2\alpha\xi''(c^2-v^2)}{c(c^2-v^2)\sigma-c^2\alpha\xi''+v\xi''k}$$

$$+\frac{6v\xi''(c^2-v^2)\,k^3}{(c^2\alpha^2-k^2)\,[c(c^2-v^2)\,\sigma-c^2\alpha\xi''+v\xi''k]}$$

$$+\frac{c^2\xi''^2\,(c^2-v^2)(c^2\alpha^2-k^2)}{[c\,(c^2-v^2)\,\sigma-c^2\alpha\xi''+v\xi'']^2}$$

$$+\frac{2(c^2-v^2)^2\,k\,[(v\xi'''+\xi''^2)\,k-c^2\alpha\xi''-2v\sigma\xi'']}{[c(c^2-v^2)\,\sigma-c^2\alpha\xi''+v\xi''k]^2},$$

and a relation of type (5b) is satisfied with $\gamma=1$.

2. The transformations of coordinates (x, y, z, t) which can be used to pass from one electromagnetic field to another in a formal manner are of interest from both the philosophical and practical point of view. From a practical standpoint the aim is to obtain transformations which can be of service in solving the boundary problems of mathematical physics. Inversion and conformal transformations in a plane have already been used in this manner with great success, but so far very little use has been made of transformations in four variables, the principal success having been obtained with the Lorentzian transformation.

A transformation of coordinates from (x, y, z, t) to (x', y', z', t') can be applied to an elementary field specified by parameters α, β, σ, τ, when we can obtain two representations

$$\left.\begin{aligned} & dx^2+dy^2+dz^2-c^2dt^2 \\ & \qquad = A\,d\alpha^2+2C\,d\alpha\,d\beta+B\,d\beta^2+S\,d\sigma^2+2R\,d\sigma\,d\tau+T\,d\tau^2 \\ & dx'^2+dy'^2+dz'^2-c^2dt'^2 \\ & \qquad = A'd\alpha^2+2C'd\alpha\,d\beta+B'd\beta^2+S'd\sigma^2+2R'd\sigma\,d\tau+T'd\tau^2 \end{aligned}\right\}, \tag{7}$$

where $$AB-C^2+ST-R^2=0, \qquad A'B'-C'^2+S'T'-R'^2=0, \tag{8}$$

and A, B, C, S, T, R, A', B', C', S', T', R' are functions of α, β, σ, τ.

This representation may be possible in a number of ways, and so there may be a class of electromagnetic fields to which a given transformation is applicable. When this class is very extensive the transformation is of particular interest, and it is, in fact, possible to classify electromagnetic fields according to the transformations which they admit. Just how far such a classification can be useful in the solution of boundary problems is not known. Judging from particular examples, the tendency is for a transformation to lead to a classification of fields according to the nature of their singularities. Thus the fields with a single moving pole and the simple radiant fields with singular rays issuing from this pole with velocity c belong to one class.

The transformations for which

$$dx'^2+dy'^2+dz'^2-c^2dt'^2 = \lambda^2[dx^2+dy^2+dz^2-c^2dt^2] \tag{9}$$

can be applied to any type of electromagnetic field. These transformations may be regarded as conformal transformations of a space of four dimensions in which x, y, z and ict are rectangular coordinates. For brevity we shall call this space S_4.

In endeavouring to find these and other transformations we may either use the analysis of quadratic differential forms or we may work directly with the six differential equations of type (4) in which γ is a constant. In this paper we shall use chiefly the latter method, as it has the advantage of directness and it leads to the development of an interesting type of analysis.

3. Let us first of all consider a simple case of equations (4) in which $\alpha = \sigma$, $\beta = \tau$. The pair of equations

$$\frac{\partial(\sigma, \tau)}{\partial(y, z)} = -\frac{\gamma}{c}\,\frac{\partial(\alpha, \beta)}{\partial(x, t)}, \quad \frac{1}{c}\,\frac{\partial(\sigma, \tau)}{\partial(x, t)} = \gamma\,\frac{\partial(\alpha, \beta)}{\partial(y, z)}, \tag{10}$$

gives $\gamma^2 = -1$, and the six equations reduce to three equations of type

$$\frac{\partial(\alpha, \beta)}{\partial(y, z)} = \frac{i}{c}\,\frac{\partial(\alpha, \beta)}{\partial(x, t)}, \tag{11}$$

which have already been solved in a simple manner by taking x, y, α and β as new independent variables.* The result may be written in the form

$$\left.\begin{aligned} \alpha' &= \phi(\alpha, \beta)+\beta'\theta(\alpha, \beta) \\ a'\theta(\alpha, \beta) &= \psi(\alpha, \beta)-b' \end{aligned}\right\}, \tag{12}$$

where $\alpha' = z-ct$, $\beta' = x+iy$, $a' = z+ct$, $b' = x-iy$. This may be regarded as the fundamental theorem of the present type of analysis. It may be noticed incidentally that if α is a real and β a complex quantity, we obtain an interesting type of solution of equations (4) in which $\tau = \alpha$. The associated electromagnetic fields have been called *simple radiant fields*.

Let us now apply a transformation of variables from (x, y, z, t) to (x', y', z', t') to the general equations (4). Expanding the Jacobians by means of formulæ of type

$$\begin{aligned} \frac{\partial(\sigma, \tau)}{\partial(y, z)} = \frac{\partial(\sigma, \tau)}{\partial(y', z')}\,\frac{\partial(y', z')}{\partial(y, z)} &+ \frac{\partial(\sigma, \tau)}{\partial(z', x')}\,\frac{\partial(z', x')}{\partial(y, z)} + \frac{\partial(\sigma, \tau)}{\partial(x', y')}\,\frac{\partial(x', y')}{\partial(y, z)} \\ &+ \frac{\partial(\sigma, \tau)}{\partial(x', t')}\,\frac{\partial(x', t')}{\partial(y, z)} + \frac{\partial(\sigma, \tau)}{\partial(y', t')}\,\frac{\partial(y', t')}{\partial(y, z)} \\ &+ \frac{\partial(\sigma, \tau)}{\partial(z', t')}\,\frac{\partial(z', t')}{\partial(y, z)}, \end{aligned} \tag{13}$$

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* H. Bateman, *Bull. Amer. Math. Soc.*, March 1915; *Amer. Jour. Math.*, April 1915.

we obtain from equations (10) a pair of equations of type

$$B'_x\frac{\partial(y',z')}{\partial(y,z)}+B'_y\frac{\partial(z',x')}{\partial(y,z)}+B'_z\frac{\partial(x',y')}{\partial(y,z)}+cE'_x\frac{\partial(x',t')}{\partial(y,z)}+cE'_y\frac{\partial(y',t')}{\partial(y,z)}+cE'_z\frac{\partial(z',t')}{\partial(y,z)}$$

$$=-\frac{1}{c}\left[D'_x\frac{\partial(y',z')}{\partial(x,t)}+D'_y\frac{\partial(z',x')}{\partial(x,t)}+D'_z\frac{\partial(x',y')}{\partial(x,t)}-cH'_x\frac{\partial(x',t')}{\partial(x,t)}\right.$$
$$\left.-cH'_y\frac{\partial(y',t')}{\partial(x,t)}-cH'_z\frac{\partial(z',t')}{\partial(x,t)}\right]$$

$$\frac{1}{c}\left[B'_x\frac{\partial(y',z')}{\partial(x,t)}+B'_y\frac{\partial(z',x')}{\partial(x,t)}+B'_z\frac{\partial(x',y')}{\partial(x,t)}+cE'_x\frac{\partial(x',t')}{\partial(x,t)}\right.$$
$$\left.+cE'_y\frac{\partial(y',t')}{\partial(x,t)}+cE'_z\frac{\partial(x',t')}{\partial(x,t)}\right]$$

$$=D'_x\frac{\partial(y',z')}{\partial(y,z)}+D'_y\frac{\partial(z',x')}{\partial(y,z)}+D'_z\frac{\partial(x',y')}{\partial(y,z)}-cH'_x\frac{\partial(x',t')}{\partial(y,z)}$$
$$-cH'_y\frac{\partial(y',t')}{\partial(y,z)}-cH'_z\frac{\partial(z',t')}{\partial(y,z)}, \tag{14}$$

where

$$\left.\begin{aligned}B'_x&=\frac{\partial(\sigma,\tau)}{\partial(y',z')}, & cE'_x&=\frac{\partial(\sigma,\tau)}{\partial(x',t')}\\ D'_x&=\gamma\frac{\partial(\alpha,\beta)}{\partial(y',z')}, & cH'_x&=-\gamma\frac{\partial(\alpha,\beta)}{\partial(x',t')}\end{aligned}\right\}. \tag{15}$$

If the transformation is applicable to any electromagmetic field in a specified medium, the above equations should be satisfied identically in virtue of the structural relations connecting the vectors D', E', B', H'. In the case when these relations are $D'=\pm E'$, $B'=\pm H'$, the above equations are satisfied identically when the system of eighteen differential equations

$$\left.\begin{array}{ll}\dfrac{\partial(y',z')}{\partial(y,z)}=\pm\dfrac{\partial(x',t')}{\partial(x,t)}, & c^2\dfrac{\partial(x',t')}{\partial(y,z)}=\mp\dfrac{\partial(y',z')}{\partial(x,t)}\\ \cdots\quad\cdots\quad\cdots & \cdots\quad\cdots\quad\cdots\\ \dfrac{\partial(x',y')}{\partial(y,z)}=\pm\dfrac{\partial(z',t')}{\partial(x,t)}, & c^2\dfrac{\partial(z',t')}{\partial(y,z)}=\mp\dfrac{\partial(x',y')}{\partial(x,t)}\\ \cdots\quad\cdots\quad\cdots & \cdots\quad\cdots\quad\cdots\end{array}\right\} \tag{16}$$

are satisfied, either the upper or the lower sign being taken throughout.

These equations have already been solved* by a method which leads directly to a relation of type (9), and the general equations of a conformal transformation may then be deduced in the usual manner. There is, however, another method of solution which leads directly to a very convenient set of formulæ for a conformal transformation, and which belongs to the type of analysis at present under consideration.

Taking the upper sign in each of the equations (16) and writing

$$\alpha = z' + ct', \qquad \beta = x' + iy', \tag{17}$$

we obtain three equations of type (11) and these lead to equations of type (12). We next notice that equations (16) may be written in the alternative form

$$\frac{\partial(x, t)}{\partial(x', t')} = \pm \frac{\partial(y, z)}{\partial(y', z')}, \qquad c^2 \frac{\partial(x, t)}{\partial(y', z')} = \mp \frac{\partial(y, z)}{\partial(x', t')}. \tag{18}$$
$$\cdots \quad \cdots \quad \cdots \qquad \cdots \quad \cdots \quad \cdots$$

This may be seen most easily by writing

$$\frac{\partial(y', z')}{\partial(y, z)} = \frac{\partial(y', z', x, t)}{\partial(y, z, x, t)}, \quad \text{\&c.},$$

and multiplying all the equations by the Jacobian $\dfrac{\partial(x, y, z, t)}{\partial(x', y', z', t')}$.

From equations (18) we may deduce that

$$\left.\begin{aligned} \alpha &= \Phi(\alpha', \beta') + \beta\Theta(\alpha', \beta') \\ a\Theta(\alpha', \beta') &= \Psi(\alpha', \beta') - b \end{aligned}\right\}, \tag{19}$$

where $a = z' - ct'$, $b = x' - iy'$, and Θ, Φ, Ψ are arbitrary functions of α' and β'.

Since in general we can expect there to be only one relation between the four quantities α, β, α' and β', we must conclude that the two equations

$$\left.\begin{aligned} \alpha' &= \phi(\alpha, \beta) + \beta'\theta(\alpha, \beta) \\ \alpha &= \Phi(\alpha', \beta') + \beta\Theta(\alpha', \beta') \end{aligned}\right\} \tag{20}$$

are the same. To determine the form of the single equation to which both are equivalent, we regard θ and ϕ as auxiliary variables and write

$$\alpha = F(\theta, \phi), \qquad \beta = G(\theta, \phi),$$

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* H. Bateman, *Proc. London Math. Soc.*, Ser. 2, Vol. 8 (1910), p. 223.

then since $\alpha' = \phi + \beta'\theta$, we must have the identity

$$F(\theta,\ \alpha' - \beta'\theta) \equiv \Phi(\alpha',\ \beta') + G(\theta,\ \alpha' - \beta'\theta)\,\Theta(\alpha',\ \beta').$$

Putting $\theta = 0$ and eliminating Φ, we obtain

$$\Theta(\alpha',\ \beta') = \frac{F(\theta,\ \alpha' - \beta'\theta) - F(0,\ \alpha')}{G(\theta,\ \alpha' - \beta'\theta) - G(0,\ \alpha')}.$$

Since Θ is independent of θ, we may make $\theta \to 0$ in the above expression and obtain expressions of type

$$\Theta(\alpha',\ \beta') = \frac{F_1(\alpha') - \beta' F_2(\alpha')}{G_1(\alpha') - \beta' G_2(\alpha')},$$

$$\Phi(\alpha',\ \beta') = F(0,\ \alpha') - G(0,\ \alpha')\,\frac{F_1(\alpha') - \beta' F_2(\alpha')}{G_1(\alpha') - \beta' G_2(\alpha')},$$

provided that the partial derivatives of the functions F and G exist and are continuous at $\theta = 0$, $\phi = \alpha'$.

The relation between α, β, α', β' is now seen to be of the form

$$[\alpha - F(0,\ \alpha')][G_1(\alpha') - \beta' G_2(\alpha')] = [\beta - G(0,\ \alpha')][F_1(\alpha') - \beta' F_2(\alpha')],$$

and is of the first degree in each of the variables α, β, β'. In a similar way it may be shown to be of the first degree in each of the variables α', β, β'; hence we conclude that it must be of the first degree in each of the four variables α, α', β, β', and must accordingly be of the form

$$\alpha'(l\alpha + m\beta + n) = p\alpha + q\beta + r + \beta'(u\alpha + v\beta + w), \tag{21}$$

where l, m, n, p, q, r, u, v, w are constants.

If we interpret α, β, α', β' as the coordinates of a point in a space of four dimensions, the theorem that has just been proved tells us that if a variety possess ∞^2 generators in planes $\alpha =$ const., $\beta =$ const., and also ∞^2 generators in planes $\alpha' =$ const., $\beta' =$ const., it is a quadric.

The relation (7) may also be written in the form

$$\alpha(l\alpha' - u\beta' - p) = -n\alpha' + w\beta' + r + \beta(-m\alpha' + v\beta' + q),$$

and the additional equations in relations (12) and (19) may now be written down as

$$\alpha'(u\alpha + v\beta + w) = e\alpha + f\beta + g - b'(l\alpha + m\beta + n),$$

and

$$\alpha(-m\alpha' + v\beta' + q) = h\alpha' + j\beta' + k - b(l\alpha' - u\beta' - p),$$

where e, f, g, h, j, k are new constants. Writing these equations in the forms

$$\alpha(ua'+lb'-e) = -wa'-nb'+g-\beta(a'v+mb'-f),$$

$$\alpha'(-ma+lb-h) = -qa+pb+k-\beta'(av-ub-j),$$

we may associate with them a relation of type

$$a(a'v+mb'-f) = ja'-hb'+s+b(ua'+lb'-e),$$

where s is a new constant, this equation combined with either of the preceding equations forming a pair of equations of type (12) or (19). We now have four independent relations

$$\left.\begin{aligned} \alpha(l\alpha'-u\beta'-p) &= -n\alpha'+w\beta'+r+\beta(-m\alpha'+v\beta'+q) \\ \alpha(ua'+lb'-e) &= -wa'-nb'+g-\beta(va'+mb'-f) \\ a(-m\alpha'+v\beta'+q) &= h\alpha'+j\beta'+k-b(l\alpha'-u\beta'-p) \\ a(va'+mb'-f) &= ja'-hb'+s+b(ua'+lb'-e) \end{aligned}\right\}, \qquad (22)$$

which enable us to express x', y', z', t' in terms of x, y, z, t or *vice versa*. These equations are simpler than the usual equations* for a conformal transformation in four variables, the latter being derivable from the above equations, by solving with respect to x', y', z', t'.

The derivation of equations (22) from the usual equations is not so simple. It has, no doubt, been known for some time that the equations of a conformal transformation can be expressed in this form, for the general theory of a transformation expressed by means of bilinear equations is well known, at any rate for the case of three variables.

The above equations indicate clearly that the transformation depends on fifteen arbitrary constants and is birational. The transformation represented by these equations has a positive Jacobian. To obtain a transformation with a negative Jacobian, we must take the lower sign in all of the equations (16). The solution for this case may be obtained by simply writing $-t'$ in place of t' in formulæ (22), *i.e.* by interchanging a and α.

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* S. Lie, *Gött. Nachr.* (1871); *Math. Ann.*, Vol. 5 (1872), p. 186; G. Darboux, *Leçons sur les systèmes orthogonaux*, Vol. 1 (1898); P. F. Smith, *Trans. Amer. Math. Soc.*, Vol. 1 (1900), p. 375.

It should be noticed incidentally that we may obtain a solution of equations (10) by writing $x' = \sigma$, $t' = \tau$, $y' = \alpha$, $z' = \beta$, $c\gamma = 1$, where x', y', z', ict' are connected with x, y, z, itc by a conformal transformation.

It is important to notice that the conformal transformation transforms an elementary field into an elementary field, $d\alpha\delta\beta - d\beta\delta\alpha$ and $d\sigma\delta\tau - d\tau\delta\sigma$ being invariants. The relations between the two sets of field vectors are given by equations (13) in conjunction with (2), (3) and (15). These relations are identical with those used by Volterra in his work on the generalization of analytic functions.*

4. We shall now study the effect of the general conformal transformation on the simple type of radiant field specified by expressions of type

$$Q_x \equiv H_x + iE_x = f(\alpha_0, \beta_0)\frac{\partial(\alpha_0, \beta_0)}{\partial(y, z)} = \frac{i}{c}f(\alpha_0, \beta_0)\frac{\partial(\alpha_0, \beta_0)}{\partial(x, t)}, \qquad (23)$$

where α_0 and β_0 are solutions of the equations

$$\left.\begin{aligned} z - ct &= \phi(\alpha_0, \beta_0) + (x + iy)\,\theta(\alpha_0, \beta_0) \\ (z + ct)\,\theta(\alpha_0, \beta_0) &= \psi(\alpha_0, \beta_0) - (x - iy) \end{aligned}\right\}. \qquad (24)$$

The formulæ of transformation for the components of the field vectors indicate that the field is transformed into another field of the same type, and that the quantities α_0 and β_0 can be regarded as invariants. These quantities will, then, be solutions of equations of type

$$\left.\begin{aligned} z' - ct' &= \phi'(\alpha_0, \beta_0) + (x' + iy')\,\theta'(\alpha_0, \beta_0) \\ (z' + ct')\,\theta'(\alpha_0, \beta_0) &= \psi'(\alpha_0, \beta_0) - (x' - iy') \end{aligned}\right\}, \qquad (25)$$

and our object now is to express the functions θ', ϕ', ψ' in terms of θ, ϕ, ψ.

Writing the last pair of equations in the form

$$a = \phi' + \theta'\beta, \qquad \theta' a = \psi' - b,$$

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* V. Volterra, *The Rice Institute Pamphlet*, Vol. 4 (Jan. 1917), No. 1.

and substituting in (22) we obtain the relations

$$\begin{aligned}
&\phi'(-ma'+v\beta'+q)+\theta'\beta(-ma'+v\beta'+q)\\
&\qquad\qquad = ha'+j\beta'+k-b(la'-u\beta'-p),\\
&\theta'(-na'+w\beta'+r)+\theta'\beta(-ma'+v\beta'+q)\\
&\qquad\qquad = \psi'(la'-u\beta'-p)-b(la'-u\beta'-p),\\
&\phi'(va'+mb'-f)+\theta'\beta(va'+mb'-f)\\
&\qquad\qquad = ja'-hb'+s+b(ua'+lb'-e),\\
&-\theta'(wa'+nb'-g)-\theta'\beta(va'+mb'-f)\\
&\qquad\qquad = \psi'(ua'+lb'-e)-b(ua'+lb'-e).
\end{aligned}$$

Eliminating b and β, we obtain two relations which may be written in the form

$$a' = \phi+\theta\beta', \qquad \theta a' = \psi-b',$$

where

$$\left.\begin{aligned}
\chi\theta &= w\theta'-v\phi'+u\psi'+j\\
\chi\phi &= r\theta'-q\phi'+p\psi'+k\\
\chi\psi &= g\theta'-f\phi'+e\psi'-s\\
\chi &= n\theta'-m\phi'+l\psi'-h
\end{aligned}\right\}. \tag{26}$$

The quantities (θ', ϕ', ψ') are thus connected with (θ, ϕ, ψ) by the equations of a projective transformation in a space of three dimensions, and the above analysis gives a precise formal proof of the well known geometrical theorem that the conformal group in a space of four dimensions is isomorphic with the projective group in a space of three dimensions.*

In the particular case when $l = 0$, $m = 0$, $u = 0$, $v = 0$, the conformal transformation becomes a linear transformation and θ' depends only on θ; we have, in fact,

$$\theta = \frac{w\theta'+j}{n\theta'-h}.$$

The fact that the transformation of an elementary field of type (23) is associated with one of the most important properties of a conformal transformation may be regarded as significant. It leads us to again emphasize the importance of the simple radiant fields of type (23). It is known that all the fields that have so far been found to be of importance in mathematical physics can be built up from fields of type (23) by superposition. Even though a field of type (23) generally possesses singular rays, it is possible to obtain a cancellation of singularities when certain fields are superposed.* In fact if we use the symbol $Q^{(r)}$ to denote the vector $H^{(r)}+iE^{(r)}$ derived from the function $f^{(r)}(\alpha_0, \beta_0)$, we can in some cases cancel the singularities in a field of type (23) by means of singularities in a field specified by an equation of type

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* F. Klein, *Höhere Geometrie*, Vol. 1, p. 487; S. Lie, *Theorie der Transformationsgruppen*, Vol. 3, Leipzig (1893), pp. 281, 357; J. E. Wright, *Trans. Amer. Math. Soc.* (1907), p. 391.

$$Q=\frac{\partial Q^{(1)}}{\partial x}+\frac{\partial Q^{(2)}}{\partial y}+\frac{\partial Q^{(3)}}{\partial z}+\frac{\partial Q^{(4)}}{\partial t}.$$

The field of a moving electric pole can be derived from simple radiant fields in this manner in a variety of ways. This is an important result, since it leaves room for the introduction of the methods of statistical mechanics into electromagnetic theory, a thing that seems to be needed on account of the growing importance of the quantum theory of radiation. The result is also interesting from a philosophical point of view, because it suggests that in electromagnetism we are concerned not so much with an æther (in the ordinary sense) as the fundamental substance as with *radiation* of a semi-material nature, matter itself being associated with regions of space where this radiation is transformed. The more familiar types of electromagnetic fields, such as electrostatic fields, are produced by the *interference* of simple radiant fields, and this results in a localisation of energy analogous, except in distribution, to that which occurs when waves of light interfere.

5. It is known that constant values of θ, ϕ and ψ specify a point which travels along a straight line with the velocity of light c. If we make θ, ϕ and ψ functions of a single parameter τ, we obtain a series of "light-particles" which generally lie on a moving curve. In some cases this curve may be a line of electric induction in an electromagnetic field, and it is then important to know the way in which θ, ϕ and ψ vary with τ.

Let us consider the case of the field of a moving electric pole whose coordinates at time τ are $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$. For each line of electric induction one light-particle starts from the pole at each instant, and its parameters (θ, ϕ, ψ) satisfy the Riccatian equations

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1919), p. 95.

$$2(c^2-v^2)\frac{d\theta}{d\tau} = \Phi'\Psi''-\Phi''\Psi',$$

$$2(c^2-v^2)\left(\frac{d\phi}{d\tau}-\Phi'\right) = (\xi+i\eta)(\Phi''\Psi'-\Phi'\Psi''),$$

$$2(c^2-v^2)\left(\frac{d\psi}{d\tau}-\Psi'\right) = (\zeta+c\tau)(\Phi'\Psi''-\Phi''\Psi'),$$

where $$v^2 = \left(\frac{d\xi}{d\tau}\right)^2+\left(\frac{d\eta}{d\tau}\right)^2+\left(\frac{d\zeta}{d\tau}\right)^2,$$

$$\Phi' = \frac{d\zeta}{d\tau}-c-\theta\left(\frac{d\xi}{d\tau}+i\frac{d\eta}{d\tau}\right), \qquad \Phi'' = \frac{d^2\zeta}{d\tau^2}-\theta\left(\frac{d^2\xi}{d\tau^2}+i\frac{d^2\eta}{d\tau^2}\right),$$

$$\Psi' = \theta\left(\frac{d\zeta}{d\tau}+c\right)+\frac{d\xi}{d\tau}-i\frac{d\eta}{d\tau}, \qquad \Psi'' = \theta\frac{d^2\zeta}{d\tau^2}+\frac{d^2\xi}{d\tau^2}-i\frac{d^2\eta}{d\tau^2},$$

$$\phi = \zeta-c\tau-\theta(\xi+i\eta), \qquad \psi = \theta(\zeta+c\tau)+\xi-i\eta.$$

In the second equation Φ', Ψ', Φ'' and Ψ'' may be expressed as linear functions of ϕ, while in the third equation they may be expressed as linear functions of ψ.

By means of the transformation (26) this set of Riccatian equations is transformed into another set of Riccatian equations of the same type. Since the general Riccatian equation is equivalent to the general linear differential equation of the second order, this result is of some interest in connection with the theory of linear differential equations, especially as our theorem enables us to associate various linear differential equations with a moving pole.

From a physical point of view the chief value of this result is that the conformal transformations, and in particular the transformations of the theory of relativity, can be used to deduce the form of the lines of electric induction for various types of motion of an electric pole when the lines of induction are known for a particular type of motion. Thus from the known equations of the lines of induction for uniform motion in a circle,* we can deduce the equations of the lines of induction for uniform motion in a helix. From the known lines of force of a stationary electric pole we can, by applying the general conformal transformation, obtain the lines of force for an electric pole in "hyperbolic motion," *i.e.* in a type of motion represented in S_4 by a circle.[*] Since the lines of force of the stationary pole are represented in S_4 by planes through a line, the lines of force of the moving pole will be represented in S_4 by spheres through the circle.

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* F. D. Murnaghan, *Amer. Jour. of Math.* (April 1917).

6. The Riccatian equations admit of certain other transformations of considerable interest which arise from a succession of local conformal transformations in S_4, each transformation being applied only to a variety[†] and not to the whole of the space, the variety being different for different transformations of the series.

A transformation of coordinates

$$\left.\begin{aligned} x_0 &= \int_0^\tau [x-\xi(s)]\, f'(s)\, ds, & y_0 &= \int_0^\tau [y-\eta(s)]\, f'(s)\, ds \\ z_0 &= \int_0^\tau [z-\zeta(s)]\, f'(s)\, ds, & t_0 &= \int_0^\tau [t-s]\, f'(s)\, ds \end{aligned}\right\}, \qquad (27)$$

$$[x-\xi(\tau)]^2+[y-\eta(\tau)]^2+[z-\zeta(\tau)]^2 = c^2(t-\tau)^2 \quad (\tau \leqslant t),$$

considered in a former paper,[‡] gives

$$\xi_0 = \int_0^\tau f(s)\, \xi'(s)\, ds, \qquad \tau_0 = \int_0^\tau f(s)\, ds,$$

$$\theta_0 = \theta, \quad \phi_0 = \zeta_0 - c\tau_0 - \theta(\xi_0 + i\eta_0), \quad \psi_0 = \theta(\zeta_0 + c\tau_0) + \xi_0 - i\eta_0,$$

where $f(0) = 0$. The velocities at corresponding points are the same, and the differential equation satisfied by θ_0 is the same as that satisfied by θ. The lines of induction for an electric pole moving along a plane curve with constant speed can now be derived from the known equations for the lines of induction of an electric pole moving uniformly in a circle. Other examples in which the transformation can be used with advantage are easily devised.

In this case the succession of conformal transformations is obtained by giving different constant values to τ, the varieties to which they apply being represented by $\tau =$ constant. The transformation (27) can also be applied to all the simple radiant fields having singular rays issuing from the moving electric pole, and so there is quite an extensive class of fields to which the transformation is applicable.

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[*] This includes uniformly accelerated motion. The lines of force for this type of motion of an electric pole have been found by Leigh Page, *Amer. Jour. Sci.* (1914).

[†] *I.e.* a 3-way.

[‡] *Proc. London Math. Soc.*, Ser. 2. Vol. 10 (1911), p. 96.

This example suggests that transformations applicable to a class of electromagnetic fields may be constructed most easily by forming a one-parameter group of conformal transformations in S_4, and applying each transformation of the group to a single variety. The determination of the relation which must exist between the group of transformations and the system of varieties is an interesting problem which has not yet been solved in a general manner. Several examples of classes of fields associated with particular transformations are already known.

There are some transformations applicable to our Riccatian equations which have not yet been associated with transformations of the coordinates x, y, z, t. For instance, if we make the transformation

$$\theta_0 = \frac{\Phi'}{\Psi'}, \quad \xi_0 = \xi, \quad \eta_0 = -\eta, \quad \zeta_0 = \zeta, \quad \tau_0 = \tau,$$

$$\phi_0 = \zeta_0 - c\tau_0 - \theta_0(\xi + i\eta_0), \quad \psi_0 = \theta_0(\zeta_0 + c\tau_0) + \xi_0 - i\eta_0,$$

the differential equations satisfied by θ_0, ϕ_0, ψ_0 are of exactly the same form in the new coordinates as those satisfied by θ, ϕ, ψ in the old. This transformation makes a moving electric pole correspond to its image in the mirror $y = 0$, but a line of electric induction corresponds to the image of the associated line of electric induction,* and not to its own image.

7. The problem of determining classes of electromagnetic fields is associated with the following problem relating to quadratic differential forms:—

> Given two quadratic differential forms of type (7) whose coefficients satisfy equations (8), the problem is to find a transformation of parameters from α, β, σ, τ to α_0, β_0, σ_0, τ_0 which will transform the two quadratic differential forms into two other forms of the same type.

In this general form the problem has a wider interest than would appear at first sight from the text, because it is not necessary to restrict the forms to be reducible to the forms on the left. We may, for instance, consider forms reducible to quadratic differential forms of the type occurring in Einstein's theory of gravitation and obtain a transformation theory for lines of induction in Einsteinian electromagnetic fields.

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* For the definition of associated lines of electric induction, see F. D. Murnaghan, *loc. cit.*, and H. Bateman, *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1919), p. 95.

EXPANSIONS AND FUNCTIONS REDUCED TO ZERO BY THE OPERATOR sinh $D-cD$

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[Received October 13th, 1921.—Read November 17th, 1921.]

1. Operators such as $e^D-e^{-D}-2cD$ play an important part in the theory of elastic plates* bounded by circular arcs or straight lines.

In §§ 2–4 of the present paper the expansion is discussed of an arbitrary function of x in the form $\Sigma A e^{\lambda x}$, where $e^{\lambda}-e^{-\lambda}-2c\lambda=0$. The main result is (3 . 5) or a modification of it (4 . 51) which is slightly wider, and even more curious in form.

In §§ 5, 6 the theory of functions that are reduced to zero by the operator is investigated by analogy with periodic functions.

In §§ 7, 8 an extension analogous to the theory of doubly periodic functions is given.

The special case $c=1$ is considered in § 9.

The points of convergency that arise are held over for discussion in § 10.

2. Let c be a constant other than 1 or 0, and let (R) denote the circumference of a circle about the origin of radius R, (∞) being written instead of (R) when R is increased indefinitely according to some law.

Then the following lemmas may be proved (see § 10 below):—

LEMMA 1.—If q is real and between ± 1, exclusive, then

$$\int_{(\infty)} \frac{\exp \lambda q \, d\lambda}{\sinh \lambda - c\lambda} \quad \text{and} \quad \int_{(\infty)} \frac{\exp \lambda q \, d\lambda}{\lambda(\sinh \lambda - c\lambda)}$$

are both zero, and are still zero when each element of the integrals is taken in absolute value. (2 . 11)

If $q=\pm 1$, the first integral is infinite and the second is $2\iota\pi q$. (2 . 12)

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 19, pp. 373-386. See also J. Dougall (*Trans. R. S. E.*, Vol. 41) on the theory of an infinite thick plate, and L. N. G. Filon (*Proc. London Math. Soc.*, Ser. 2, Vol. 4, p. 422).

Lemma 2.—Let $f(q)$ be a function which with its differential coefficients of all orders is limited and continuous throughout the range of integration, whatever it may be; then

$$\int_{(\infty)}\int_{-u}^{v}\frac{f(q)\exp\lambda q\,dq\,d\lambda}{\sinh\lambda-c\lambda}\quad\text{and}\quad\int_{(\infty)}\int_{-u}^{v}\frac{f(q)\exp\lambda q\,dq\,d\lambda}{\lambda(\sinh\lambda-c\lambda)}$$

are zero if u, v are between ± 1. The second of these is also zero if 1 is put for u or v, and in all the cases the value is still zero if the elements are taken in absolute value. (2.13)

If in the first integral of Lemma 2 we change the upper limit to 1, we add

$$\int_{(\infty)}\int_{v}^{1}\frac{f(q)\exp\lambda q\,dq\,d\lambda}{\sinh\lambda-c\lambda}$$

$$=\int_{(\infty)}\frac{f(1)\exp\lambda-f(v)\exp\lambda v}{\lambda(\sinh\lambda-c\lambda)}\,d\lambda-\int_{(\infty)}\int_{v}^{1}\frac{f'(q)\exp\lambda q\,dq\,d\lambda}{\lambda(\sinh\lambda-c\lambda)},$$

which is $2\iota\pi f(1)$, or more definitely $2\iota\pi f(1-0)$. (2.14)

Similarly $$\int_{(\infty)}\int_{-1}^{-u}\frac{f(q)\exp\lambda q\,dq\,d\lambda}{\sinh\lambda-c\lambda}=2\iota\pi f(-1+0). \qquad (2.15)$$

Again $$\int_{(\infty)}\int_{-u}^{v}\frac{f(q)\,\lambda\exp\lambda q\,dq\,d\lambda}{\sinh\lambda-c\lambda}$$

$$=\int_{(\infty)}\frac{f(v)\exp\lambda v-f(-u)\exp(-\lambda u)}{\sinh\lambda-c\lambda}\,d\lambda-\int_{(\infty)}\int_{-u}^{v}\frac{f'(q)\exp\lambda q\,dq\,d\lambda}{\sinh\lambda-c\lambda},$$

which is zero, u, v being still between ± 1. (2.21)

$$\int_{(\infty)}\int_{v}^{1}\frac{f(q)\,\lambda\exp\lambda q\,dq\,d\lambda}{\sinh\lambda-c\lambda}=-2\iota\pi f'(1-0),\ \text{if}\ f(1-0)=0, \qquad (2.22)$$

and

$$\int_{(\infty)}\int_{-1}^{-u}\frac{f(q)\,\lambda\exp\lambda q\,dq\,d\lambda}{\sinh\lambda-c\lambda}=-2\iota\pi f'(-1+0),\ \text{if}\ f(-1+0)=0. \qquad (2.23)$$

To extend the range of values for q, we have

$$\int_{(\infty)}\int_{1}^{1+v}\frac{f(q)\exp\lambda q\,dq\,d\lambda}{\sinh\lambda-c\lambda}$$

$$=\int_{(\infty)}\int_{1}^{1+v}f(q)\left\{\frac{\exp\lambda(q-2)+2c\lambda\exp\lambda(q-1)}{\sinh\lambda-c\lambda}+2\exp\lambda(q-1)\right\}dq\,d\lambda,$$

and, since $$\int_{(R)}\int_{1}^{1+v} f(q)\exp\lambda(q-1)\,dq\,d\lambda = 0$$

for all values of R, this gives

$$\int_{(\infty)}\int_{-1}^{v-1}\frac{f(q+2)\exp\lambda q}{\sinh\lambda-c\lambda}\,dq\,d\lambda+2c\int_{(\infty)}\int_{0}^{v}\frac{\lambda f(q+1)\exp\lambda q}{\sinh\lambda-c\lambda}\,dq\,d\lambda,$$

which is $2\iota\pi f(1+0)$, if $v<1$, (2.31)

but $2\iota\pi\{f(1+0)-2cf'(2-0)\}$, if $v=1$ and if $f(2-0)=0$. (2.32)

So also $$\int_{(\infty)}\int_{-1-v}^{-1}\frac{f(q)\exp\lambda q\,dq\,d\lambda}{\sinh\lambda-c\lambda} = 2\iota\pi f(-1-0),\text{ if } v<1; \qquad (2.33)$$

$$= 2\iota\pi\{f(-1-0)+2cf'(-2+0)\},\text{ if } v=1,\text{ and if } f(-2+0)=0. \qquad (2.34)$$

3. Now let a be any real constant, and in the above put $t-x\pm1$ for q, $f(q+x\mp1)$ for $f(q)$; suppose that the interval between $a\pm1$ can be divided into a finite number of sub-intervals in each of which $f(x)$ and its differential coefficients of all orders are limited and continuous; then, by combining the above results for the different sub-intervals, we have, if $a-1<x<a$,

from (2.14), $$\int_{(\infty)}\int_{a-1}^{x}\frac{f(t)\exp\lambda(t-x+1)}{\sinh\lambda-c\lambda}\,dt\,d\lambda = 2\iota\pi f(x-0), \qquad (3.11)$$

from (2.31), $$\int_{(\infty)}\int_{x}^{a}\frac{f(t)\exp\lambda(t-x+1)}{\sinh\lambda-c\lambda}\,dt\,d\lambda = 2\iota\pi f(x+0), \qquad (3.12)$$

from (2.13), $$\int_{(\infty)}\int_{a}^{a+1}\frac{f(t)\exp\lambda(t-x-1)}{\sinh\lambda-c\lambda}\,dt\,d\lambda = 0; \qquad (3.13)$$

but if $a<x<a+1$, then

from (2.13), $$\int_{(\infty)}\int_{a-1}^{a}\frac{f(t)\exp\lambda(t-x+1)}{\sinh\lambda-c\lambda}\,dt\,d\lambda = 0, \qquad (3.21)$$

from (2.33), $$\int_{(\infty)}\int_{a}^{x}\frac{f(t)\exp\lambda(t-x-1)}{\sinh\lambda-c\lambda}\,dt\,d\lambda = 2\iota\pi f(x-0), \qquad (3.22)$$

from (2.15), $$\int_{(\infty)}\int_{x}^{a+1}\frac{f(t)\exp\lambda(t-x-1)}{\sinh\lambda-c\lambda}\,dt\,d\lambda = 2\iota\pi f(x+0), \qquad (3.23)$$

and so, by addition, if $a-1 < x < a+1$,

$$\tfrac{1}{2}f(x+0)+\tfrac{1}{2}f(x-0)$$
$$= \frac{1}{4\iota\pi}\left[\int_{(\infty)}\int_{a-1}^{a}\frac{f(t)\exp\lambda(t-x+1)}{\sinh\lambda-c\lambda}\,dt\,d\lambda+\int_{(\infty)}\int_{a}^{a+1}\frac{f(t)\exp\lambda(t-x-1)}{\sinh\lambda-c\lambda}\,dt\,d\lambda\right]. \tag{3 . 3}$$

If $x = a$, this result follows from (2 . 14) and (2 . 15).

When $x = a-1$, the value of the right-hand expression in (3 . 3) is

$$\tfrac{1}{2}f(a-1+0)-cf'(a-0)+\tfrac{1}{2}f(a+1-0) \text{ if } f(a-0)=0 \text{ by } (2.32) \text{ and } (2.14), \tag{3 . 31}$$

when $x = a+1$, the value is, by (2 . 15) and (2 . 34),

$$\tfrac{1}{2}f(a-1+0)+\tfrac{1}{2}f(a+1-0)+cf'(a+0) \text{ if } f(a+0)=0. \tag{3 . 32}$$

These values agree with the form (3 . 3) if we suppose that

$$f(x+1)-f(x-1)-2cf'(x) = 0: \tag{3 . 4}$$

this relation gives $f(x)$ for all real values of x if it is known for values between $a\pm 1$.

In (3 . 3) the results of the integration with respect to t are analytical functions of λ, and therefore by applying Cauchy's theorem to the contour integrals, we have the desired *expansion of the arbitrary function* $f(x)$ *in the form* $\Sigma A e^{\lambda x}$ *where* $\sinh\lambda = c\lambda$, namely,

$$f(x) = \tfrac{1}{2}\Sigma e^{\lambda x}\left[\int_{a-1}^{a}\frac{f(t)\exp(-\lambda t-\lambda)}{\cosh\lambda-c}\,dt+\int_{a}^{a+1}\frac{f(t)\exp(-\lambda t+\lambda)}{\cosh\lambda-c}\,dt\right], \tag{3 . 5}$$

where it is assumed that $f(x\pm 0)$ have a common value $f(x)$, and the sign of λ has been changed. *The summation extends over all the roots of the equation* $\sinh\lambda = c\lambda$.

The expansion has been established under the following conditions :—

(1) x lies between $a\pm 1$, both limits being excluded.

(2) $f(x)$ is limited between $a\pm 1$, and the range between $a\pm 1$ can be divided into a finite number of intervals in each of which $f(x)$ has limited differential coefficients* of all orders.

(3) c is not 1 or 0 : the case $c = 1$ is discussed below (§ 9) ; when

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* The differential coefficients will be different on the two sides of any point which separates two intervals, but at other points they will be the same on both sides.

$c=0$ the expansion is that of Fourier, and the proof does not apply in all particulars.

(4) If $f(x+0)$ and $f(x-0)$ are different, the sum of the series is their arithmetic mean; it will be seen in § 10 that this conclusion depends on the form of the λ-path, and therefore holds when the terms in (3 . 5) are arranged so that the value of $|\lambda|$ increases or does not decrease from each term to the next, but does not necessarily hold for other arrangements of the terms.

With respect to values of x not between $a\pm 1$ it is to be noted that the several terms of the expansion satisfy the functional equation (3 . 4), and the sum of the series will therefore satisfy the same equation if term-by-term differentiation is legitimate.

For the values $a\pm 1$ themselves see (3 . 31) and (3 . 32).

The existence of limited differential coefficients of all orders has been assumed for simplicity, but clearly it goes far beyond what is necessary: in fact, in (2 . 13), (3 . 13), (3 . 21), it is sufficient that $f(q)$ be limited; in (2 . 14), (2 . 15), (2 . 21), (2 . 31), (2 . 33), (3 . 11), (3 . 12), (3 . 22), (3 . 23), (3 . 3), (3 . 5) it is sufficient that f have a limited and continuous differential coefficient throughout each sub-interval; in (2 . 22), (2 . 23), (2 . 32), (2 . 34), (3 . 31), (3 . 32) it is sufficient that f have limited and continuous differential coefficients of the first and second orders throughout each sub-interval. The existence and continuity of the second differential coefficient are only necessary to the proof in the sub-interval containing $a+0$ when $x=a+1$, and in that containing $a-0$ when $x=a-1$. In each case the existence of a continuous first differential coefficient in the sub-intervals containing $a-1+0$ and $a+1-0$ is also assumed. Otherwise the existence of a continuous first differential coefficient in the sub-interval or sub-intervals containing $x\pm 0$ suffices, since by (2 . 13) the validity of the expansion in one sub-interval is not affected by the lack of a differential coefficient in another sub-interval so long as f is limited and integrable throughout, except as just stated when $x=a\pm 1$.

4. If the given function $f(x)$ is the sum of a finite number of terms such as $B\exp\lambda x$, in each of which $\sinh\lambda = c\lambda$, its expansion should be valid for all real values of x between $a\pm 1$, but not for these extreme values unless $\Sigma B\exp\lambda a = 0$. It is interesting to verify this by taking a single such term $e^{\mu x}$.

The coefficient of $\exp\lambda x$ is now, when $\lambda \neq \mu$,

$$\left\{\int_{a-1}^{a}\exp(\mu t-\lambda t-\lambda)\,dt+\int_{a}^{a+1}\exp(\mu t-\lambda t+\lambda)\,dt\right\}\Big/ 2(\cosh\lambda - c),$$

or $\exp(\mu a - \lambda a)\,\{\exp(-\lambda) - \exp(-\mu) + \exp\mu - \exp\lambda\}/2(\mu-\lambda)(\cosh\lambda - c)$,

or
$$c\exp(\mu a - \lambda a)/(\cosh\lambda - c),$$

since
$$\sinh\lambda = c\lambda, \quad \sinh\mu = c\mu.$$

The coefficient of $\exp\mu x$ itself is

$$\cosh\mu/(\cosh\mu - c).$$

The expansion is therefore

$$\exp\mu x + c\exp\mu a\,\sigma(x-a), \tag{4.1}$$

where $\sigma(x)$ denotes
$$\Sigma\exp\lambda x/(\cosh\lambda - c).$$

In particular the expansion of 1 is $1 + c\sigma(x-a)$. (4 . 2)

The expansion of $\exp\mu(x-a) - 1$ is $\exp\mu(x-a) - 1$, and that of $\Sigma B\exp\lambda x$ is

$$\Sigma B\exp\lambda x + c\sigma(x-a)\,\Sigma B\exp\lambda a. \tag{4.3}$$

This affords the desired verification, for $\sigma(x)$ may be written

$$\frac{1}{2\iota\pi}\int_{(\infty)}\frac{\exp\lambda x\,d\lambda}{\sinh\lambda - c\lambda},$$

and according to (2 . 11) is zero when x is between ± 1 but infinite when $x = \pm 1$.

The expansion of $f(x)$ may be written*

$$\tfrac{1}{2}\left\{\int_{a-1}^{a} f(t)\,\sigma(x-t-1)\,dt + \int_{a}^{a+1} f(t)\,\sigma(x-t+1)\,dt\right\}. \tag{4.4}$$

It is formed by integrating the product of $f(x)$ and a function σ which is generally zero but occasionally infinite.

The function $\sigma(x)$ is the difference between two expressions for another function, the two being valid in two different regions separated by the line on which the expansion theorem holds good.

This other function is a multiple of $\varpi(x)$, which is discussed in the following paragraphs.

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* We may evade the condition $f(a) = 0$ by using the expansion

$$f(a) + \tfrac{1}{2}\left[\int_{a-1}^{a}\{f(t) - f(a)\}\,\sigma(x-t-1)\,dt + \int_{a}^{a+1}\{f(t) - f(a)\}\,\sigma(x-t+1)\,dt\right], \tag{4.5}$$

provided that $f(a\pm 0)$ have a common value.

Another form of (4 . 5) is

$$\tfrac{1}{2}\left[\int_{a-1}^{a} f(t)\,\sigma(x-t-1)\,dt+\int_{a}^{a+1} f(t)\,\sigma(x-t+1)\,dt-2cf(a)\,\sigma(x-a)\right], \quad (4.51)$$

which follows by substitution from (4 . 2) for the expansion of a constant.

The Function $\varpi(z)$.

5. The improper integral $\int_0^{+\infty} \frac{\sinh \lambda z\, d\lambda}{\sinh \lambda - c\lambda}$, where $z = x+\iota y$, and λ is real, is clearly convergent so long as $-1 < x < 1$, and in the strip so defined represents an analytical odd function of z: let this function be denoted by

$$-\varpi(z). \qquad (5 . 0)$$

$\varpi(z)$ may be continued over the wider strip $-2 < x < 2$ by writing it in the form

$$-\tfrac{1}{2}\pi \tan \tfrac{1}{2}\pi z - \int_0^{\infty} \frac{c\lambda \sinh \lambda z\, d\lambda}{\sinh \lambda (\sinh \lambda - c\lambda)}, \qquad (5 . 1)$$

since the second part of this expression similarly converges, and represents an analytical function, in the wider strip, while in the former strip it is known that

$$\int_0^{\infty} \frac{\sinh \lambda z}{\sinh \lambda}\, d\lambda = \tfrac{1}{2}\pi \tan \tfrac{1}{2}\pi z. \qquad (5 . 11)$$

Hence $\varpi(z)$ has a pole, with residue 1, at each of the points ± 1, and has no other singularity within the wider strip.

In the strip $1 < x < 2$, we have

$$\tfrac{1}{2}\pi \tan \tfrac{1}{2}\pi z = \int_0^{\infty} \frac{\sinh \lambda (z-2)}{\sinh \lambda}\, d\lambda, \qquad (5 . 12)$$

and thus after reduction

$$\varpi(z) = -\int_0^{\infty} \frac{\sinh \lambda(z-2) + 2c\lambda \cosh \lambda (z-1)}{\sinh \lambda - c\lambda}\, d\lambda$$

$$= \varpi(z-2) + 2c\varpi'(z-1).$$

Hence
$$\varpi(z+1) - \varpi(z-1) - 2c\varpi'(z) = 0, \qquad (5 . 2)$$

first in the strip $0 < x < 1$, and then universally by successive continuation from strip to strip: thus $\varpi(z)$ is uniform in the whole plane, and has singularities at ± 1, ± 2, ± 3, ... only.

Other expressions for $\varpi(z)$ may be given, for

$$\varpi(z) \text{ or } -\int_0^\infty \frac{\sinh \lambda z\, d\lambda}{\sinh \lambda - c\lambda} = -\tfrac{1}{2}\int_{-\infty}^\infty \frac{\sinh \lambda z\, d\lambda}{\sinh \lambda - c\lambda} \tag{5 . 3}$$

$$= -\tfrac{1}{2}\int_{-\infty}^\infty \frac{\exp \lambda z\, d\lambda}{\sinh \lambda - c\lambda}, \tag{5 . 31}$$

if in the last we take the average for the two paths, the one above and the other below the origin.

Now when $-1 < x < 1$ and y is positive the last integral taken over the arc of a semicircle of radius R about the origin in the first two quadrants tends to 0 when $R \to \infty$.

Hence
$$\varpi(z) = -\tfrac{1}{2} \lim \int \frac{\exp \lambda z\, d\lambda}{\sinh \lambda - c\lambda} - \frac{\iota\pi}{2(1-c)}, \tag{5 . 4}$$

where now the path of integration consists of the semicircular arc and its diameter altered slightly so as to pass above the origin. When y is negative we take the semicircle in the two lower quadrants, but still pass above the origin.

By Cauchy's theorem for contour integrals

$$\varpi(z) = -\frac{\iota\pi}{2(1-c)} - \iota\pi\Sigma \frac{\exp \lambda z}{\cosh \lambda - c}, \tag{5 . 5}$$

when $-1 < x < 1$ and $y > 0$, the summation applying to all values of λ in the first two quadrants satisfying the equation $\sinh \lambda = c\lambda$, but

$$\varpi(z) = \frac{\iota\pi}{2(1-c)} + \iota\pi\Sigma \frac{\exp(-\lambda z)}{\cosh \lambda - c}, \tag{5 . 51}$$

when $-1 < x < 1$ and $y < 0$, the summation applying to the same values of λ.

When $y = 0$ and $-1 < x < 1$, both expressions are good and their difference is $\iota\pi\sigma(z)$.

On account of (5 . 2), the expression (5 . 5) or (5 . 51) holds at internal points of the other strips: the restriction to internal points is unnecessary, since for a fixed value of y the convergence of (5 . 5) or (5 . 51) is uniform over any finite range of values of x (§ 10), so that these series represent a continuous function of x.

6. Other uniform functions having the property (3 . 4) and having assigned algebraic singularities can be formed by means of $\varpi(z)$ and its

derivatives in a way which is quite familiar. The following treatment gives some useful results, relating especially to the uniqueness of such functions.

Let $f(z)$, $g(z)$ be two uniform functions with the property (3.4); take a rectangle* bounded by the lines $x = \pm 1$, $y = a, b$ $(a > b)$, and divided into two by the axis of y, and add $\int f(z)\, g(z-1)\, dz$ round the right half to $\int f(z)\, g(z+1)\, dz$ round the left half: it is the same to take

$$\int \{f(z)\, g(z-1) + f(z-1)\, g(z)\}\, dz, \tag{6.1}$$

round the right half.

If within the whole rectangle $f(z)$ has only simple poles at points α, with residues A; and $g(z)$ has only simple poles, at points β, with residues B, then the value of the integral is

$$2\iota\pi \{\Sigma A g(\alpha \pm 1) + \Sigma B f(\beta \pm 1)\}, \tag{6.2}$$

the signs in $\alpha \pm 1$ and $\beta \pm 1$ being so chosen that all these points fall within the rectangle.

The contribution of the vertical sides is

$$\int_{\iota b}^{\iota a} \{f(z+1)\, g(z) + f(z)\, g(z+1) - f(z)\, g(z-1) - f(z-1)\, g(z)\}\, dz, \tag{6.11}$$

that is, on account of the property (3.4),

$$2c\, \{f(\iota a)\, g(\iota a) - f(\iota b)\, g(\iota b)\}. \tag{6.12}$$

Hence $$2\iota\pi \{\Sigma A g(\alpha \pm 1) + \Sigma B f(\beta \pm 1)\} = F(b) - F(a), \tag{6.3}$$

where

$$F(y) = \int_0^1 \{f(x+\iota y)\, g(x+\iota y-1) + f(x+\iota y-1)\, g(x+\iota y)\}\, dx - 2cf(\iota y)\, g(\iota y).$$

If then all the singularities of f and g are outside the rectangle, $F(b) = F(a)$ and $F(y)$ is, in fact, constant for values of y from b to a inclusive. (6.31)

If $g(z) = \varpi(z - Z)$, Z being within the rectangle, then $\beta \pm 1$ has the one value Z, and if all the singularities of f are outside,

$$F(b) - F(a) = 2\iota\pi f(Z). \tag{6.4}$$

Now on the side $y = b$, $\varpi(z - Z)$ can be expanded by the formula (5.51), and on the side $y = a$ by formula (5 . 5), so that writing z for Z we have the following expansion, which includes that of Laurent for a periodic function $(c = 0)$:—

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* Or any parallelogram whose base is parallel to the axis of x and of length 2.

$$f(z) = \tfrac{1}{2}\Sigma \frac{e^{\lambda z}}{\cosh\lambda - c}\left[\int_0^1 \{f(t+\iota b)e^{-\lambda(t+\iota b-1)} + f(t+\iota b-1)\,e^{-\lambda(t+\iota b)}\}\,dt \right. \\ \left. -2cf(\iota b)\,e^{-\lambda\iota b}\right], \quad (6\,.\,5)$$

the summation being over *all* values of λ such that $\sinh\lambda = c\lambda$.

In the coefficient a may be written for b [see (6 . 31)], and the expansion holds within the rectangle $-1 < x < 1$, $b < y < a$, if all the singularities of f are outside the rectangle. By the property (3 . 4) or by the use of a parallelogram with oblique instead of vertical sides, it can be extended to all finite values of z in the strip $b < y < a$. It agrees with (4 . 51) and shows when it is safe to extend (4 . 51) laterally from the real axis.

If all the singularities lie below the line $y = b$, then in (6 . 5) we may increase b to any extent in the positive direction, and if $f(x+\iota y)$ is limited when $y \to +\infty$ and $-1 \leqslant x \leqslant 1$, then for values of λ below the real axis the coefficient of $e^{\lambda z}$ in (6 . 5) can only be zero. For values of λ above the real axis, when $y \to +\infty$, $e^{\lambda z}$ itself tends to zero. Hence, unless there are real values of λ other than 0, $f(x+\iota y)$ is not only limited but tends to a definite limit independent of x when $y \to +\infty$. The same applies to negative values of y, if $e^{-\lambda z}$ is read for $e^{\lambda z}$.

If $f(\pm\iota\infty)$ are both definite, then by putting 1 for $g(z)$ in (6.3), we find

$$f(+\iota\infty) - f(-\iota\infty) \\ = -\frac{\iota\pi}{1-c} \times \text{sum of residues of } f \text{ in the strip } -1 < x < 1. \quad (6\,.\,6)$$

If in the strip f has no singularities and is limited, the expansion contains only the terms in which λ is real: that is, *$f(z)$ is constant,* unless there are real values of λ other than 0.

Again, *if two uniform functions with the property* (3 . 4) *have the same singularities, and are both limited towards infinity in the strip, they can only differ by a constant,* unless there are real values of λ other than 0, which happens when c is real, positive and > 1.

If there is a finite quantity k such that $|f(x+\iota y)|$ does not exceed e^{ky} when y is above a certain positive quantity, then since only a finite number of values of λ lie in the strip $-k < y < k$, there will only be a finite number of terms in (6 . 5) with λ on the lower side of the real axis, the coefficients of the rest being zero, just as when $f(+\iota\infty)$ was limited. The same thing holds with the signs of y and λ changed.

The Function $\zeta(z)$.

7. Just as functions with the property (3 . 4) include periodic functions in particular, so there are functions with two such properties including doubly periodic functions in particular.

Let κ be a complex or purely imaginary quantity, and to fix the ideas suppose it to fall above the real axis. Let γ, as well as c, be a constant other than 1.

Let $\zeta(z)$ denote $\frac{1}{4}\int_{(1)} \frac{\exp \lambda z \, d\lambda}{(\sinh \lambda - c\lambda)(\sinh \kappa\lambda - \gamma\kappa\lambda)}$, (7 . 1)

where (1) indicates a path consisting of two branches of hyperbolic type crossing the lateral axis above and below the origin and leading in from infinity in the second and fourth quadrants out to infinity in the first and third respectively. The zeros of $\sinh \kappa\lambda - \gamma\kappa\lambda$ are all to be between the two branches, those of $\sinh \lambda - c\lambda$ other than the origin in the other part of the plane.

We may also put $\zeta(z) = \frac{1}{2}\int_{(2)} \frac{\sinh \lambda z \, d\lambda}{(\sinh \lambda - c\lambda)(\sinh \kappa\lambda - \gamma\kappa\lambda)}$, (7 . 11)

where (2) indicates the branch of (1) lying in the first two quadrants.

So long as z lies within the parallelogram formed by $\pm 1 \pm \kappa$, these integrals converge absolutely and denote an analytical odd function of z. The same integrals along any arc of a circle about the origin as centre tend discontinuously to zero when the radius increases indefinitely; the discontinuous changes happen when the arc passes over a zero of the denominator.

Hence the path in (7 . 1) may also be taken as (3) the limit of a closed dumb-bell curve formed by (1) and two circular arcs at a great distance crossing the real axis, this path being described in the negative sense (7 . 12), or (4) the limit of two separate closed curves formed by (1) and two circular arcs at a great distance crossing the lateral axis (7 . 13).

Taking the integral over (4) we have formally

$$\zeta(z+1) - \zeta(z-1) - 2c\zeta'(z) = \frac{1}{2}\int_{(4)} \frac{\exp \lambda z \, d\lambda}{\sinh \kappa\lambda - \gamma\kappa\lambda},$$

which vanishes because (4) encloses no zeros of the denominator. Thus the expression (7 . 13) converges for $z \pm 1$, if for $z \mp 1$ and z; the function

ζ exists in the strip bounded by the horizontal lines through $\pm\kappa$, and is such that

$$\zeta(z+1)-\zeta(z-1)-2c\zeta'(z) = 0. \tag{7.2}$$

Similarly within the strip between the other two sides of the parallelogram ζ exists, and

$$\zeta(z+\kappa)-\zeta(z-\kappa)-2\gamma\kappa\zeta'(z) = \tfrac{1}{2}\int_{(3)} \frac{\exp\lambda z\, d\lambda}{\sinh\lambda - c\lambda} = -\iota\pi/(1-c), \tag{7.3}$$

since (3) encloses the origin but no other zero of $\sinh\lambda - c\lambda$ and is described negatively.

To find the singularities of $\zeta(z)$ write it in the form

$$\iota\pi\Sigma \frac{\sinh\lambda z}{(\cosh\lambda - c)(\sinh\kappa\lambda - \gamma\kappa\lambda)}. \tag{7.4}$$

where the summation covers all the roots of the equation $\sinh\lambda = c\lambda$ above the real axis.

This series converges in the strip between the horizontal lines through $\pm\kappa$. It may also be written

$$-\iota\pi\Sigma\left\{\exp\lambda(z+\kappa)-\exp\lambda(\kappa-z)\right\}/(\cosh\lambda-c)$$

$$+\tfrac{1}{2}\iota\pi\Sigma\frac{\exp\lambda(z+2\kappa)-2\gamma\kappa\lambda\exp\lambda(z+\kappa)-\exp\lambda(2\kappa-z)+2\gamma\kappa\lambda\exp\lambda(\kappa-z)}{(\cosh\lambda-c)(\sinh\kappa\lambda-\gamma\kappa\lambda)}, \tag{7.5}$$

where the second part converges and represents a function without singularity in the wider strip between the horizontal lines through $\pm 2\kappa$, while the first part is $\varpi(z+\kappa)+\varpi(z-\kappa)$ [(5 . 5) and (5 . 51)].

Hence the only singularities of ζ inside the wider strip are those of $\varpi(z+\kappa)+\varpi(z-\kappa)$: it has then simple poles with unit residues at the points $\pm 1 \pm \kappa$ and no other singularity within the parallelogram formed by these points or on its sides.

Hence ζ is a uniform function everywhere and has the properties (7 . 2) and (7 . 3) everywhere.

8. Take any two uniform functions $f(z)$, $g(z)$ with such properties, say

$$f(z+1)-f(z-1)-2cf'(z) = f_1, \tag{8.11}$$

$$f(z+\kappa)-f(z-\kappa)-2\gamma\kappa f'(z) = f_2, \tag{8.12}$$

$$g(z+1)-g(z-1)-2cg'(z) = g_1, \tag{8.13}$$

$$g(z+\kappa)-g(z-\kappa)-2\gamma\kappa g'(z) = g_2, \tag{8.14}$$

and evaluate

$$\int \{f(z)\,g(z-1-\kappa)+f(z-1)\,g(z-\kappa)+f(z-\kappa)\,g(z-1)+f(z-1-\kappa)\,g(z)\}\,dz, \tag{8.2}$$

round the parallelogram formed by 0, 1, $1+\kappa$, κ, which is a quarter of the *fundamental* parallelogram formed by $\pm 1 \pm \kappa$.

The result is $\quad 2\iota\pi\,\{\Sigma A g(\alpha\pm 1\pm\kappa)+\Sigma B f(\beta\pm 1\pm\kappa)\}, \qquad (8.21)$

if, within the fundamental parallelogram, f has only simple poles at points α with residues A, and g has only simple poles at points β with residues B, and the ambiguous signs are so decided that $\alpha\pm 1\pm\kappa$, $\beta\pm 1\pm\kappa$ all lie within the fundamental parallelogram.

This quantity then is equal to

$$\begin{aligned}&\int_0^1 \{f(z)\,g(z-1-\kappa)+f(z-1)\,g(z-\kappa)+f(z-\kappa)\,g(z-1)+f(z-1-\kappa)\,g(z)\\ &\qquad -f(z+\kappa)g(z-1)-f(z+\kappa-1)g(z)-f(z)g(z+\kappa-1)-f(z-1)g(z+\kappa)\}\,dz\\ &+\int_0^\kappa \{f(z+1)\,g(z-\kappa)+f(z)\,g(z+1-\kappa)+f(z+1-\kappa)\,g(z)+f(z-\kappa)\,g(z+1)\\ &\qquad -f(z)g(z-1-\kappa)-f(z-1)g(z-\kappa)-f(z-\kappa)g(z-1)-f(z-1-\kappa)g(z)\}\,dz,\end{aligned} \tag{8.22}$$

which in virtue of (8.11), etc., reduces to

$$\begin{aligned}&f_1\left\{\int_{-\kappa}^{\kappa} g(z)\,dz-2\gamma\kappa g(0)\right\}+g_1\left\{\int_{-\kappa}^{\kappa} f(z)\,dz-2\gamma\kappa f(0)\right\}\\ &-f_2\left\{\int_{-1}^{1} g(z)\,dz-2cg\,(0)\right\}-g_2\left\{\int_{-1}^{1} f(z)\,dz-2cf(0)\right\},\end{aligned}$$

or, say, $\qquad f_1 G_2+g_1 F_2-f_2 G_1-g_2 F_1. \qquad (8.23)$

As one special case let $g(z)=1$, then

$$g_1=g_2=0, \qquad G_1=2\,(1-c), \qquad G_2=2\kappa(1-\gamma),$$

so that $\qquad 2\iota\pi\Sigma A = 2\kappa f_1(1-\gamma)-2f_2(1-c). \qquad (8.3)$

If $g(z)=z$, then

$$g_1=2(1-c), \qquad g_2=2\kappa(1-\gamma), \qquad G_1=G_2=0,$$

so that $\qquad 2\iota\pi\Sigma A\,(\alpha\pm 1\pm\kappa)=2(1-c)\,F_2-2\kappa(1-\gamma)\,F_1. \qquad (8.4)$

Again, if $g(z) = \zeta(z-Z)$, and Z is in the fundamental parallelogram, then

$$g_1 = 0, \quad g_2 = -\iota\pi/(1-c), \quad \beta \pm 1 \pm \kappa = Z, \quad B = 1,$$

so that $2\iota\pi f(Z) + 2\iota\pi \Sigma A \zeta(\alpha \pm 1 \pm \kappa - Z) = f_1 G_2 - f_2 G_1 + \iota\pi F_1/(1-c)$.

But $$\frac{\partial G_1}{\partial Z} = \int_{-1}^{1} \frac{\partial}{\partial Z} \zeta(z-Z)\, dz + 2c\zeta'(Z) = 0 \text{ by } (7\,.\,2),$$

and $$\frac{\partial G_2}{\partial Z} = \int_{-\kappa}^{\kappa} \frac{\partial}{\partial Z} \zeta(z-Z)\, dz + 2\gamma\kappa\zeta'(Z) = \iota\pi/(1-c) \text{ by } (7\,.\,3).$$

Hence, writing z for Z, we have

$$f(z) = \Sigma A \zeta\{z-(\alpha \pm 1 \pm \kappa)\} + f_1 z/2(1-c) + \text{const.}, \tag{8 . 5}$$

just as in the theory of elliptic functions, where $c = \gamma = 0$.*

The Case $c = 1$.

9. Much of the above work still applies to the case $c = 1$, in spite of the triple root of the equation $\sinh\lambda = \lambda$ at the origin. No change is needed in § 2 or § 3 before (3 . 5), where the triple root gives rise to three terms in the expansion.

The expansion of $\exp \lambda q/(\sinh\lambda - \lambda)$ in ascending powers of λ is

$$\frac{6}{\lambda^3} + \frac{6q}{\lambda^2} + \frac{3q^2}{\lambda} - \frac{3}{10\lambda} + \dots,$$

and thus the terms in $\sigma(x)$ are $3x^2 - \frac{3}{10}$; in (3 . 5) or (4 . 4),

$$\tfrac{3}{2}\int_{a-1}^{a} f(t)\{(x-t-1)^2 - \tfrac{1}{10}\}\, dt + \tfrac{3}{2}\int_{a}^{a+1} f(t)\{(x-t+1)^2 - \tfrac{1}{10}\}\, dt;$$

and in (4 . 51),

$$\tfrac{3}{2}\int_{a-1}^{a} f(t)\{(x-t-1)^2 - \tfrac{1}{10}\}\, dt + \tfrac{3}{2}\int_{a}^{a+1} f(t)\{(x-t+1)^2 - \tfrac{1}{10}\}\, dt$$
$$-3f(a)\{(x-a)^2 - \tfrac{1}{10}\}.$$

The last expression reduces to $f(x)$ if $f(x) = A + Bx + Cx^2$, and to 0 if $f(x) = \exp \mu x$, where $\sinh\mu = \mu$. In this way the verification of § 4 is completed.

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* When $c = \gamma = 0$, (7 . 1) is a compact expression for an elliptic function of the second kind.

At the opening of § 5 we take

$$\varpi(z) = -\int_0^\infty \left(\frac{\sinh \lambda z}{\sinh \lambda - \lambda} - \frac{6z}{\lambda^2}\right) d\lambda,$$

and make the same change in (5 . 1). There is no change in (5 . 2) or (5 . 31), but in (5 . 4), (5 . 5), (5 . 51), we must write

$$\tfrac{3}{2}\iota\pi(z^2-\tfrac{1}{10}) \quad \text{for} \quad \tfrac{1}{2}\iota\pi/(1-c)$$

in the constant terms.

The only change in § 6 is in (6 . 5) which is still the same as (4 . 51) with ιb written for a.

In §§ 7, 8 the supposition $\gamma = 1$ would not affect the work unless we took $c = 1$ also, but if $c = 1$ the right side of (7 . 3) must be changed to $-3\iota\pi(z^2-\frac{1}{10})$. In (8 . 5),

$$f(z) - \Sigma A \zeta\{z-(\alpha \pm 1 \pm \kappa\} = \text{a cubic in } z,$$

and f_1 cannot vanish unless ΣA does so by (8 . 3), but f_2 is arbitrary.

If $c = \gamma = 1$, we have by (8 . 3) and (8 . 4),

$$\Sigma A = 0, \quad \Sigma A(\alpha \pm 1 \pm \kappa) = 0,$$

so that a function with the properties (8 . 11) and (8 . 12) must have at least three poles, distinct or not. For instance, $\zeta''(z-\alpha)$ is a function with these properties, having one triple pole in the fundamental parallelogram. Other such functions are

$$\zeta(z-\alpha) - \zeta(z-\beta) + (\alpha-\beta)\,\zeta'(z-\alpha),$$

with a double pole at $\alpha \pm 1 \pm \kappa$ and a single one at $\beta \pm 1 \pm \kappa$, and

$$(\beta-\theta)\,\zeta(z-\alpha) + (\theta-\alpha)\,\zeta(z-\beta) + (\alpha-\beta)\,\zeta(z-\theta),$$

with three simple poles.

Appendix on Convergency.

10. In the foregoing it has been assumed that certain improper integrals and infinite series converge. In the proof of this the main point is to establish inferior limits for the absolute value of $\sinh \lambda - c\lambda$ when λ is great. We have to avoid the zeros of this function which all lie in a region that stretches to infinity along the lateral axis both ways

Writing $\xi+\iota\eta$ for λ, we exclude from consideration all values of λ within the parabolas $\xi^2=\pm\eta$, except those for which η is an odd multiple of $\frac{1}{2}\pi$.

Outside or upon the parabolas, λ being great, ξ must be great, and thus $\sinh\lambda-c\lambda$ is of the same order of magnitude as $\exp|\xi|$.

Inside the parabolas $\lambda=\pm(n+\frac{1}{2})\iota\pi+\xi$, n being a positive integer, and, if $c=c_1+\iota c_2$,

$$\sinh\lambda-c\lambda=\pm\iota(-1)^n\cosh\xi-(c_1+\iota c_2)\{\pm(n+\tfrac{1}{2})\iota\pi+\xi\}$$
$$=\{\pm(n+\tfrac{1}{2})c_2\pi-c_1\xi\}\pm\iota\{(-1)^n\cosh\xi-(n+\tfrac{1}{2})c_1\pi\mp c_2\xi\}.$$

Now $\xi^2<(n+\frac{1}{2})\pi$, and thus if $n>4c_2^2/\pi c_1^2$ and is odd or even as c_1 is positive or negative, the real part may be ignored, and

$$|\sinh\lambda-c\lambda|>\cosh\xi+\tfrac{1}{2}n\pi|c_1|.$$

Hence both within, upon, and without the parabolas

$$\left|\frac{\exp(\pm\lambda)}{\sinh\lambda-c\lambda}\right| \quad\text{and}\quad \left|\frac{\lambda}{\sinh\lambda-c\lambda}\right|{}^{*}$$

are limited, say $<K$, for the great values of λ considered: the same is true even when $c_1=0$, for then within the parabolas the real part is of the order of n and the lateral part of the order of $\cosh\xi$ (if ξ is great). Thus we have excluded all the distant zeros of $\sinh\lambda-c\lambda$, since they lie within the parabolas, but we have left bridges by which to cross the strip in which the zeros lie.

Where a circular path has been prescribed for λ, as for instance in § 2, the radius of the circle will be taken as

$$R=\{(n+\tfrac{1}{2})^2\pi^2+(n+\tfrac{1}{2})\pi\}^{\frac{1}{2}},$$

n being a positive integer, odd, if c_1 is positive, even, if c_1 is negative, so that the circle has two of the bridges as chords, and these chords will be taken in as parts of the path instead of their minor arcs.

Now in the first integral of (2 . 11), which is the limit of

$$\int_{(R)}\frac{\exp\lambda q\,d\lambda}{\sinh\lambda-c\lambda},$$

$$|\exp\lambda q|=\exp\xi q,\qquad |\sinh\lambda-c\lambda|>\exp|\xi|\div K,$$

---

[*] This part of the statement is not true when $c=0$, so that the conditions are less favourable in the Fourier case than in general.

and also $$|\sinh\lambda-c\lambda|>|\lambda|\div K>|\eta|\div K.$$

On the path outside the parabolas in the first and fourth quadrants

$$\xi>\sqrt{(n\pi+\tfrac{1}{2}\pi)},$$

and thus the subject of integration is less in absolute value than

$$K\exp\xi(q-1)<K\exp(q-1)\sqrt{n},$$

where $q-1$ is negative; the length of path is less than πR which is of the order of n. Thus the contribution of this part of the path tends to zero when R is increased if $q<1$.

On the chords or bridges

$$d\lambda=d\xi,\quad |\eta|=(n+\tfrac{1}{2})\pi,\quad |\xi|<\sqrt{|\eta|}:$$

we separate the parts of the path on which $|\xi|>\log|\eta|$ from the rest. Thus in the first or fourth quadrant the contribution is less in absolute value than

$$\int_0^{\log|\eta|}\frac{K}{|\eta|}\exp\xi q\,d\xi+\int_{\log|\eta|}^{\sqrt{|\eta|}}K\exp(\xi q-\xi)\,d\xi,$$

and therefore than $$\frac{K}{|\eta|}\,\frac{|\eta|^q+1}{q}+\frac{K}{1-q}|\eta|^{q-1}.$$

This quantity tends to zero when R is increased indefinitely if $q<1$, and therefore the whole contribution of the first and fourth quadrants also tends to zero.

The second and third quadrants give expressions which differ from those for the first and fourth only in the sign of q, so that their contribution tends to zero if $q>-1$, and the whole tends to zero if $-1<q<1$ as stated. The second part of (2 . 11) follows at once.

If $q=1$, as in (2 . 12), the contribution of the second and third quadrants has been proved to vanish. In the first and fourth write

$$\frac{\exp\lambda q}{\lambda(\sinh\lambda-c\lambda)}=\frac{\exp\lambda}{\lambda(\sinh\lambda-c\lambda)}=\frac{2}{\lambda}+\frac{\exp(-\lambda)+2c\lambda}{\lambda(\sinh\lambda-c\lambda)}.$$

The contribution of the second part tends to zero as before, and the first part taken through the two quadrants gives $2\iota\pi$,* which is therefore the value of the second integral in (2 . 11) when $q = 1$. Similarly when $q = -1$, the value is $-2\iota\pi$.* The first integral does not converge.

The convergency in (2 . 11) is uniform so long as q lies in an interval internal to that between ± 1. Hence the first two parts of (2 . 13) follow by integration from (2 . 11).

In dealing with
$$\int_{(R)}\int_{v}^{1} \frac{f(q)\exp\lambda q\,dq\,d\lambda}{\lambda(\sinh\lambda - c\lambda)},$$
suppose that $|f(q)| < G$, a constant. Then
$$\int_{v}^{1} |f(q)\exp\lambda q|\,dq < G\int_{v}^{1}\exp\xi q\,dq.$$

In the second and third quadrants, ξ being negative, this expression is less than
$$G(1-v)\exp\xi v,$$
and the contribution from these quadrants tends to zero if $1 > v > -1$ as in the case of (2 . 11).

In the first and fourth quadrants, where ξ is positive, it follows that
$$\int_{v}^{1} |f(q)\exp\lambda q|\,dq < G\exp\xi/\xi \text{ and also } < G(1-v)\exp\xi,$$
the latter expression being required within the parabolas. Outside the parabolas the absolute value of the subject of the second integration, with respect to λ, is less than
$$G\exp\xi/\xi \times K/R\exp\xi \quad\text{or}\quad GK/\xi R \quad\text{or}\quad GK/R\sqrt{n},$$
while the length of path is less than πR.

Hence this part of the double integral is less than $\pi GK/\sqrt{n}$, and tends to zero when n increases.

The parts of the path inside the parabolas contribute less than
$$2\int_{0}^{\sqrt{|\eta|}} G(1-v)\exp\xi\,.\,K/|\lambda|\exp\xi\,.\,d\xi$$

---

* If the path of integration crosses the lateral axis at unequal distances $+R_1$, $-R_2$ from the origin, these values must be changed to $\pm 2\iota\pi + 2\log R_1/R_2$. This affects (2 . 12) and the results depending on it; for instance, on the left of (3 . 3) we must add
$$\{f(x-0) - f(x+0)\}\,(\lim\log R_1/R_2) \div 2\iota\pi,$$
but (3 . 5) is not affected.

or

$$\frac{2GK}{|\eta|}\int_0^{\sqrt{|\eta|}}(1-v)\,d\xi.$$

or

$$2GK(1-v)/\sqrt{|\eta|},$$

which also tends to zero. Hence all the statements in (2 . 13) are established.

In the contour-integral at (5 . 4) we know from (2 . 11) that

$$\int_{(R)}\left|\frac{\exp\lambda x\,d\lambda}{\sinh\lambda-c\lambda}\right|\to 0,$$

and that

$$|\exp\iota\lambda y|<1,$$

when λ is the first two quadrants and y positive. Hence

$$\int\frac{\exp\lambda z\,dz}{\sinh\lambda-c\lambda},$$

over the arc of the semicircle, tends to zero.

At the end of § 5 the question of uniform convergency arises for the series (5 . 5) and (5 . 51). Now, if c changes, the number of roots of the equation $\sinh\lambda=c\lambda$ within any contour can only change when one of the roots falls on the contour: thus the number of roots in the area between the curve $\eta=\xi^2$ and the two bridges $\eta=(n+\frac{1}{2})\pi$ and $\eta=(n+\frac{5}{2})\pi$ cannot change when n is above a certain limit depending on the range of values of c. When $c=0$, there are two such roots, $\iota\pi(n+1)$ and $\iota\pi(n+2)$. Hence in general there are two roots such as

$$\iota\pi(n+\tfrac{3}{2})+\xi+\iota\beta,$$

where β is between $\pm\pi$ and ξ is not of higher order than $\sqrt{n}$.

In these two terms, the denominator $\cosh\lambda-c$ is of order n being equal to $\sqrt{(1+c^2\lambda^2)}-c$; the absolute value of the numerator $\exp\lambda z$ is

$$\exp\{-y\pi(n+\tfrac{3}{2})+x\xi-y\beta\}.$$

So then, if x is within any finite range and y exceeds a positive quantity b, ultimately the n-th term of the series is less than $\exp(-n\pi b')$ where b' is a positive quantity less than b and depending on b, not on x or y. Thus the series (5 . 5) and similarly (5 . 51) are uniformly convergent, as assumed at the end of § 5 and also at (6 . 5).

In dealing with the integrals of § 7 we may put $z=X+\kappa Y$ where X, Y are real, so that z lies in the parallelogram if X, Y are both between

± 1. Then

$$\frac{\exp \lambda z}{(\sinh \lambda - c\lambda)(\sinh \kappa\lambda - \gamma\kappa\lambda)} = \frac{\exp \lambda X}{\sinh \lambda - c\lambda} \times \frac{\exp \kappa\lambda Y}{\sinh \kappa\lambda - \gamma\kappa\lambda}.$$

We exclude from consideration the same points as before and also those inside a second pair of parabolas for the factor $\sinh \kappa\lambda - \gamma\kappa\lambda$; this second region is also to be crossed by bridges. We have seen that on a bridge the corresponding fraction

$$\frac{\exp \lambda X}{\sinh \lambda - c\lambda} \quad \text{or} \quad \frac{\exp \kappa\lambda Y}{\sinh \kappa\lambda - \gamma\kappa\lambda}$$

is limited, while the other fraction is exponentially small: outside the parabolas both factors are exponentially small. Hence the integral, and also its formal derivatives with respect to z, converge whichever of the paths (1), (2), (3), (4) is taken.

In (7 . 4) and (7 . 5) we may write $-\frac{1}{2}\exp(-\kappa\lambda)$ for $\sinh \kappa\lambda - \gamma\kappa\lambda$ without affecting the convergency, and then the statements made follow from the discussion of (5 . 5) in this paragraph.

RECIPROCAL CORRESPONDENCE OF DIFFERENCES AND SUMS

By W. F. SHEPPARD.

[Received August 22nd, 1921.—Read November 17th, 1921.]

1. This paper deals with (1) the indefinite sums corresponding to "divided" or "adjusted" differences; (2) the definite set of sums (in an extended sense) corresponding to the differences, on any system, of a particular set of quantities; (3) the differences (in an extended sense) corresponding to symmetrical sums of a set of quantities.

(1) *Indefinite sums corresponding to divided or adjusted differences.*

2. In the ordinary theory of differences, summing is the inverse of differencing. In the advancing difference notation, for instance, if $\ldots u_{r-1}, u_r, u_{r+1}, \ldots$ is a set of quantities, taken in this order, Δu_r means $u_{r+1}-u_r$, and Σu_r is a function of r such that

$$\Delta \Sigma u_r = u_r.$$

If the extreme u's are u_0 and u_l, we can write Σu_r in either of the forms

$$\Sigma u_r = \text{const.} + u_0 + u_1 + \ldots + u_{r-2} + u_{r-1}$$

or

$$\Sigma u_r = \text{const.} - u_r - u_{r+1} - \ldots - u_l.$$

Similarly in the central difference notation $\delta u_{r+\frac{1}{2}}$ means $u_{r+1}-u_r$, and the operator σ is such that

$$\delta \sigma u_r = u_r,$$

so that

$$\sigma u_{r+\frac{1}{2}} = \text{const.} + u_0 + u_1 + \ldots + u_{r-1} + u_r$$

$$= \text{const.} - u_{r+1} - u_{r+2} - \ldots - u_l.$$

The successive sums $\Sigma^2 u_r$, $\Sigma^3 u_r$, ... or $\sigma^2 u_r$, $\sigma^3 u_{r+\frac{1}{2}}$, ... are formed on the same principle, a new arbitrary constant being introduced for each successive set of sums. We want to find what corresponds to these sums in the systems of divided and of adjusted differences.

3. These systems are applicable when the tabulated quantities correspond to values of some variable taken at irregular intervals and not necessarily in order of magnitude. We will take divided differences first. Let the values of the variable be $x_0, x_1, x_2, \ldots, x_l$, and let the corresponding tabulated quantities be $u_0, u_1, u_2, \ldots, u_l$. Then the *divided* differences, of successive orders, are obtained successively in the same way as advancing differences, with the exception that at each step the difference found in this way is divided by the difference of the x's corresponding to the extreme u's involved. If we denote the divided differences of orders 1, 2, etc., by $\nabla u_0, \nabla u_1, \ldots, \nabla^2 u_0, \nabla^2 u_1, \ldots$, etc., we have

$$\left.\begin{array}{ll} \nabla u_0 \equiv (u_1-u_0)/(x_1-x_0), & \nabla u_1 \equiv (u_2-u_1)/(x_2-x_1), \ \ldots \\ \nabla^2 u_0 \equiv (\nabla u_1-\nabla u_0)/(x_2-x_0), & \nabla^2 u_1 \equiv (\nabla u_2-\nabla u_1)/(x_3-x_1), \ \ldots \\ \text{etc.} & \end{array}\right\}, \quad (3.1)$$

The general relation is

$$\nabla^f u_q \equiv (\nabla^{f-1} u_{q+1} - \nabla^{f-1} u_q)/(x_{q+f}-x_q); \quad (3.2)$$

and it will be found that

$$\nabla^f u_0 = \frac{u_0}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_f)} + \frac{u_1}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_f)} + \ldots + \frac{u_f}{(x_f-x_0)(x_f-x_1)\ldots(x_f-x_{f-1})}, \quad (3.3)$$

$$u_q = u_0 + (x_q-x_0)\nabla u_0 + (x_q-x_0)(x_q-x_1)\nabla^2 u_0 + \ldots + (x_q-x_0)(x_q-x_1)\ldots(x_q-x_{q-1})\nabla^q u_0. \quad (3.4)$$

It should be noticed that, though the particular u's involved in $\nabla^q u_0$ are determined by the order of the u's as a whole, the formula for $\nabla^q u_0$ is independent of the order in which these particular u's are taken; *e.g.* for $\nabla^2 u_0$ the first three u's might be u_0, u_1, u_2, or u_1, u_2, u_0, etc. This is not the case in the ordinary systems. If, for instance, $abcd$ are four u's, the resulting third difference in the ordinary advancing difference system is $d-3c+3b-a$ if they are taken in the order $abcd$, but $a-3d+3c-b$ if they are taken in the order $bcda$.

4. To find the sums corresponding to divided differences, we take Ξ to be an operator inverse to ∇, *i.e.* an operator such that

$$\nabla \Xi = 1. \quad (4.1)$$

This is not the same thing as $\mathfrak{Z}\nabla = 1$; but to give an interpretation to $\mathfrak{Z}$ we can try $\nabla = \mathfrak{Z}^{-1}$. Substituting for this in (3 . 2), we get

$$\mathfrak{Z}^{-f} u_q = (\mathfrak{Z}^{-f+1} u_{q+1} - \mathfrak{Z}^{-f+1} u_q)/(x_{q+f} - x_q);$$

and therefore, replacing $-f$ by g,

$$\mathfrak{Z}^{g} u_q = (\mathfrak{Z}^{g+1} u_{q+1} - \mathfrak{Z}^{g+1} u_q)/(x_{q-g} - x_q), \tag{4 . 2}$$

which may be written

$$\mathfrak{Z}^{g+1} u_{q+1} = \mathfrak{Z}^{g+1} u_q - (x_q - x_{q-g})\, \mathfrak{Z}^{g} u_q. \tag{4 . 3}$$

5. In the advancing difference system the relation of differences and sums is as shown in the following diagram, in which each quantity is the difference (lower *minus* upper) of the two quantities adjacent to it on the left.

$$\begin{array}{ccccccc}
\Sigma^3 u_q & & \Sigma u_{q-1} & & \Delta u_{q-2} & & \Delta^3 u_{q-3} \\
& \Sigma^2 u_q & & u_{q-1} & & \Delta^2 u_{q-2} & \\
\Sigma^3 u_{q+1} & & \Sigma u_q & & \Delta u_{q-1} & & \Delta^3 u_{q-2} \\
& \Sigma^2 u_{q+1} & & u_q & & \Delta^2 u_{q-1} & \\
\Sigma^3 u_{q+2} & & \Sigma u_{q+1} & & \Delta u_q & & \Delta^3 u_{q-1} \\
& \Sigma^2 u_{q+2} & & u_{q+1} & & \Delta^2 u_q & \\
\Sigma^3 u_{q+3} & & \Sigma u_{q+2} & & \Delta u_{q+1} & & \Delta^3 u_q \\
& \Sigma^2 u_{q+3} & & u_{q+2} & & \Delta^2 u_{q+1} &
\end{array}$$

The diagram implies that $\Delta^0 u_q = \Sigma^0 u_q = u_q$. In the divided difference system, supposing the Δ's and Σ's in the above diagram to be replaced by ∇'s and $\mathfrak{Z}$'s, we may take as typical relations, from (3 . 2) and (4 . 2),

$$\nabla^3 u_{q-1} = (\nabla^2 u_q - \nabla^2 u_{q-1})/(x_{q+2} - x_{q-1}),$$

$$\mathfrak{Z}^2 u_{q+1} = (\mathfrak{Z}^3 u_{q+2} - \mathfrak{Z}^3 u_{q+1})/(x_{q-1} - x_{q+1}).$$

These two formulæ correspond exactly; in each case the suffixes of the two x's in the denominator on the right-hand side of the formula are those of the u's found by drawing lines in the diagram from the quantity on the left of the formula through the two quantities in the numerator on the right, these being the quantities which in the diagram are adjacent to it on its left. We therefore take this as the definition of the operation ∇, in place of the more limited definition given in § 3, and we then have, from (3 . 2) and (4 . 2),

$$\nabla \cdot \nabla^{f-1} u_q = \nabla^f u_q, \tag{5 . 1}$$

$$\nabla \,.\, \mathfrak{Z}^{g+1} u_q = \mathfrak{Z}^g u_q. \tag{5 . 2}$$

We can therefore treat the values of $\varSigma^{g+1}u_q$ which satisfy (4 . 2), for the various values of q, as the sums of order $g+1$.

6. When, however, we come to build up the $\varSigma$'s by means of (4 . 3), which is the equivalent of (4 . 2), we find a difficulty. The process of constructing sums of orders 2, 3, ... from those of order 1 is simple enough, (4 . 3) giving

$$\varSigma^2 u_{q+1} = \varSigma^2 u_q - (x_q - x_{q-1})\,\varSigma u_q,$$

$$\varSigma^3 u_{q+1} = \varSigma^3 u_q - (x_q - x_{q-2})\,\varSigma^2 u_q,$$

etc.

The difficulty arises when we want to construct the table of $\varSigma u_q$; for, putting $g = 0$ in (4 . 3), we have

$$\begin{aligned}\varSigma u_{q+1} &= \varSigma u_q - (x_q - x_q)\,\varSigma^0 u_q \\ &= \varSigma u_q - 0 \,.\, \varSigma^0 u_q. \qquad (6 \,.\, 1)\end{aligned}$$

There are only two interpretations. The one is that $\varSigma u_{q+1} = \varSigma u_q$, which means that the u's do not enter into the $\varSigma$'s at all, except in so far as we can introduce them successively as arbitrary constants: the other is that $\varSigma^0 u_q$ is infinite, so that it cannot be identical with u_q.

7. The explanation of this difficulty can be seen by taking the x's to be successive integers. We then have, from (3 . 2) and (3 . 3),

$$\nabla^f u_q = (\nabla^{f-1} u_{q+1} - \nabla^{f-1} u_q)/f,$$

$$\nabla^f u_0 = \Delta^f u_0 / f!,$$

so that the successive differences, beginning with ∇u_0, are equal to Δu_0, $\Delta^2 u_0/2!$, $\Delta^3 u_0/3!$, ... By analogy we should expect that the successive sums would be equal to $\Sigma u_0/(-1)!$, $\Sigma^2 u_0/(-2)!$, ..., *i.e.* would be zero.

8. Let us see whether we can get rid of the difficulty by using *adjusted* differences. The system of adjusted differences differs from that of divided differences in that, in forming the successive differences, the division by $x_{q+f} - x_q$ is corrected by multiplication by f; *i.e.* if the

successive adjusted differences of u_0 are Θu_0, $\Theta^2 u_0$, ..., we have

$$\left.\begin{array}{ll} \Theta u_0 \equiv (u_1-u_0)/(x_1-x_0), & \Theta u_1 \equiv (u_2-u_1)/(x_2-x_1), \; \dots \\ \Theta^2 u_0 \equiv 2(\Theta u_1-\Theta u_0)/(x_2-x_0), & \Theta^2 u_1 \equiv 2(\Theta u_2-\Theta u_1)/(x_3-x_1), \; \dots \\ \qquad \text{etc.} & \end{array}\right\}, \qquad (8\,.\,1)$$

The formulæ (3 . 2), (3 . 3), (3 . 4) are replaced by

$$\Theta^f u_q = f(\Theta^{f-1} u_{q+1} - \Theta^{f-1} u_q)/(x_{q+f}-x_q), \qquad (8\,.\,2)$$

$$\Theta^f u_q = \nabla^f u_q \times f!, \qquad (8\,.\,3)$$

$$u_q = u_0 + (x_q-x_0)\,\Theta u_0/1! + (x_q-x_0)(x_q-x_1)\,\Theta^2 u_0/2! + \dots$$
$$+ (x_q-x_0)(x_q-x_1)\dots(x_q-x_{q-1})\,\Theta^q u_0/q!, \qquad (8\,.\,4)$$

where, in (8 . 3), $\nabla^f u_q$ means the value given for this by (3 . 3). The system becomes equivalent to the advancing difference system when the x's are successive integers.

9. If Z denotes the operator for adjusted differences which corresponds to Ƹ for divided differences, we must have

$$\Theta Z = 1. \qquad (9\,.\,1)$$

Proceeding as before, we shall find that (4 . 2) and (4 . 3) are replaced by

$$Z^g u_q = g(Z^{g+1} u_{q+1} - Z^{g+1} u_q)/(x_q - x_{q-g}), \qquad (9\,.\,2)$$

$$Z^{g+1} u_{q+1} = Z^{g+1} u_q + (x_q - x_{q-g})/g \,.\, Z^g u_q\,; \qquad (9\,.\,3)$$

and (5 . 1) and (5 . 2) by

$$\Theta \,.\, \Theta^{f-1} u_q = \Theta\, u_q, \qquad (9\,.\,4)$$

$$\Theta \,.\, Z^{g+1} u_q = Z^g u_q. \qquad (9\,.\,5)$$

If we put $g=0$ in (9 . 3), we get

$$Z u_{q+1} = Z u_q + 0/0 \,.\, Z^0 u_q. \qquad (9\,.\,6)$$

This is consistent with $Z^0 u_q$ being equal to u_q, but it leaves $Z u_{q+1} - Z u_q$ indeterminate. For further elucidation we must consider the definite sums corresponding to adjusted differences (see § 19).

(2) *Definite sums corresponding to differences.**

---

* This is based on part of a paper recently published in the *Phil. Trans.* (A. 221, at pp. 204–207).

10. The sums so far considered are indefinite sums; *i.e.* in passing to the sums of each order from those of the preceding order we introduce a new arbitrary constant. When we are dealing with a definite set of quantities $u_0, u_1, u_2, \ldots, u_l$, the differences of which are constructed in a definite way, it is convenient to have definite values for the corresponding complete sums of the 1st, 2nd, ... order.

11. A principle for obtaining such values is given by the theory of reciprocal relation, such as holds in the case of contragredient quantities. If

$$u_\mu \equiv u_0, u_1, \ldots, u_l, \qquad y_\mu \equiv y_0, y_1, \ldots, y_l,$$

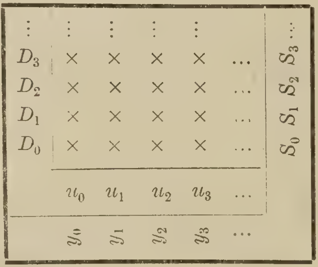
$$D_\mu \equiv D_0, D_1, \ldots, D_l, \qquad S_\mu \equiv S_0, S_1, \ldots, S_l,$$

are four sets of variable quantities, each of the D's being a linear function of the u's, and each of the S's a linear function of the y's, then the relation of the u's to the D's is the *reciprocal* of the relation of the y's to the S's if the coefficients in the linear functions are such that these relations are of the form ($p = 0, 1, 2, \ldots, l$)

$$u_p = \sum_t a_{pt} D_t, \qquad S_p = \sum_t a_{tp} y_t,$$

where $\sum_t$ denotes summation for $t = 0, 1, 2, \ldots, l$; or, in the tensor notation, if

$$u_\mu / D_\nu = S_\nu / y_\mu.$$



$\vdots$
D_3 × × × × ...
D_2 × × × × ...
D_1 × × × × ...
D_0 × × × × ...
u_0 u_1 u_2 u_3 ...
y_0 y_1 y_2 y_3 ...
S_0 S_1 S_2 S_3 ...

The relation is as shown in the diagram in the margin, in which the crosses represent coefficients of the D's in the u's and of the y's in the S's, or of the u's in the D's and of the S's in the y's. We may describe the u's and the y's, or the D's and the S's, as corresponding reciprocally to each other.

The important fact, for our purpose, is that, when the D's are differences or linear functions of differences of the u's, the S's are linear functions of sums of the y's; and conversely that we can obtain a set of D's to be linear functions of the u's and to correspond in this way to a set of arbitrary linear functions of the sums of the y's. We can quite properly describe these S's and D's as, in an extended sense, sets of sums and differences.

I will take first the simple case of the definite sums corresponding to advancing differences, where the u's are treated as if they corresponded to successive integral values of some variable; then the definite sums corresponding to divided or adjusted differences—central sums being a particular case of these; and then the differences corresponding to moments, which are particular linear functions of sums. In connection with the second of these, we can see what light is thrown on the meaning of indefinite sums for adjusted differences.

12. When we have found two sets which correspond reciprocally, in the sense described above, we sometimes want to know how one is altered by making slight changes in the other. For our present purpose we only require the following simple cases.*

(i) If a term D_q is altered to cD_q, where c is a constant, then S_q is altered to S_q/c.

(ii) If D_r is altered to D_r+cD_t, then S_t is altered to S_t-cS_r.

Any replacement of the D's by linear functions of the D's is equivalent to a combination of processes of these two kinds.

(a) *Definite sums corresponding to advancing differences.*

13. In the ordinary theory the summation operator Σ is the inverse (with an arbitrary constant) of Δ, *i.e.* $\Delta\Sigma u_q = u_q$; and, except for arbitrary constants, the table of differences and sums of the u's is the same as that which would be obtained from the set of differences or sums, of the u's, of any order. The diagram in § 5, for instance, might be regarded as giving part of a differenced table of ... $\Sigma^3 u_q$, $\Sigma^3 u_{q+1}$, ..., or of a summed table of ... $\Delta^3 u_{q-3}$, $\Delta^3 u_{q-2}$, ...

Taking the u's to be $u_0, u_1, u_2, \ldots, u_l$, the advancing differences

$$D_0 \equiv u_0, \quad D_1 \equiv \Delta u_0, \quad D_2 \equiv \Delta^2 u_0, \quad \ldots, \quad D_l \equiv \Delta^l u_0,$$

are connected with them by the relations

$$\left.\begin{aligned} u_0 &= D_0, \\ u_1 &= D_0+D_1, \\ u_2 &= D_0+2D_1+D_2, \\ u_3 &= D_0+3D_1+3D_2+D_3, \\ &\text{etc.} \end{aligned}\right\} \tag{13 . 1}$$

---

* The general theorem, expressed in the tensor notation, is that, if D_μ and S_μ correspond reciprocally, then so also do $p_{\mu\nu} D_\nu$ and $p^{\mu\nu} S_\nu$; where, if P is the determinant whose elements are $p_{\mu\nu}$, and P_{rt} is the cofactor of p_{rt}, then $p^{rt} \equiv P_{rt}/P$.

Entering the coefficients in the diagram of § 11, we find that

$$\left.\begin{aligned} S_0 &= y_0+y_1+\ y_2+\ y_3+\ y_4+\ldots+y_l, \\ S_1 &= \qquad y_1+2y_2+3y_3+4y_4+\ldots+ly_l, \\ S_2 &= \qquad\qquad y_2+3y_3+6y_4+\ldots+\tfrac{1}{2}l(l-1)\,y_l, \\ &\vdots \\ S_g &= \sum_t (t,\,g)\,y_t, \end{aligned}\right\} \qquad (13\,.\,2)$$

where $\sum_t$ means summation for $t = 0, 1, 2, \ldots, l$.

These S's cannot be obtained by any simple summation in the ordinary way; for, if we took successive sums y_0, y_0+y_1, $y_0+y_1+y_2$, ..., the last of which is S_0, the sums of the next order would be

$$y_0,\ 2y_0+y_1,\ 3y_0+2y_1+y_2,\ \ldots,$$

and the last but one of these would be

$$ly_0+(l-1)y_1+(l-2)\,y_2+\ldots+1\,.\,y_{l-1}.$$

The coefficients in this latter expression are the same as the coefficients in S_1, but in the reverse order. Examining S_2, S_3, ... in the same way, however, it will be seen that the S's can be obtained by successive summations in the direction y_l to y_0. If we introduce an operator Σ'' such that

$$\left.\begin{aligned} \Sigma'' y_r &\equiv y_r+y_{r+1}+\ldots+y_l, \\ \Sigma''^2 y_r &\equiv \Sigma'' y_r+\Sigma'' y_{r+1}+\ldots+\Sigma'' y_l, \\ &\text{etc.,} \end{aligned}\right\} \qquad (13\,.\,3)$$

so that

$$\Delta\Sigma'' = -1, \qquad (13\,.\,4)$$

$$\Sigma''^p y_{l+1} = 0, \qquad (13\,.\,5)$$

we shall have

$$\Sigma''^{g+1} y_r = y_r+(g+1,\,g)\,y_{r+1}+(g+2,\,g)\,y_{r+2}+\ldots+(g+l-r,\,g)\,y_l, \quad (13\,.\,6)$$

and

$$S_g = \Sigma''^{g+1} y_g. \qquad (13\,.\,7)$$

14. By taking Σ'' to be such that $\Delta\Sigma'' = -1$, we are breaking away from the continuity of the diagram in § 5. We can, however, preserve continuity by beginning our summations with $-S_0$, $+S_1$, $-S_2$, $+S_3$, ..., instead of ending with $+S_0$, $+S_1$, $+S_2$, $+S_3$, ... For six y's, for in-

stance, we should have a diagram as follows, in which the crosses represent intermediate sums.

$$
\begin{array}{cccccccccccc}
 & & & & & -S_0 & & & & & & \\
 & & & & S_1 & & y_0 & & & & & \\
 & & & -S_2 & & \times & & \Delta y_0 & & & & \\
 & & S_3 & & \times & & y_1 & & \Delta^2 y_0 & & & \\
 & -S_4 & & \times & & \times & & \Delta y_1 & & \Delta^3 y_0 & & \\
S_5 & & \times & & \times & & y_2 & & \Delta^2 y_1 & & \Delta^4 y_0 & \\
 & \times & & \times & & \times & & \Delta y_2 & & \Delta^3 y_1 & & \Delta^5 y_0 \\
0 & & \times & & \times & & y_3 & & \Delta^2 y_2 & & \Delta^4 y_1 & \\
 & 0 & & \times & & \times & & \Delta y_3 & & \Delta^3 y_2 & & \\
 & & 0 & & \times & & y_4 & & \Delta^2 y^3 & & & \\
 & & & 0 & & \times & & \Delta y_4 & & & & \\
 & & & & 0 & & y_5 & & & & & \\
 & & & & & 0 & & & & & &
\end{array}
$$

The values of the S's are determined by the position of the 0's, as shown.

15. Corresponding to the relations (13 . 2), we have

$$
\left.\begin{aligned}
y_0 &= S_0 - S_1 + S_2 - S_3 + S_4 - \ldots + (-1,\ l) S_l, \\
y_1 &= S_1 - 2S_2 + 3S_3 - 4S_4 + \ldots + (-2,\ l-1) S_l, \\
y_2 &= S_2 - 3S_3 + 6S_4 - \ldots + (-3,\ l-2) S_l, \\
y_3 &= S_3 - 4S_4 + \ldots + (-4,\ l-3) S_l, \\
&\vdots \\
y_r &= \sum_g (-r-1,\ g-r) S_g.
\end{aligned}\right\} \qquad (15\,.\,1)
$$

This last formula may seem to conflict with (13 . 7), which, taken with (13 . 4), looks as if it ought to give $y_r = (-)^r \Delta^{r+1} S_r$. But what (13 . 4) gives is $y_r = (-)^{t+1} \Delta^{t+1} \Sigma''^{t+1} y_r$, which is a different thing.

(*b*) *Definite sums corresponding to divided or adjusted differences.*

16. (*a*) For divided differences we take D_0, D_1, D_2, ... to be u_0, ∇u_0, $\nabla^2 u_0$, ... The values of the u's in terms of the D's are then given by (3 . 4). Substituting in the diagram of § 11, we find that

$$
\left.\begin{aligned}
S_0 &= y_0 + y_1 + y_2 + \ldots + y_l, \\
S_1 &= (x_1 - x_0) y_1 + (x_2 - x_0) y_2 + \ldots + (x_l - x_0) y_l \\
&= \sum_{t=1}^{t=l} (x_t - x_0) y_t, \\
&\quad \text{etc.}
\end{aligned}\right\} \qquad (16\,.\,1)
$$

the general formula being

$$S_g = \sum_{t=g}^{t=l} (x_t - x_0)(x_t - x_1) \ldots (x_t - x_{g-1}) y_t. \tag{16.2}$$

Conversely, by (3 . 3), the y's are given in terms of the S's by

$$\left.\begin{array}{ll}
y_0 = S_0 + \dfrac{S_1}{x_0 - x_1} + \dfrac{S_2}{(x_0-x_1)(x_0-x_2)} + \dfrac{S_3}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + \ldots, \\
y_1 = \qquad \dfrac{S_1}{x_1 - x_0} + \dfrac{S_2}{(x_1-x_0)(x_1-x_2)} + \dfrac{S_3}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + \ldots, \\
y_2 = \qquad \dfrac{S_2}{(x_2-x_0)(x_2-x_1)} + \dfrac{S_3}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \ldots, \\
y_3 = \qquad \dfrac{S_3}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} + \ldots, \\
\qquad \text{etc.}
\end{array}\right\} \tag{16.3}$$

(b) Similarly for adjusted differences we find that

$$S_g = \sum_{t=g}^{t=l} (x_t - x_0)(x_t - x_1) \ldots (x_t - x_{g-1}) y_t / g!. \tag{16.4}$$

We might have obtained (16 . 4) from (16 . 2) by means of (i) of § 12.

17. As an example, take the case of central differences for an odd number of u's. We will call these $u_{-n}, u_{-n+1}, \ldots, u_n$, but will take them in the order $u_0, u_1, u_{-1}, u_2, u_{-2}, \ldots, u_n, u_{-n}$. For the D's (divided differences) we notice that, as has been pointed out in § 3, the value of $\nabla^r u_0$ is independent of the order in which the u's in it are taken, so that, *e.g.*, the fourth divided difference is the same as that of $u_{-2}, u_{-1}, u_0, u_1, u_2$, *i.e.* is $\delta^4 u_0/4!$. The S's are obtained from (16 . 2) by replacing $x_0, x_1, x_2, x_3, x_4, \ldots$ by $0, 1, -1, 2, -2, \ldots$. Hence we find that the reciprocally corresponding sets are

$$u_0, \quad \delta u_{\frac{1}{2}}/1!, \quad \delta^2 u_0/2!, \quad \delta^3 u_{\frac{1}{2}}/3!, \quad \delta^4 u_0/4!, \quad \ldots,$$

and $\Sigma y_r, \quad \Sigma r y_r, \quad \Sigma r(r-1) y_r, \quad \Sigma r(r^2-1^2) y_r, \quad \Sigma r(r^2-1^2)(r-2) y_r, \quad \ldots.$

From this, by means of § 12, it is easily deduced that, if the D's are

$$u_0, \; \mu\delta u_0, \; \delta^2 u_0, \; \mu\delta^3 u_0, \; \ldots,$$

then the S's are given by

$$S_{2h} = \sum_{r=-n}^{r=n} [r,\, 2h) y_r, \qquad S_{2h-1} = \sum_{r=-n}^{r=n} (r,\, 2h-1] y_r.$$

This might have been obtained directly from the formula

$$u_r = u_0 + (r,\, 1] \mu\delta u_0 + [r,\, 2)\delta^2 u_0 + (r,\, 3]\mu\delta^3 u_0 + [r,\, 4)\delta^4 u_0 + \ldots .$$

18. We have next to express the S's, as found in § 16, in terms of sums, definite or indefinite, of the y's. We will take adjusted differences and begin with the definite sums corresponding to those introduced in § 13.

(*a*) In the case of advancing differences we took

$$\Sigma''^{g+1} y_r \equiv \Sigma''^{g} y_r + \Sigma''^{g} y_{r+1} + \ldots + \Sigma''^{g} y_l, \qquad \Sigma''^{0} y_r \equiv y_r,$$

and found that

$$\Sigma''^{g+1} y_r = \sum_{t=r}^{t=l} (t-r+g,\, g)\, y_t$$

$$= \sum_{t=r}^{t=l} (t-r+g)(t-r+g-1)\ldots(t-r+1) y_t/g!,$$

$$S_g = \Sigma''^{g+1} y_g = \sum_{t=g}^{t=l} t(t-1)\ldots(t-g+1) y_t/g!.$$

(*b*) For adjusted differences we have, by (16 . 4),

$$S_g = \sum_{t=g}^{t=l} (x_t - x_0)(x_t - x_1)\ldots(x_t - x_{g-1}) y_t/g!.$$

This suggests that, with a corresponding notation, we should take

$$Z''^{g+1} y_r = \sum_{t=r}^{t=l} (x_t - x_{r-g})(x_t - x_{r-g+1})\ldots(x_t - x_{r-1}) y_t/g!, \tag{18.1}$$

so that

$$S_g = Z''^{g+1} y_g. \tag{18.2}$$

(*c*) This would give

$$Z''^{g+1} y_{r+1} - Z''^{g+1} y_r = (x_{r-g} - x_r) \sum_{t=r}^{t=l} (x_t - x_{r-g+1})(x_t - x_{r-g+2})\ldots(x_t - x_{r-1}) y_t/g!$$

$$= (x_{r-g} - x_r)/g \,.\, Z''^{g} y_r, \tag{18.3}$$

$$Z''^{g+1} y_{l+1} = 0, \tag{18.4}$$

so that

$$Z''^{g+1}y_r = \{(x_r - x_{r-g})Z''^g y_r + (x_{r+1} - x_{r-g+1})Z''^g y_{r+1} + \dots + (x_l - x_{l-g})Z''^g y_l\}/g. \quad (18.5)$$

This is for values of g exceeding 0. For $g = 0$, (18 . 1) would still hold, and would give

$$Z''y_r = \sum_{t=r}^{t=l} y_t = y_r + y_{r+1} + \dots + y_l. \quad (18.6)$$

We may take (18 . 6) and (18 . 5) as defining $Z''^{g+1}y_r$ and as leading to (18 . 3), (18 . 4) and (18 . 1). The value of S_g is then given by (18 . 2).

(*d*) We can write (18 . 3) in the form

$$Z''^g y_r = -\Theta Z''^{g+1} y_r, \quad (18.7)$$

so that

$$\Theta Z'' = -1. \quad (18.8)$$

19. For indefinite sums, (18 . 6) suggests that we should take

$$Zy_r = \text{const.} + y_0 + y_1 + y_2 + \dots + y_{r-1}, \quad (19.1)$$

so that

$$Zy_{r+1} - Zy_r = y_r. \quad (19.2)$$

This, taken with (9 . 3), in the form

$$Z^{g+1}y_{r+1} = Z^{g+1}y_r + (x_r - x_{r-g})/g \,.\, Z^g y_r, \quad (19.3)$$

enables us to build up the successive sums, introducing a new arbitrary constant into those of each order. The sums and differences will then form a system similar to that of § 5, each quantity in the diagram being the adjusted difference, as defined in §§ 5 and 8, of the two adjacent to it on its left. The principle of the diagram in § 14 will still hold: if the arbitrary constants are chosen so as to produce 0's as shown in that diagram, the resulting sums $-S_0$, $+S_1$, $-S_2$, $+S_3$, ... will be as shown, the S's being the quantities which correspond reciprocally to the adjusted differences.

(3) *Differences corresponding to symmetrical sums.*

20. We are accustomed to think of differences as being found successively, so that, if we start with u_0, the first difference involves u_0 and u_1, the second involves these and u_2, and so on. These differences are

unsymmetrical, and so also are the corresponding sums. In the case of divided or adjusted differences, the asymmetry is mitigated by the fact that the expression for any difference is independent of the order in which the particular u's which enter into it are arranged; but, even so, it is only the last difference that is completely symmetrical. On the other hand, we are familiar with moments, which are symmetrical sums, in the extended sense in which we use this word. The question therefore arises whether we cannot find a set of quantities which shall correspond to a set of symmetrical sums, such as the moments. If we can, we may properly call these quantities differences; the case of central differences, in which the first difference may cover three u's, or the second difference four u's, supplies a precedent.

21. Where the number of u's is small, we could do this by means of the two theorems of § 12. Suppose, for instance, that we have u_0, u_1, u_2, corresponding to x_0, x_1, x_2. Then the differences are u_0, ∇u_0, $\nabla^2 u_0$, and the corresponding sums—denoting the ordinary moments of the y's by M_0, M_1, M_2—are

$$S_0 = y_0+y_1+y_2 = M_0, \qquad S_1 = (x_1-x_0)\,y_1+(x_2-x_0)\,y_2 = M_1-x_0M_0,$$

$$S_2 = (x_2-x_0)(x_2-x_1)y_2 = M_2-(x_0+x_1)M_1+x_0x_1M_0\,;$$

and hence we find that the quantities corresponding reciprocally to M_0, M_1, M_2, are

$$u_0-x_0\nabla u_0+x_0x_1\nabla^2u_0, \quad \nabla u_0-(x_0+x_1)\nabla^2u_0, \quad \nabla^2u_0.$$

Denoting these by ∇_0', ∇_1', ∇_2', we find that

$$\nabla_0' = \frac{x_1x_2}{(x_0-x_1)(x_0-x_2)}u_0+\frac{x_0x_2}{(x_1-x_0)(x_1-x_2)}u_1+\frac{x_0x_1}{(x_2-x_0)(x_2-x_1)}u_2,$$

$$\nabla_1' = -\frac{x_1+x_2}{(x_0-x_1)(x_0-x_2)}u_0-\frac{x_0+x_2}{(x_1-x_0)(x_1-x_2)}u_1-\frac{x_0+x_1}{(x_2-x_0)(x_2-x_1)}u_2,$$

$$\nabla_2' = \frac{1}{(x_0-x_1)(x_0-x_2)}u_0+\frac{1}{(x_1-x_0)(x_1-x_2)}u_1+\frac{1}{(x_2-x_0)(x_2-x_1)}u_2.$$

This process, however, is tedious, and we require a general formula.

22. (*a*) For convenience of printing, we take the x's to be $\alpha, \beta, \gamma, \ldots, \eta, \theta$;

and we denote the moments of the y's by M_0, M_1, M_2, ..., M_l, so that

$$\begin{aligned} M_0 &= y_0 + y_1 + y_2 + \dots + y_{l-1} + y_l, \\ M_1 &= \alpha y_0 + \beta y_1 + \gamma y_2 + \dots + \eta y_{l-1} + \theta y_l, \\ M_2 &= \alpha^2 y_0 + \beta^2 y_1 + \gamma^2 y_2 + \dots + \eta^2 y_{l-1} + \theta^2 y_l, \\ &\vdots \\ M_l &= \alpha^l y_0 + \beta^l y_1 + \gamma^l y_2 + \dots + \eta^l y_{l-1} + \theta^l y_l. \end{aligned}$$

Let the quantities which correspond reciprocally to the M's be

$$\Delta_0,\ \Delta_1,\ \Delta_2,\ \dots,\ \Delta_l.$$

Then, by substitution in the diagram of § 11, it will be seen that the Δ's are given by the equations

$$\left.\begin{aligned} u_0 &= \Delta_0 + \alpha\Delta_1 + \alpha^2\Delta_2 + \dots + \alpha^l\Delta_l, \\ u_1 &= \Delta_0 + \beta\Delta_1 + \beta^2\Delta_2 + \dots + \beta^l\Delta_l, \\ u_2 &= \Delta_0 + \gamma\Delta_1 + \gamma^2\Delta_2 + \dots + \gamma^l\Delta_l, \\ &\vdots \\ u_{l-1} &= \Delta_0 + \eta\Delta_1 + \eta^2\Delta_2 + \dots + \eta^l\Delta_l, \\ u_l &= \Delta_0 + \theta\Delta_1 + \theta^2\Delta_2 + \dots + \theta^l\Delta_l. \end{aligned}\right\} \qquad (22.1)$$

(*b*) The solution of these equations is

$$(-)^{l-f}\Delta_f = \frac{S_{l-f,\alpha}}{(\alpha-\beta)(\alpha-\gamma)\dots(\alpha-\theta)}u_0 + \frac{S_{l-f,\beta}}{(\beta-\alpha)(\beta-\gamma)\dots(\beta-\theta)}u_1 + \dots$$
$$+ \frac{S_{l-f,\theta}}{(\theta-\alpha)(\theta-\beta)\dots(\theta-\eta)}u_l, \qquad (22.2)$$

where $S_{r,\epsilon}$ denotes the sum of the products, r at a time, of $\alpha, \beta, \gamma, \dots, \theta$, with the omission of ϵ, no letter occurring more than once in any product; $S_{0,\epsilon}$ being 1.

(*c*) If we write

$$v_0 \equiv \frac{u_0}{(\alpha-\beta)(\alpha-\gamma)\dots(\alpha-\theta)}, \quad v_1 \equiv \frac{u_1}{(\beta-\alpha)(\beta-\gamma)\dots(\beta-\theta)}, \quad \dots$$
$$v_l \equiv \frac{u_l}{(\theta-\alpha)(\theta-\beta)\dots(\theta-\eta)},$$

then
$$(-)^{l-f}\Delta_f = S_{l-f,\alpha}v_0 + S_{l-f,\beta}v_1 + \dots + S_{l-f,\theta}v_l. \qquad (22.3)$$

If S_r denotes the sum of the products, as above, of all the $l+1$ x's, then

$$S_{r,\epsilon} = S_r - \epsilon S_{r-1} + \epsilon^2 S_{r-2} - \ldots + (-)^r \epsilon^r S_0.$$

Hence, if we denote the moments of the v's by $N_0, N_1, N_2, \ldots, N_l$, so that

$$\begin{aligned} N_0 &\equiv v_0 + v_1 + v_2 + \ldots + v_{l-1} + v_l, \\ N_1 &\equiv \alpha v_0 + \beta v_1 + \gamma v_2 + \ldots + \eta v_{l-1} + \theta v_l, \\ N_2 &\equiv \alpha^2 v_0 + \beta^2 v_1 + \gamma^2 v_2 + \ldots + \eta^2 v_{l-1} + \theta^2 v_l, \\ &\vdots \\ N_l &\equiv \alpha^l v_0 + \beta^l v_1 + \gamma^l v_2 + \ldots + \eta^l v_{l-1} + \theta^l v_l, \end{aligned}$$

then $$\Delta_f = S_0 N_{l-f} - S_1 N_{l-f-1} + S_2 N_{l-f-2} + \ldots + (-)^{l-f} S_{l-f} N_0. \quad (22.4)$$

NEUER BEWEIS DES ALLGEMEINEN KRONECKERSCHEN APPROXIMATIONSSATZES

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Communicated by G. H. HARDY.

[Received December 23rd, 1921.—Read January 12th, 1922.]

I. *Eindimensional.*

1. a sei eine irrationale Zahl. A sei der Punkt 0, B der Punkt 1 einer X-Achse. Wir markieren auf der Einheitsstrecke AB die Punkte P_n mit den Koordinaten $na-[na]$, $n=0, 1, 2, \ldots$, und bezeichnen die unendliche Punktfolge $P_0=A$, P_1, P_2, ... als entstanden "durch zirkuläre Abtragung der Zahl a auf der Strecke AB von A aus". (Denn denken wir die Strecke AB zu einem Kreis zusammengebogen, sodass B an A geheftet ist, so entsteht diese Punktmenge durch beständige und in gleichbleibendem Sinne vorgenommene Abtragung der Länge $|a|$ von A aus auf der Peripherie.)

2. Da a irrational ist, fällt niemals ein Punkt der Menge auf einen schon vorhandenen; ausser A fallen alle Punkte der Menge ins Innere der Strecke AB.

3. Die Strecke P_nP_{n+r} $(n \geqslant 0,\ r > 0)$ werde als Vektor aufgefasst. Setzt man diesen Vektor an irgend einen Punkt P_m der Menge oder an B im richtigen Sinne an (greift er über A oder B hinaus, so ist das Reststück an B bzw. A anzusetzen), so gibt die Spitze des Vektors wieder einen Punkt der Menge an, nämlich den Punkt P_{m+r}.

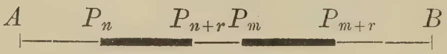

FIG. 1.

Beweis: Aus der Deutung auf dem Kreise ist klar, dass das Vorrücken, welches durch r Abtragungen bewirkt wird, unabhängig davon ist, bei welchem Punkt begonnen wurde.

4. *Jeder Punkt der Strecke AB ist Häufungspunkt der Menge.*

Beweis: $\delta > 0$ vorgegeben. Nach 2. hat die Menge, weil beschränkt, mindestens einen Häufungspunkt; es gibt also einen Vektor $P_n P_{n+r}$, dessen Länge $< \delta$ ist. Diesen Vektor setzen wir an A an, wenn er nach rechts, an B, wenn er nach links gerichtet ist, und sodann beständig wieder an sich selbst, bis das ganze Intervall AB durchschritten ist. Dadurch ist AB mit Punkten der Menge so belegt, dass in jedem Teilintervall der Länge δ mindestens ein Punkt der Menge liegt. Da δ beliebig klein genommen werden kann, kommt man auf diese Weise jedem Punkte des Intervalles AB mit Punkten der Menge beliebig nahe.

II. *Zweidimensional.*

1. Die reellen Zahlen a und b seien linear unabhängig, d.h. es bestehe keine Beziehung der Form

$$m_1 a + m_2 b + m_3 = 0$$

mit ganzen, nicht sämtlich verschwindenden m_1, m_2, m_3. In einem rechtwinkligen Koordinatensystem (X, Y) mit dem Anfangspunkt A tragen wir auf der Einheitsstrecke AB_1 der X-Achse die Zahl a, auf der Einheitsstrecke AB_2 der Y-Achse die Zahl b zirkulär ab (im Sinne von I, 1.) und markieren zugleich im Einheitsquadrat $AB_1 CB_2$ die Punkte P_n, deren Projektionen die n-ten Punkte jener beiden Abtragungen sind ($n = 0, 1, 2, \ldots$).

2. Nach I, 2. fällt niemals ein Punkt der Menge auf einen schon vorhandenen; ausser $P_0 = A$ fallen alle Punkte P_n ins Innere des Einheitsquadrates.

3. Da für die Projektionen der Punktmenge auf die beiden Achsen I, 3. gilt, folgt dieselbe Aussage für Vektoren in der Ebene, mit der Modifikation, dass Vektoren, die über den Rand des Einheitsquadrates hinaus-

greifen, von den gegenüberliegenden Seiten her so fortzusetzen sind, als ob AB_1 an B_2C und AB_2 an B_1C geheftet sei.

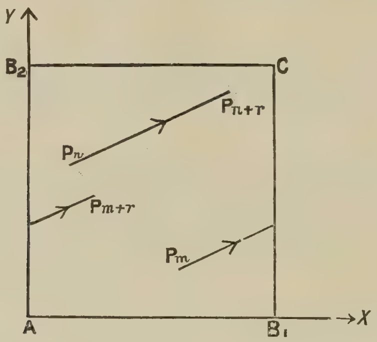

Fig. 2.

Zusatz : Speziell kann ein Vektor, der, an A angesetzt, nicht nach dem Inneren des Quadrates gehen würde, an demjenigen der Punkte B_1, C oder B_2 angesetzt werden, von welchem aus er nach innen geht.

4. *Jeder Punkt des Einheitsquadrates ist Häufungspunkt der Menge.*

Beweis : $\delta > 0$ vorgegeben. Nach II, 2. hat die Menge, weil beschränkt, mindestens einen Häufungspunkt; also gibt es unendlich viele Vektoren $P_n P_{n+r}$ von der Länge $< \delta$, wo ausserdem noch r beliebig gross genommen werden kann.

1. *Fall :* Alle diese Vektoren sind parallel, sei es gleich- oder ungleichsinnig. Wir tragen sie alle an P_1 an und haben dann eine Gerade durch P_1, auf welcher Punkte der Menge mit beliebig hohem Index liegen. P_x und P_y seien zwei von ihnen ($x > 1$, $y > 1$).

Anmerkung : Zur Vereinfachung der folgenden Formeln sei $0 < a < 1$ und $0 < b < 1$ vorausgesetzt; dies ist keine Einschränkung der Allgemeinheit, da die Zahlen $a' = a - [a]$ und $b' = b - [b]$ dieselben Voraussetzungen wie a und b erfüllen und die identische Punktmenge erzeugen.

Ferner sei $\delta <$ Min. $(a, b, 1-a, 1-b)$, damit alle diese an P_1 angesetzten Vektoren ungebrochen seien.

Es folgt

$$\begin{vmatrix} a & b & 1 \\ xa-[xa] & xb-[xb] & 1 \\ ya-[ya] & yb-[yb] & 1 \end{vmatrix} = 0$$

oder

$$\begin{vmatrix} a & b & 1 \\ [xa] & [xb] & x-1 \\ [ya] & [yb] & y-1 \end{vmatrix} = 0.$$

Der Koeffizient von a muss nach II, 1. verschwinden:

$$\begin{vmatrix} [xb] & x-1 \\ [yb] & y-1 \end{vmatrix} = 0$$

oder

$$\begin{vmatrix} [xb] & x-1 \\ \frac{[yb]}{y-1} & 1 \end{vmatrix} = 0.$$

Hierin $y \rightarrow \infty$:

$$\begin{vmatrix} [xb] & x-1 \\ b & 1 \end{vmatrix} = 0,$$

im Widerspruch mit der Irrationalität von b. Der 1. Fall kann also nicht eintreten.

2. *Fall:* Es gibt zwei nicht parallele Vektoren von der Länge $< \delta$. Wir tragen beide an P_1 an und konstruieren durch beständige Ansetzung nach II, 3. aus ihnen ein *Parallelogrammnetz* mit Seitenlängen $< \delta$, dessen Ecken Punkte der Menge sind und welches nach endlich vielen Ansetzungen das Quadrat ganz überdeckt (vgl. hiezu die Erklärung in III, 4, 2. Fall). Da δ beliebig klein genommen werden kann, kommt man auf diese Weise jedem Punkte des Einheitsquadrates mit Punkten der Menge beliebig nahe.

III. *Mehrdimensional.*

Das Beweisverfahren von II lässt sich (ohne Induktion) auf mehrdimensionale Räume übertragen.

1. Die reellen Zahlen $a_1, \ldots, a_p$ $(p \geqslant 2)$ seien linear unabhängig; d.h. es bestehe keine Beziehung der Form

$$m_1 a_1 + \ldots + m_p a_p + m_{p+1} = 0$$

mit ganzen, nicht sämtlich verschwindenden $m_1, \ldots, m_{p+1}$. In einem rechtwinkligen Koordinatensystem $(X_1, \ldots, X_p)$ des p-dimensionalen Raumes tragen wir auf der Einheitsstrecke AB_ν der X_ν-Achse die Zahl a_ν zirkulär ab (im Sinne von I, 1.) für $\nu = 1, \ldots, p$, und markieren zugleich im Einheitswürfel $0 \leqslant X_1, X_2, \ldots, X_p < 1$ die Punkte P_n, deren Projektionen die n-ten Punkte jener p Abtragungen sind ($n = 0, 1, 2, \ldots$).

2. Niemals fällt ein Punkt der Menge auf einen schon vorhandenen; ausser $P_0 = A$ fallen alle Punkte P_n ins Innere des Einheitswürfels.

3. Da für die Projektionen der Punktmenge I, 3. gilt, folgt dieselbe Aussage für Vektoren des p-dimensionalen Raumes mit sinngemässer Uebertragung der in II, 3. ausgesprochenen Modifikation.

4. *Jeder Punkt des p-dimensionalen Einheitswürfels ist Häufungspunkt der Menge.*

Beweis: $\delta > 0$ vorgegeben. Nach III, 2. hat die Menge, weil beschränkt, mindestens einen Häufungspunkt; es gibt also unendlich viele Vektoren von der Länge $< \delta$; wir tragen sie alle an P_1 an.

1. *Fall:* Sie liegen alle in einem (linearen) $(p-1)$-dimensionalen Raum durch P_1. Wir setzen wie in II, 4. $0 < a_\nu < 1$ und $\delta <$ Min. $(a_\nu, 1-a_\nu)$ für $\nu = 1, \ldots, p$ voraus. Dann besteht die Beziehung

$$\begin{vmatrix} a_1 & \ldots & a_p & 1 \\ x_1 a_1 - [x_1 a_1] & \ldots & x_1 a_p - [x_1 a_p] & 1 \\ \ldots & \ldots \quad \ldots & \ldots & \ldots \\ x_p a_1 - [x_p a_1] & \ldots & x_p a_p - [x_p a_p] & 1 \end{vmatrix} = 0$$

oder

$$\begin{vmatrix} a_1 & \ldots & 1 \\ [x_1 a_1] & \ldots & x_1 - 1 \\ \ldots & \ldots & \ldots \\ [x_p a_1] & \ldots & x_p - 1 \end{vmatrix} = 0$$

für gewisse natürliche Zahlen $x_1, \ldots, x_p$, die alle beliebig gross genommen werden können. Der Koeffizient von a_1 muss nach III, 1. verschwinden:

$$\begin{vmatrix} [x_1 a_2] & \ldots & x_1 - 1 \\ \ldots & \ldots & \ldots \\ [x_p a_2] & \ldots & x_p - 1 \end{vmatrix} = 0.$$

Hierin dividieren wir die erste Zeile durch $x_1 - 1$ und machen sodann den

Grenzübergang $x_1 \rightarrow \infty$ unter Festhaltung von $x_2, \ldots, x_p$; dies ergibt:

$$\begin{vmatrix} a_2 & \ldots & a_p & 1 \\ [x_2 a_2] & \ldots & [x_2 a_p] & x_2 - 1 \\ \ldots & \ldots & \ldots & \ldots \\ [x_p a_2] & \ldots & [x_p a_p] & x_p - 1 \end{vmatrix} = 0.$$

Jetzt wird genau dasselbe Verfahren so oft wiederholt, bis dasteht:

$$\begin{vmatrix} a_p & 1 \\ [x_p a_p] & x_p - 1 \end{vmatrix} = 0,$$

im Widerspruch zur Irrationalität von a_p. Der 1. Fall kann also nicht eintreten.

2. *Fall:* Es gibt unter den an P_1 angetragenen Vektoren der Länge $< \delta$ p solche, welche nicht in demselben $(p-1)$-dimensionalen Raume liegen. Ihre Verlängerungen zu Halbstrahlen bilden ein von P_1 ausgehendes p-dimensionales p-Kant. Wir setzen diese Vektoren beständig an sich selbst an in der Weise, dass an jeden so erhaltenen Punkt sämtliche p Vektoren wieder angesetzt werden. Dadurch wird das ganze Innere des p-Kantes *mit einem Netz von* (p*-dimensionalen*) *parallelogrammartigen Zellen überzogen,* deren sämtliche Eckpunkte Mengenpunkte sind (bzw. modulo 1 zu solchen kongruent, wenn die Ansetzungen aus dem Einheitswürfel herausgeführt haben) und deren sämtliche Kantenlängen $< \delta$ sind.

Wo dieses Netz aus dem Einheitswürfel austritt, soll es von der betr. Gegenseite her (d.h. modulo 1) wieder in ihn hinein fortgesetzt werden. Nach endlich vielen Ansetzungen ist der ganze Einheitswürfel mit diesem Netz überdeckt. Diese Tatsache ergibt sich aus folgender Ueberlegung:

Es ist gleichgültig, ob (wie bisher) der Einheitswürfel festgehalten wird und die Ansetzungen beim Heraustreten immer wieder an die Gegenseite überspringen, oder ob die Ansetzungen beständig in dem Raume des wahren p-Kantes weiter vor sich gehen und statt des Einheitswürfels der jeweils erreichte Gitterpunktswürfel betrachtet wird. Aus dem Begriff des p-Kantes ist aber klar, dass (von einer gewissen Entfernung von P_1 an) ganze Gitterpunktswürfel in seinem Innern liegen.

Da δ beliebig klein genommen werden kann, kommt man auf diese Weise jedem Punkte des Einheitswürfels mit Punkten der Menge beliebig nahe.

5. Mit der Aussage III, 4. ist der allgemeine Approximationssatz von Kronecker gewonnen, welcher bekanntlich so lautet:

Die reellen Zahlen $a_1, a_2, \ldots, a_p$ $(p \geqslant 1)$ seien linear unabhängig; d.h. es bestehe keine Beziehung der Form

$$m_1 a_1 + m_2 a_2 + \ldots + m_p a_p + m_{p+1} = 0$$

mit ganzen, nicht sämtlich verschwindenden $m_1, m_2, \ldots, m_{p+1}$. $b_1, b_2, \ldots, b_p$ seien reelle Zahlen; ω und ϵ positive Zahlen. Dann gibt es eine ganze Zahl $t > \omega$ und ganze Zahlen $n_1, n_2, \ldots, n_p$ derart, dass folgende Ungleichungen erfüllt sind:

$$\begin{gathered} |ta_1 - b_1 - n_1| < \epsilon, \\ |ta_2 - b_2 - n_2| < \epsilon, \\ \ldots \quad \ldots \quad \ldots \quad \ldots \\ |ta_p - b_p - n_p| < \epsilon. \end{gathered}$$

IV. *Der Fall linearer Abhängigkeit.*

Es sei noch der im Vorhergehenden ausgeschlossene Fall erörtert, dass die Zahlen $a_1, \ldots, a_p$ *nicht* linear unabhängig sind. Damit ist jetzt zugelassen, dass einige oder alle von ihnen rational sind; denn Rationalsein von a_μ ist gleichbedeutend mit einer Beziehung der Form

$$m_1 a_\mu + m_2 = 0,$$

wo m_1 und m_2 ganz und nicht beide $= 0$ sind.

Zur Abkürzung bezeichnen wir mit L_ν (-Menge) einen ν-dimensionalen linearen Unterraum unseres p-dimensionalen Raumes (also mit L_p letzteren selbst); als "L_ν-Stück des Einheitswürfels" das Schnittgebilde desselben mit einer L_ν; mit L_ν^* eine Gesamtheit endlich vieler paralleler und äquidistanter L_ν-Stücke des Einheitswürfels; als "geschlossene L_ν^*" eine solche L_ν^*, welche bei Fortsetzung modulo 1 im Einheitswürfel (vergl. III, 4.) keine neuen Stücke mehr liefert. (Beispiel mit $p = 2$, $\nu = 1$, s. Fig. 3.)

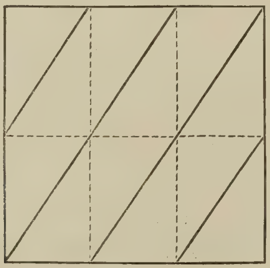

FIG. 3.

Dann gilt über die Verteilung der Punktmenge P_n im Einheitswürfel der folgende allgemeine Satz:

Bestehen zwischen den reellen Zahlen $a_1, \ldots, a_p$ *genau* $q \, (0 \leqslant q \leqslant p)$ *von einander unabhängige Beziehungen der Form*

$$m_1 a_1 + \ldots + m_p a_p + m_{p+1} = 0,$$

jede mit ganzen, nicht sämtlich verschwindenden $m_1, \ldots, m_{p+1}$, *so liegen die Punkte* $P_n \, (n \geqslant 0)$ *alle auf einer geschlossenen* L^*_{p-q} *und zwar überall dicht auf ihr.*

Der Fall $q = 0$ ist der Kroneckersche Satz. Das Rationalsein einiger a_μ beeinflusst den Charakter der Verteilung nicht, sondern hat lediglich zur Folge, dass die L_{p-q}-Stücke auf den betreffenden X_μ-Achsen senkrecht stehen.

Im Fall $q = p$ sind die $a_1, \ldots, a_p$ alle rational und L^*_0 ist eine endliche Punktmenge.

Beweis:

$$m_1 a_1 + \ldots + m_p a_p + m_{p+1} = 0 \tag{1}$$

sei irgend eine der q linearen Beziehungen. Wir legen für jedes $n \geqslant 0$ durch den Punkt P_n die L_{p-1}:

$$m_1 \{X_1 - (na_1 - [na_1])\} + \ldots + m_p \{X_p - (na_p - [na_p])\} = 0.$$

Das konstante Glied dieser Gleichung ist für alle n beschränkt, nämlich absolut

$$< |m_1| + \ldots + |m_p|,$$

und wegen (1) eine ganze Zahl, kann also nur endlich viele Werte annehmen. Dies besagt: Alle P_n liegen auf endlich vielen parallelen L_{p-1}-Stücken.

Dies gilt für jede der q unabhängigen linearen Beziehungen, woraus folgt, dass alle P_n auf endlich vielen parallelen L_{p-q}-Stücken liegen.

Um zu zeigen, dass auf jedem dieser Stücke die Punkte P_n überall dicht liegen, genügt es darzutun, dass nicht alle an P_1 angetragenen Vektoren der Länge $< \delta$ (dasselbe δ wie in III) in einer einzigen L_{p-q-1} durch P_1 liegen. Angenommen dies sei der Fall. Dann folgt durch Projektion in den $(X_1, \ldots, X_{p-q})$-Raum (indem ohne Beschränkung der Allgemeinheit $0 \leqslant a_1, \ldots, a_p < 1$ angenommen ist), dass die Beziehung

$$\begin{vmatrix} a_1 & \ldots & a_{p-q} & 1 \\ x_1 a_1 - [x_1 a_1] & \ldots & x_1 a_{p-q} - [x_1 a_{p-q}] & 1 \\ \ldots & \ldots & \ldots & \ldots \\ x_{p-q} a_1 - [x_{p-q} a_1] & \ldots & x_{p-q} a_{p-q} - [x_{p-q} a_{p-q}] & 1 \end{vmatrix} = 0$$

für gewisse natürliche Zahlen $x_1, \ldots, x_{p-q}$ besteht, die alle beliebig gross genommen werden können. Daraus folgt mindestens eine lineare Beziehung (immer ist gemeint: mit ganzen, nicht sämtlich verschwindenden Koeffizienten) zwischen $a_1, \ldots, a_{p-q}$; denn bestünde keine solche, so wären $a_1, \ldots, a_{p-q}$ sicher irrational und man käme durch das Verfahren von III, 4, 1. Fall, zu einem Widerspruch.

a_{μ_1} komme in der Beziehung zwischen $a_1, \ldots, a_{p-q}$ wirklich vor. Dann erhalten wir ebenso durch Projektion in den $(X_1, \ldots, X_{p-q+1}$ ohne $X_{\mu_1})$-Raum eine lineare Beziehung zwischen $a_1, \ldots, a_{p-q+1}$ ohne a_{μ_1}. In dieser komme a_{μ_2} wirklich vor. Dann erhalten wir durch Projektion in den $(X_1, \ldots, X_{p-q+2}$ ohne $X_{\mu_1}, X_{\mu_2})$-Raum eine lineare Beziehung zwischen $a_1, \ldots, a_{p-q+2}$ ohne a_{μ_1} und a_{μ_2}.

Durch Fortsetzung dieses Verfahrens erhalten wir schliesslich $q+1$ sicher von einander unabhängige lineare Beziehungen zwischen $a_1, \ldots, a_p$, was der Voraussetzung des Satzes widerspricht. Also liegen die Punkte P_n auf den endlich vielen parallelen L_{p-q}-Stücken überall dicht.

Diese Stücke müssen äquidistant sein; denn sind P_r und P_s $(r < s)$ zwei Punkte auf verschiedenen Stücken, so muss der Vektor $P_r P_s$, an welchen Punkt P_n man ihn auch ansetzt, stets wieder in eines der Stücke münden. Also bilden die Stücke eine L^*_{p-q}. Dieselbe muss notwendig geschlossen sein; denn würde sie bei Fortsetzung modulo 1 ein neues Stück liefern, so könnte man auch durch geeignete Vektorenansetzung in dieses Stück hineingelangen, während doch alle Punkte P_n bereits auf den Stücken von L^*_{p-q} liegen.

Anmerkung zu I–IV.

Mit derselben elementaren Methode lässt sich auch beweisen, dass die Punktmenge P_n im Einheitswürfel (bzw. bei IV in der L^*_{p-q}) nicht nur überall dicht, sondern sogar *gleichmässig* (*gleichverteilt*) liegt. Für den Fall der linearen Unabhängigkeit (I–III) ist dies von H. Weyl bewiesen worden [*Math. Ann.*, Vol. 77 (1916), S. 319].

ANOTHER PROOF OF KRONECKER'S THEOREM

By H. BOHR.

[Received March 31st, 1922.—Read April 20th, 1922.]

MANY different proofs of Kronecker's Theorem have been given, among which the proof of Dr. Lettenmeyer, based on simple geometrical considerations, is a particularly beautiful one. In this note I give one more. This, like Weyl's proof of his extension of Kronecker's Theorem*, is based on the use of the function $e^{2\pi i x}$, the "invariant of the number x to modulus 1"; and has a simple connection with the applications I have made of Kronecker's Theorem to the theory of Dirichlet's series.†

Kronecker's Theorem may be stated as follows. Suppose that $\lambda_1 \dots \lambda_N$ are N linearly independent real numbers, $\phi_1 \dots \phi_N$ are N arbitrary real numbers, and ϵ is positive. Then there exist N integers $h_1 \dots h_N$ and a real number t such that

$$|t\lambda_n - \phi_n - h_n| < \epsilon \quad (n = 1, 2, \dots, N),$$

i.e. such that the N complex numbers

$$e^{2\pi i (t\lambda_n - \phi_n)} \quad (n = 1, 2, \dots, N)$$

all differ by less than ϵ_1 from $e^0 = 1$. In other words, the upper limit L_F of the numerical value of the function

$$F(t) = 1 + e^{2\pi i (t\lambda_1 - \phi_1)} + e^{2\pi i (t\lambda_2 - \phi_2)} + \dots + e^{2\pi i (t\lambda_N - \phi_N)},$$

for all real values of t, is equal to the upper limit $L_G = N+1$ of the numerical value of the function

$$G(x_1, x_2, \dots, x_N) = 1 + e^{2\pi i x_1} + e^{2\pi i x_2} + \dots + e^{2\pi i x_N},$$

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* H. Weyl, "Über die Gleichverteilung von Zahlen mod. Eins", *Math. Ann.*, Vol. 77 (1916), pp. 313–352.

† See, for example, H. Bohr, "Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der Dirichletschen Reihen $\Sigma a_n n^{-s}$", *Göttinger Nachrichten* (1913), pp. 441–488.

where $x_1, x_2, \ldots, x_N$ vary *independently* through the interval $0 \leqslant x \leqslant 1$. Evidently it is enough to prove that $F_F \geqslant L_G = N+1$.

We consider the power

$$\text{(1)} \qquad \{F(t)\}^p = \{1+e^{2\pi i (t\lambda_1-\phi_1)}+\ldots+e^{2\pi i (t\lambda_N-\phi_N)}\}^p,$$

where p is an arbitrary positive integer. In the polynomial development of (1) no two terms fall together, since—on account of the linear independence of the λ's—no two terms contains the same exponential $e^{2\pi i a t}$. In other words, the development (1) contains the same number of terms as the polynomial development of

$$\text{(2)} \qquad \{G(x_1, \ldots, x_N)\}^p = \{1+e^{2\pi i x_1}+\ldots+e^{2\pi i x_N}\}^p,$$

and these terms have the same numerical coefficients. Therefore, by a classical argument, the two mean values

$$F_p = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\{F(t)\}^p|^2 \, dt$$

and

$$G_p = \int_0^1 \int_0^1 \ldots \int_0^1 |\{G(x_1, x_2, \ldots, x_N)\}^p|^2 \, dx_1 \, dx_2 \ldots dx_N$$

are equal [their common value being the sum of the squares of the numerical coefficients in the development of (1) and (2)].

Now evidently (for reasons of continuity)

$$G_p^{1/2p} \to L_G (= N+1),$$

when $p \to \infty$, since the function $|G(x_1, x_2, \ldots, x_N)|$ really assumes its upper limit L_G in the point $(0, 0, \ldots, 0)$ of the N-dimensional unit-cube $0 \leqslant x_1 \leqslant 1, \ldots, 0 \leqslant x_N \leqslant 1$; and since $F_p = G_p$, we have

$$F_p^{1/2p} \to L_G.$$

But it is evident that $F_p^{1/2p} \leqslant L_F$ for all p; and therefore

$$L_F \geqslant L_G,$$

which proves the theorem.

It would of course be possible to prove that $F_p^{1/2p} \to N+1$ more directly, by a calculation based on the actual values of the coefficients; but the argument above avoids all calculation and brings out the real point more clearly.

ON A CLASS OF SURFACES IN EUCLIDEAN SPACE WHICH GENERATE AN EXPRESSION FOR THE SPACE TIME INTERVAL IN EINSTEIN'S GEOMETRY OF A PARTICULAR FORM

By J. E. CAMPBELL.

[Received December 12th, 1921.—Read January 12th, 1922.]

Introduction.

In an address, which I had the honour of giving to the London Mathematical Society, on "Einstein's Theory of Gravitation as an Hypothesis in Differential Geometry," I showed that in the fourfold continuum of time space, which I regarded as a four-way space, there were at every point four curves going out from the point. These curves have the property of being invariant for any transformation which leaves

$$a_{ik}\,dx_i\,dx_k$$

invariant.

I made the hypothesis that these curves are the intersections of a quadruply infinite system of higher surfaces—three-way loci. I then considered Einstein's Law of Gravitation, and combined it with my hypothesis, and showed that the space time interval could then be taken as

$$a^2\partial x_1^2+b^2\partial x_2^2+c^2\partial x_3^2+d^2\partial x_4^2,$$

where a, b, c, d are functions of the variables x_1, x_2, x_3, x_4 which satisfy the differential equations

$$a_{23}=a_2\frac{b_3}{b}+a_3\frac{c_2}{c},\quad a_{34}=a_3\frac{c_4}{c}+a_4\frac{d_3}{d},\quad a_{42}=a_4\frac{d_2}{d}+a_2\frac{b_4}{b};$$

$$b_{31}=b_3\frac{c_1}{c}+b_1\frac{a_3}{a},\quad b_{14}=b_1\frac{a_4}{a}+b_4\frac{d_1}{d},\quad b_{43}=b_4\frac{d_3}{d}+b_3\frac{c_4}{c};$$

$$c_{12}=c_1\frac{a_2}{a}+c_2\frac{b_1}{b},\quad c_{24}=c_2\frac{b_4}{b}+c_4\frac{d_2}{d},\quad c_{41}=c_4\frac{d_1}{d}+c_1\frac{a_4}{a};$$

$$d_{23}=d_2\frac{b_3}{b}+d_3\frac{c_2}{c},\quad d_{31}=d_3\frac{c_1}{c}+d_1\frac{a_3}{a},\quad d_{12}=d_1\frac{a_2}{a}+d_2\frac{b_1}{b};$$

$$bcK_{23}+\left(\frac{b_3}{c}\right)_3+\left(\frac{c_2}{b}\right)_2+\frac{b_1c_1}{a^2}+\frac{b_4c_4}{d^2}=0,$$

$$caK_{31}+\left(\frac{c_1}{a}\right)_1+\left(\frac{a_3}{c}\right)_2+\frac{a_2c_2}{b^2}+\frac{a_4c_4}{d^2}=0,$$

$$abK_{12}+\left(\frac{a_2}{b}\right)_2+\left(\frac{b_1}{a}\right)_1+\frac{a_3b_3}{c^2}+\frac{a_4b_4}{d^2}=0,$$

$$adK_{14}+\left(\frac{a_4}{d}\right)_4+\left(\frac{d_1}{a}\right)_1+\frac{a_2d_2}{b^2}+\frac{a_3d_3}{c^2}=0,$$

$$bdK_{24}+\left(\frac{b_4}{d}\right)_4+\left(\frac{d_2}{b}\right)_2+\frac{b_1d_1}{a^2}+\frac{b_3d_3}{c^2}=0,$$

$$cdK_{34}+\left(\frac{c_4}{d}\right)_4+\left(\frac{d_3}{c}\right)_3+\frac{c_1d_1}{a^2}+\frac{c_2d_2}{b^2}=0.$$

The suffixes denote differentiation when applied to the quantities a, b, c, d, but not when applied to K. The six curvatures in Riemann's sense are

$$K_{23}=K_{14}, \quad K_{31}=K_{24}, \quad K_{12}=K_{34},$$

and
$$K_{23}+K_{31}+K_{12}=0.$$

Whether the hypothesis I have made has any physical background I do not know, but the form given for the gravitation field due to a particle at rest is a very particular solution of my system of differential equations. In this paper I give the particular solution of the system of differential equations obtained by assuming that a, b, c, d depend only on the variables x_1 and x_2.

This solution, though only a particular one as regards the general problem, is yet much more general than the solution known as that of the gravitation field due to a particle at rest, and it includes that solution.

It is worthy of remark that the solution here given depends on a linear differential of the second order which occurs in the theory of sound, and that it is deduced from the differential geometry of a surface in ordinary Euclidean space.

Consider the equation

$$2(u+v)\beta_{12}+\beta_1+\beta_2=0,$$

which occurs in the theory of the propagation of sound. The solution was obtained by Riemann in a form suitable for physical application, and it is discussed by Darboux (Vol. 2, Chapters 3 and 4).

The equation may be written

$$\big(\beta_1(u+v)\big)_2+\big(\beta_2(u+v)\big)_1=0,$$

so that, when we know any integral β, we can by quadrature find a to satisfy

$$a_1=\beta_1(u+v), \quad a_2=-\beta_2(u+v).$$

We can therefore find, again by quadrature, θ so that

$$\big(a+\beta\,(u+v)\big)^2=1-2\theta_1(u+v),$$

$$\big(a-\beta\,(u+v)\big)^2=1-2\theta_2(u+v).$$

Consider now the surface on which the element of length is given by

$$ds^2=2e^{-\theta}du\,dv.$$

Let y be defined by the consistent equations

$$2y_1=a(u+v)^{-1}+\beta, \qquad 2y_2=a(u+v)^{-1}-\beta,$$

and x by $\frac{1}{2}\log(u+v)$.

Let the measure of curvature on the surface be denoted by $e^{3\rho}$: then

$$e^{3\rho}=e^{\theta}\theta_{12}.$$

Remembering that on *any* surface when the element of length is thrown into the form

$$ds^2=2h\,du\,dv,$$

we have, whatever be the functions ϕ and ψ,

$$\Delta_2(\phi)=2h^{-1}\phi_{12}, \qquad \Delta(\phi)=2h^{-1}\phi_1\phi_2,$$

$$\nabla(\phi,\,\psi)=h^{-1}(\phi_1\psi_2+\phi_2\psi_1).$$

We find that on the surface we are considering

$$(1) \qquad \begin{cases} \Delta_2(x)+2\Delta(x)=0, \qquad \Delta_2(y)+2\nabla(xy)=0, \\ \nabla(y,\,\Delta x)+\nabla(x,\,y)\big(\Delta(y)+3\Delta(x)\big)=0, \\ \nabla(x,\,\Delta x)+3\big(\Delta(x)\big)^2+2\big(\nabla(x,\,y)\big)^2-\Delta(y)\,\Delta(x)=0, \\ \Delta(y)-\Delta(x)-e^{3\rho}=0. \end{cases}$$

On this surface draw the system of curves

$$y=\text{constant}, \qquad x+\rho=\text{constant},$$

and take, as parametric curves, the curves bisecting the angles between the curves

$$y = \text{constant}, \qquad x+\rho = \text{constant}.$$

The equation of the bisectors of any systems of curves

$$\phi = \text{constant}, \qquad \psi = \text{constant},$$

on any surface are given by

$$\frac{d\phi^2}{\Delta(\phi)} = \frac{d\psi^2}{\Delta(\psi)}.$$

The square of the element of length on our surface may now be taken

$$ds^2 = a^2du^2+b^2dv^2,$$

and from the property of the bisectors we see that

$$x_1+\rho_1 = \sigma y_1, \qquad x_2+\rho_2 = -\sigma y_2,$$

where $$\sigma^2\Delta(y) = \Delta(x+\rho).$$

From $$\Delta_2(x)+2\Delta(x) = 0,$$

we see that $$\Delta_2 e^{2x} = 0,$$

so that e^{2x} is a potential function of the surface, and the lines $x =$ constant are the isotherms.

The function y is defined by the equations (1), which are of course invariant for any system of coordinates.

Let, as usual,

$$\frac{{}_{11}Z}{a^2} \equiv \frac{Z_{11}}{a^2} - \frac{\alpha_1 Z_1}{a^2} + \frac{\alpha_2 Z_2}{b^2},$$

$$\frac{{}_{22}Z}{b^2} \equiv \frac{Z_{22}}{b^2} - \frac{\beta_2 Z_2}{b^2} + \frac{\beta_1 Z_1}{a^2},$$

$${}_{12}Z \equiv Z_{12} - \alpha_2 Z_1 - \beta_1 Z_2.$$

Remembering that

$$\Delta_2(\phi) \equiv \frac{{}_{11}\phi}{a^2} + \frac{{}_{22}\phi}{b^2},$$

$$\Delta(\phi) \equiv \frac{\phi_1^2}{a^2} + \frac{\phi_2^2}{b^2}, \qquad \nabla(\phi, \psi) \equiv \frac{\phi_1\psi_1}{a^2} + \frac{\phi_2\psi_2}{b^2},$$

we see that $$\big(\Delta(Z)\big)_1 \equiv \frac{2Z_1}{a^2}\,{}_{11}Z + \frac{2Z_2}{b^2}\,{}_{12}Z,$$

$$(\Delta Z)_2 \equiv \frac{2Z_1}{a^2}\,{}_{12}Z + \frac{2Z_2}{b^2}\,{}_{22}Z\,;$$

and therefore from

$$\nabla(y,\, \Delta x)+\nabla(x,\, y)\Big(\Delta(y)+3\Delta(x)\Big)=0,$$

$$\nabla(x,\, \Delta x)+3\Big(\Delta(x)\Big)^2+2\Big(\nabla(x,\, y)\Big)^2-\Delta(y)\,\Delta(x)=0$$

we obtain

$$\frac{y_1\Big(\Delta(x)\Big)_1}{a^2}+\frac{y_2\Big(\Delta(x)\Big)_2}{b^2}+\nabla(x,\, y)\Big(\Delta(y)+3\Delta(x)\Big)=0,$$

$$\frac{x_1\Big(\Delta(x)\Big)_1}{a^2}+\frac{x_2\Big(\Delta(x)\Big)_2}{b^2}+3\Big(\Delta(x)\Big)^2+2\Big(\nabla(x,\, y)\Big)^2-\Delta(x)\,\Delta(y)=0.$$

From these two equations and from

$$\Delta_2(x)+2\Delta(x)=0,$$

we verify by simple algebra that

$$\frac{{}_{11}x}{a^2}+\frac{3x_1^2+y_1^2}{2a^2}+\frac{x_2^2-y_2^2}{2b^2}=0,$$

$$\frac{{}_{22}x}{b^2}+\frac{x_1^2-y_1^2}{2a^2}+\frac{3x_2^2+y_2^2}{2b^2}=0,$$

$${}_{12}x+x_1x_2+y_1y_2=0.$$

It follows that

$$\Big(\Delta(x)\Big)_1+x_1\Big(3\Delta(x)+\Delta(y)\Big)-\frac{2y_2}{b^2}(x_1y_2-x_2y_1)=0,$$

$$\Big(\Delta(x)\Big)_2+x_2\Big(3\Delta(x)+\Delta(y)\Big)+\frac{2y_1}{a^2}(x_1y_2-x_2y_1)=0.$$

By differentiation of

$$\Delta(y)=\Delta(x)+e^{3\rho},$$

we therefore obtain

$$\frac{2y_1}{a^2}\,{}_{11}y+\frac{2y_2}{b^2}\,{}_{12}y+x_1\Big(3\Delta(x)+\Delta(y)\Big)-\frac{2y_2}{b^2}(x_1y_2+x_2y_1)=3e^{3\rho}\rho_1,$$

$$\frac{2y_1}{a^2}\,{}_{12}y+\frac{2y_2}{b^2}\,{}_{22}y+x_2\Big(3\Delta(x)+\Delta(y)\Big)+\frac{2y_1}{a^2}(x_1y_2+x_2y_1)=3e^{3\rho}\rho_2.$$

From these equations and from

$$\Delta_2(y)+2\nabla(x, y)=0,$$

that is, from $$\frac{{}_{11}y}{a^2}+\frac{{}_{22}y}{b^2}+2\frac{x_1y_1}{a^2}+2\frac{x_2y_2}{b^2}=0,$$

we now can at once verify that, by aid of the equations

$$x_1+\rho_1=\sigma y_1, \qquad x_2+\rho_2=-\sigma y_2,$$

we must have $$\frac{{}_{11}y}{a^2}+\frac{2x_1y_1}{a^2}-\frac{3\sigma}{2}e^{3\rho}=0,$$

$$\frac{{}_{22}y}{b^2}+\frac{2x_2y_2}{b^2}+\frac{3\sigma}{2}e^{3\rho}=0,$$

$${}_{12}y+y_1x_2+y_2x_1.$$

We can now write down the following equations :—

$$\frac{x_{11}}{a^2}+\frac{3x_1^2+y_1^2}{2a^2}+\frac{x_2^2-y_2^2}{2b^2}-\frac{\alpha_1x_1}{a^2}+\frac{\alpha_2x_2}{b^2}=0,$$

$$\frac{y_{11}}{a^2}+\frac{2x_1y_1}{a^2}-\frac{\alpha_1y_1}{a^2}+\frac{\alpha_2y_2}{b^2}-\frac{3\sigma}{2}e^{3\rho}=0,$$

$$\frac{x_{22}}{b^2}+\frac{x_1^2-y_1^2}{2a^2}+\frac{3x_2^2+y_2^2}{2b^2}-\frac{\beta_2x_2}{b^2}+\frac{\beta_1x_1}{a^2}=0,$$

$$\frac{y_{22}}{b^2}+\frac{2x_2y_2}{b^2}-\frac{\beta_2y_2}{b^2}+\frac{\beta_1y_1}{a^2}+\frac{3\sigma}{2}e^{3\rho}=0,$$

$$x_{12}+x_1x_2+y_1y_2-\alpha_2x_1-\beta_1x_2=0,$$

$$y_{12}+y_1x_2+y_2x_1-\alpha_2y_1-\beta_1y_2=0,$$

$$\frac{x_1^2-y_1^2}{a^2}+\frac{x_2^2-y_2^2}{b^2}+e^{3\rho}=0,$$

and from the definition of $e^{3\rho}$, we have

$$ab\,e^{3\rho}+\left(\frac{a_2}{b}\right)_2+\left(\frac{b_1}{a}\right)_1=0.$$

We have $$a=e^{\alpha}, \qquad b=e^{\beta},$$

let $$c = e^{x+y}, \quad d = e^{x-y}, \quad r = e^{3\rho},$$

$$p = \frac{3\sigma - 1}{2} r, \quad q = -\frac{3\sigma+1}{2} r,$$

so that $$p+q+r = 0.$$

The above equations now give the following sets :—

$$c_{12} = c_1 \frac{a_2}{a} + c_2 \frac{b_1}{b},$$

$$d_{12} = d_1 \frac{a_2}{a} + d_2 \frac{b_1}{b},$$

$$bcp + \left(\frac{c_2}{b}\right)_2 + \frac{b_1 c_1}{a^2} = 0,$$

$$caq + \left(\frac{c_1}{a}\right)_1 + \frac{a_2 c_2}{b^2} = 0,$$

$$abr + \left(\frac{a_2}{b}\right)_2 + \left(\frac{b_1}{a}\right)_1 = 0,$$

$$adp + \left(\frac{d_1}{a}\right)_1 + \frac{a_2 d_2}{b^2} = 0,$$

$$bdq + \left(\frac{d_2}{b}\right)_2 + \frac{b_1 d_1}{a^2} = 0,$$

$$cdr + \frac{c_1 d_1}{a^2} + \frac{c_2 d_2}{b^2} = 0.$$

The space-time interval

$$\delta s^2 = a^2 \partial x_1^2 + b^2 \partial x_2^2 + c^2 \partial x_3^2 + d^2 \partial x_4^2$$

will therefore satisfy all the conditions necessary for the gravitation law, and the curvatures will be

$$K_{23} = K_{14} = p, \quad K_{31} = K_{24} = q, \quad K_{12} = K_{34} = r.$$

Furthermore, it may easily be proved, by retracing the steps of this paper, that the solution we have obtained is the most general solution of the sixteen differential equations, if we make the further hypothesis that a, b, c, d are functions of x_1 and x_2 only.

We may call the surface we have been considering the generating surface of a particular kind of Einstein space-time continuum. We have seen how the surface itself is deduced from the consideration of a differential equation whose solution Riemann has given in a form suitable for physical applications. Finally, it may be noticed that on this generating surface the product of the coefficients c, d in the form

$$a^2\partial x_1^2+b^2\partial x_2^2+c^2\partial x_3^2+d^2\partial x_4^2$$

is a potential function, since it is annihilated by the operator Δ_2.

RELATION BETWEEN APOLARITY AND THE WEDDLE SURFACE

By WILLIAM P. MILNE.

[Received December 5th, 1921.—Read January 12th, 1922.]

1. *Introduction.*

In the *Proceedings of the London Mathematical Society* (Ser. 2, Vol. 17, pp. 158–171) R. A. Roberts published an interesting and important paper "On Polygons Circumscribed about a Conic and Inscribed in a Cubic." His treatment is almost entirely algebraic. In investigating the applications of Apolarity, I was led to study the same subject from the standpoint of synthetic geometry and obtained several new results which were published in my paper on "The Relation between Apolarity and a certain Porism of the Cubic Curve" (*Proc. London Math. Soc.*, Ser. 2, Vol. 20, pp. 107–122). In this paper I gave the necessary and sufficient conditions that must exist in order that corresponding pairs of triangles can be found circumscribing the conic and having the nine points of intersection of their sides on the cubic. The treatment was entirely synthetic except that the fundamental theorem was established by analytical methods. This defect is remedied in the present communication, and the mode of proof has led to several new properties of the Weddle Surface. Throughout the investigation, it is really the Weddle Envelope that presents itself, that is, the envelope of a plane which intersects six fixed planes in six lines touching a conic.

2. *Proof of Fundamental Theorem.*

In what follows we shall regard the nine points of intersection of the sides of two triangles as occupying positions corresponding to the constituents of a determinant of the third order. Triads of points corresponding to the positive terms in the expansion of the determinant we shall refer to as "positive triads," and similarly with regard to "negative triads."

We may therefore enunciate the fundamental theorem as follows:—

If a cubic curve pass through the nine points of intersection of the *sides of two triangles, so that both the "positive" and the "negative" triads are apolar to the same member of the Hessian-pencil, then the sides of the two triangles all touch the same conic.*

Consider a cubic surface Π, and let it be mapped out on a plane π according to the method of Clebsch [*Crelle*, Vol. 65 (1866)]. Let the base-points on π be A_1, A_2, A_3, A_4, A_5, A_6, and let the corresponding lines lying wholly in Π be a_1, a_2, a_3, a_4, a_5, a_6, which are known to form half of a double-six. Let S be any plane cubic curve through the base-points, to which corresponds the plane σ cutting Π in the cubic curve Σ. Then it has been shown (*Proc. London Math. Soc.*, Ser. 2, Vol. 21, pp. 134–139) that a (1, 1)-correspondence exists between the cubics S and Σ and that they possess therefore the same fundamental invariant. They are, in fact, projectively equivalent curves, and if they be regarded as the same curve, corresponding points are represented by elliptic parameters whose sum or difference is constant. Now the condition that a triangle inscribed in S be apolar to a given member of the Hessian-pencil of S involves only the differences of the elliptic parameters of the vertices, and hence the corresponding points on Σ must be apolar to the corresponding member of the Hessian-pencil of Σ. Let the Hessians of S and Σ be denoted by S' and Σ' respectively, and let $A_1A_2A_3$ and $A_4A_5A_6$ be each apolar to $S+\lambda S'$. Furthermore, let the line A_rA_s intersect S again in the point C_{rs}. Then since A_1, A_2, A_3 if joined to any point of S meets the curve S again in three points apolar to the same member of the Hessian-pencil $S+\lambda S'$, we see that all the triads defined by the terms of the expansion of the determinantal form

$$\left.\begin{array}{ccc} C_{14} & C_{26} & C_{35} \\ C_{25} & C_{34} & C_{16} \\ C_{36} & C_{15} & C_{24} \end{array}\right|$$

are each apolar to $S+\lambda S'$. Now the points A_r, C_{rs} on S correspond to those points of Σ in which the lines a_r, c_{rs} intersect the plane σ (the usual notation for the twenty-seven lines on the cubic surface being adopted). Hence the triads of points in which a_1, a_2, a_3 and a_4, a_5, a_6 and the triads of lines defined by the terms of the expansion of the determinantal form

$$\begin{vmatrix} c_{12} & c_{26} & c_{35} \\ c_{25} & c_{34} & c_{16} \\ c_{36} & c_{15} & c_{24} \end{vmatrix}$$

intersect σ are each apolar to $\Sigma+\lambda\Sigma'$. If now we state the condition that $A_1A_2A_3$ shall be apolar to $S+\lambda S'$, we obtain a condition $S_1+\lambda S_1'=0$, where S_1 is linear in the coefficients of the plane σ, and S_1' contains these same coefficients to the third degree. In fact $S_1=0$ is the condition that S shall be apolar to $A_1A_2A_3$, and $S_1'=0$ is the condition that $A_1A_2A_3$ shall be apolar to the Hessian of S. In the first case σ always passes through a fixed point F_{123}, and in the second case it envelops a surface of class 3. Similarly, let the condition that $A_4A_5A_6$ shall be apolar to $S+\lambda S'$ be $S_2+\lambda S_2'=0$. Hence the envelope of all planes σ intersecting a_1, a_2, a_3 and a_4, a_5, a_6 and the triads of lines defined by the terms of the expansion of the determinantal form

$$\Delta \equiv \begin{vmatrix} c_{14} & c_{26} & c_{35} \\ c_{25} & c_{34} & c_{16} \\ c_{36} & c_{15} & c_{24} \end{vmatrix}$$

in triads of points apolar to the same member of the Hessian-pencil of Σ is $\Psi \equiv S_1S_2'-S_2S_1'=0$, which is a surface-envelope of class 4. It now remains to show that this surface-envelope Ψ is of the Weddle type.

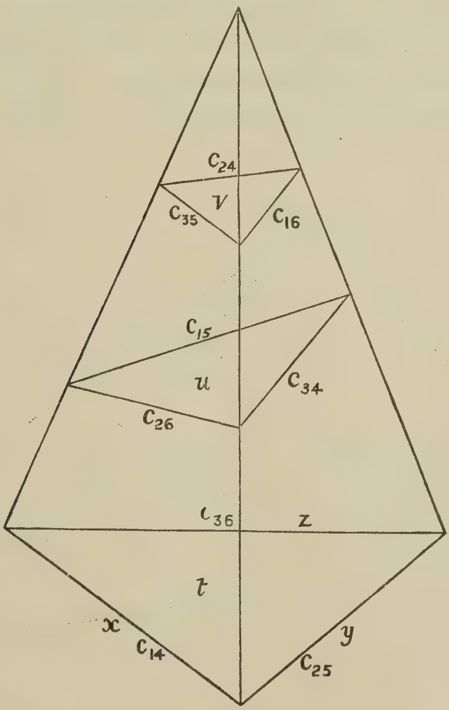

FIG. 1.

Let the diagram (Fig. 1) represent a pair of Steinerian Trihedra, whose planes are x, y, z and t, u, v. Consider any plane through the line c_{14}, intersecting the cubic surface in a cubic curve consisting of the line c_{14} and a conic. We have to show that the triads of points in which the lines $c_{14}c_{24}c_{34}$ and $c_{14}c_{15}c_{16}$ intersect the above plane are each apolar to the same member of the Hessian-pencil of the above degenerate cubic. Plainly they are each apolar to c_{14}^3. Hence any plane through c_{14} possesses the property that the six triads of the above determinantal system of points are each apolar to the same member of the Hessian-pencil of that section. The line c_{14} therefore lies completely on the above quartic envelope Ψ. Similarly we may show that the nine lines which are the mutual intersections of the above pair of Steinerian Trihedra lie wholly on Ψ.

Consider next the line yz containing the points of intersection of $c_{25}c_{36}$, $c_{34}c_{15}$, $c_{16}c_{24}$, and consider any plane passing through the line yz. The triads of points in which this plane is cut by the lines $c_{14}c_{24}c_{34}$ and $c_{14}c_{16}c_{15}$ are identical, and hence each triad is apolar to the same member of the Hessian-pencil of the curve of section. The line yz therefore lies wholly on Ψ, and similarly with regard to the lines zx, xy, uv, vt, tu.

The case of the line joining the corresponding pair of Steinerian points xyz, tuv has already been discussed by me in the *Proc. London Math. Soc.* (Ser. 2, Vol. 21, pp. 134–139). It is there proved that every plane through this line cuts the above determinantal system of lines Δ in triads of points each apolar to the curve of section Σ. Hence the line joining the points xyz, tuv lies completely on Ψ.

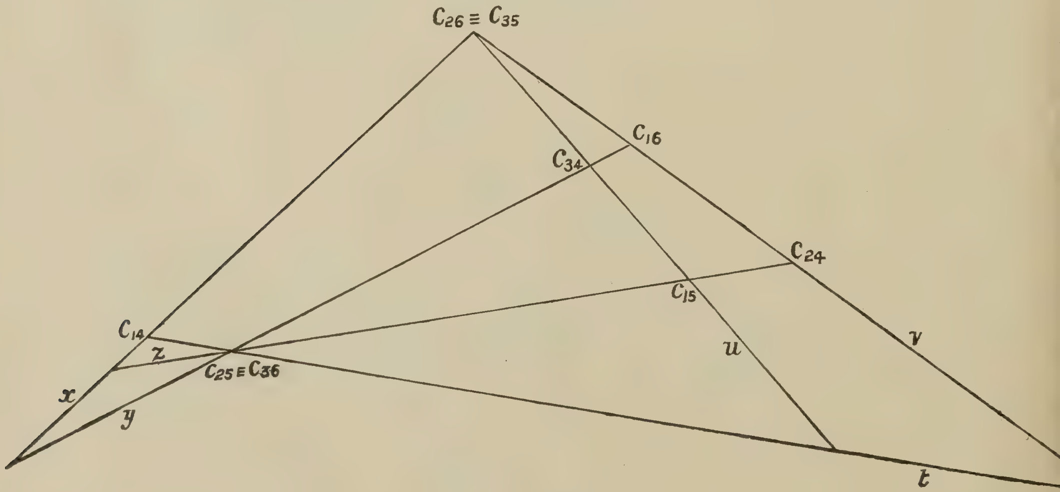

FIG. 2.

Finally, consider any plane σ through the line joining the points yzt and xuv, *i.e.* the line joining the points $c_{25}c_{36}$ and $c_{26}c_{35}$. The figure of section of the plane σ with the Steinerian Trihedra then degenerates into the figure of the diagram (Fig. 2), where x, &c. denotes the line of intersection of the plane σ with the plane x, &c., and c_{14}, &c. denotes the point of intersection of the plane σ with the line c_{14}, &c. Plainly the tangents at $c_{26}c_{35}$ and $c_{25}c_{36}$ meet at the point c_{14} which lies on the curve of section Σ. Hence, if Σ_1 be the cubic whose Hessian is Σ and which has $c_{25}c_{36}$ and $c_{26}c_{35}$ as "corresponding points," we know that any point along with $c_{25}c_{36}$ and $c_{26}c_{35}$ constitutes a triad apolar to Σ_1, and consequently in particular the triads $c_{16}c_{26}c_{36}$ and $c_{15}c_{25}c_{35}$ are each apolar to Σ_1. Thus any plane through the line joining the points yzt and xuv cuts the above determinantal form Δ in triads of points each apolar to the same member of the Hessian-pencil of Σ. The line joining yzt and xuv therefore lies wholly on Ψ, and similarly with regard to the other lines determined in a like manner.

We have therefore proved that Ψ contains all the lines defined by the intersections of the six planes x, y, z, t, u, v taken two and two, *i.e.* fifteen lines in all; and also the lines joining the points of intersection of the above planes divided into two sets of three, *i.e.* ten lines in all. The surface is therefore a Weddle Envelope, and consequently the enveloping plane σ intersects the six planes of the complementary pair of Steinerian Trihedra in six lines that touch a conic. The fundamental theorem of this paper is thus established synthetically and we state in addition the following results:—

I. *If a variable plane σ intersect a cubic surface in a cubic curve Σ such that the six points of intersection with σ of the half of a double-six constitutes two triads of points each apolar to the same member of the Hessian-pencil of Σ, then σ touches the Weddle Envelope of the six planes of the complementary pair of Steinerian Trihedra defined in the way explained above by the semi-double-six.*

II. *If a variable plane σ intersect the six planes of two complementary Steinerian Trihedra in two triangles the points of intersection of whose sides form a determinantal system of points, the six triads defined by which (as explained above) are each apolar to the same number of the Hessian-pencil of the curve of intersection Σ with the cubic surface, the plane σ generates a Weddle Envelope, and hence the six sides of the above two triangles touch the same conic.*

3. *Triads Apolar to all members of Hessian-Pencil.*

Consider now a plane σ cutting the cubic surface in a curve Σ such that the three points of section of a_1, a_2, a_3 with σ form a triad apolar not only to Σ but also to every member of its Hessian-pencil. Since the points of intersection of a_1, a_2, a_3 with σ are apolar to every member of the Hessian-pencil of Σ, they are in particular apolar to that member to which the points of intersection of a_4, a_5, a_6 are apolar. Hence σ touches the Weddle Envelope. But the points of intersection of a_1, a_2, a_3 with σ are also apolar to Σ. Hence σ passes through the point F_{123} which has been shown to lie on the line joining xyz to tuv. Thus σ envelops the tangent-cone drawn from F_{123} to the Weddle Envelope, which cone must be of the third class, since F_{123} lies on a line wholly on the surface. But if a triad of points on a cubic curve be apolar to all the members of the Hessian-pencil, we know that the tangents to the curve at the points of the triad meet at a point of the curve. Thus we have the following result:—

If the lines a_1, a_2, a_3 meet the plane σ in points, the tangents at which to Σ intersect in a point of Σ, the plane σ envelops a cone of the third class, namely, the tangent cone from F_{123} to the Weddle Envelope.

4. *A Property of the Weddle Envelope.*

The following property is now evident from the results established in the paper quoted above on the "Relation between Apolarity and a certain Porism of the Cubic Curve" (*Proc. London Math. Soc.*, Ser. 2, Vol. 20, p. 118):—

If the plane σ touch the Weddle Envelope and intersect the planes of the above Steinerian Trihedra in six lines touching the conic C, the twelve common tangents of the cubic curve Σ and the conic C have their points of contact with Σ lying three by three on four lines which touch C at the Jacobian-points of the involution defined by the points of contact of C with the planes of the complementary Steinerian Trihedra, i.e. at the points in which C is intersected by the conic circumscribing the two triangles in which σ cuts the Steinerian Trihedra.

ON CERTAIN TYPES OF PLANE UNICURSAL SEXTIC CURVES

By G. D. SADD.

[Received December 21st, 1921.—Read January 12th, 1922.]

1. The plane sextic curve of zero deficiency (*i.e.* unicursal) has in general ten double points. These may be either all distinct, or several of them may be consecutive, thus forming a singularity of more advanced type. The sextics considered here are those which have all their double points concentrated at either a double or a triple point, and are of the following types.

(1) The sextic of zero deficiency with a single double point O. This has nine double points "latent" at O, the two branches of the curve having 10-point contact there.

(2) The sextic of zero deficiency with a triple point at O and no other point-singularities. Such a sextic must necessarily have either

(i) two branches touching at O and having 8-point contact, the third branch meeting each of the others once only;

or (ii) three branches touching at O, two of which have 6-point contact;

or (iii) three branches touching at O, two of which have 4-point contact: the third branch has then 3-point contact with both.

The triple point of the curve (2) is equivalent to three double points, so that there are in each case seven double points latent at O, but in (ii) three, and in (iii) six of these latent double points are combined to form respectively one and two latent triple points.

2. The equations of the curves of § 1 can be obtained by the method of "analysis," which consists essentially in the elimination of multiple points from the curve by means of quadratic transformations. As applied to Cartesian coordinates, the method is briefly as follows: the substitution

xy for x (denoted hereafter by xy/x) in the equation of the curve, followed by division by y^k, gets rid of a k-ple point ($k \geqslant 2$) at O (the origin); if the transformed curve contains a p-ple point ($p \geqslant 2$) at O, the original curve must have, in addition to the assumed singularity, a p-ple point latent at O.

We may, by successive transformations of this type, together with transformations of the type $(y+\lambda x)/x$, eliminate from a given curve any number of multiple points at O, and by writing down the conditions for each of the transformed curves to possess a multiple point of specified order at O, we are enabled to obtain sufficient relations between the coefficients to determine the equation of the curve containing a k-ple point at O, together with a given number of latent multiple points of given order.

3. The first curve chosen for discussion is the sextic [(2), (i)] of § 1. We write down the Cartesian equation of the general sextic in the form

$$\text{(I)} \quad C_6 \equiv (a_0+a_1y+\ldots+a_6y^6)+x(A_0+A_1y+\ldots+A_5y^5)$$
$$+u\{(b_0+\ldots+b_4y^4)+x(B_0+\ldots+B_3y^3)\}$$
$$+u^2\{(c_0+c_1y+c_2y^2)+x(C_0+C_1y)\}+d_0u^3,$$

where $$u \equiv y+x^2,$$

and proceed to obtain the coefficients.

Newton's diagram for the form of the curve at the origin gives

$$a_0,\ a_1,\ a_2,\ a_3;\quad A_0,\ A_1,\ A_2;\quad b_0,\ b_1,\ b_2;\quad B_0,\ B_1;\quad c_0 = 0;\quad C_0 = 1,$$

where $u = \lambda y^2$ has been chosen as the pencil of conics having 4-point contact at O with each of the touching branches, and $x+(c_1+d_0)y = 0$ as the tangent at O to the third branch.

Now $u = 0$ meets the sextic where

$$y^8(a_4+a_5y+a_6y^2)^2+y^7(A_3+A_4y+A_5y^2)^2 = 0.$$

If $u = 0$ be taken as the conic of closest contact with (I) at O, eleven roots of this equation are zero, and therefore

$$a_4,\ a_5,\ A_3,\ A_4 = 0,$$

and we have

$$\text{(II)} \quad C_6 \equiv a_6y^6+A_5xy^5+u\{y^3(b_3+b_4y)+xy^2(B_2+B_3y)\}$$
$$+u^2\{c_1y+c_2y^2+x+C_1xy\}+d_0u^3 = 0.$$

Proceeding now by analysis, and using in succession the transformations

$$\text{(i)}\quad yx/y \quad \text{(and then dividing by } x^3\text{)},$$

$$\left.\begin{array}{ll} \text{(ii)} & (xy-y)/x \\ \text{(iii)} & xy/x \\ \text{(iv)} & xy/x \end{array}\right\} \quad \text{(dividing by } y^2 \text{ after each substitution)},$$

we obtain, by § 2, $B_2 = 0, \quad A_5 = -\frac{1}{4}b_3^2.$

Expressed in homogeneous coordinates, the sextic is now

$$\text{(III)}\quad C_6 \equiv a_6y^6 - \tfrac{1}{4}b_3^2xy^5 + uy^3\{b_3z+b_4y+B_3x\}$$
$$+u^2\{c_1yz+c_2y^2+xz+C_1xy\}+d_0u^3 = 0,$$

where O is the point (0, 0, 1), and $y = 0$ is the tangent at O to $u = 0$. The side $z = 0$ of the triangle of reference is still at our disposal.

Through the eight points of contact of the two branches which touch at O a pencil of cubics passes, each of which meets the sextic $2.8+1 = 17$ times at O, and therefore once again. But $u = 0$ has 5-point contact at O with each of these two branches of the sextic, and therefore it has 5-point contact with each cubic of the pencil. Now the most general cubic having 5-point contact with $u = 0$ at O is

$$\text{(1)}\qquad C_3 \equiv u(z+px+qy)+ay^2(x+\alpha y) = 0.$$

The cubic

$$\text{(2)}\qquad C_3' \equiv u\{x-(p+\alpha)y\}-ay^3 = 0$$

meets (1) eight times at O, so that the most general cubic having 8-point contact with $C_3 = 0$ at O is

$$\text{(3)}\qquad C_3+kC_3' = 0.$$

The substitutions

$$\{x+(p+\alpha)y\}/x, \quad \{z-2(p+\alpha)x-(p+\alpha)^2y\}/z$$

reduce equation (3) to the form

$$\text{(4)}\qquad u\{z-(p+2\alpha)x+(q-\alpha p-\alpha^2)y\}+ay^2\{x+(p+2\alpha)y\}$$
$$+k(xu-ay^3) = 0.$$

Replacing k by $k+p+2\alpha$, q by $q+\alpha p+\alpha^2$, putting y/κ for y, $z\kappa$ for z, and choosing κ properly, this becomes

$$(4') \qquad \{u(z+qy)+xy^2\}+k(xu-y^3)=0,$$

i.e.

$$(3') \qquad c_3+kc_3'=0,$$

where
$$c_3 \equiv u(z+qy)+xy^2,$$
$$c_3' \equiv xu-y^3.$$

If (3′) is taken as the equation of the pencil of cubics meeting the sextic seventeen times at O, the side $z=0$ of the triangle of reference is the tangent to the conic $u=0$ at the point (other than O) where this is met by $x=0$.

The cubic $c_3'=0$ [k infinite in (3′)] has a node at O, so that all eighteen intersections of this cubic with $C_6=0$ are coincident at O. On eliminating z between $c_3'=0$ and equation (III), by means of $u \equiv yz+x^2$, we obtain a homogeneous equation in x and y. Equating to zero the coefficients of all the terms except that in which x does not occur, we get

(α_1) $b_3=-2$ (and therefore $A_5=-1$), $c_2=1$, $a_6=c_1-B_3$, $b_4+C_1=0$.

The sextic becomes

$$\text{(IV)} \quad C_6 \equiv (c_1-B_3)\,y^6-xy^5+uy^3\{-2z-C_1y+B_3x\}$$
$$+u^2\{c_1yz+y^2+xz+C_1xy\}+d_0u^3=0.$$

Again, $k=0$ in (3′) gives the cubic $c_3=0$. The equation obtained by eliminating z between this and (IV) is homogeneous of the fifth degree in x and y (after division by y^7). All the coefficients except those of xy^4 and y^5 must vanish, since all but one of the intersections lie at O, and we obtain

(α_2) $b_4+c_1=0$, $B_3=2c_1$, $q(q-c_1)=c_1+d_0$, $(q-c_1)(1+q^3)=d_0q^2$.

Combining (α_1) and (α_2), we have

$$a_6=b_4=-c_1, \quad C_1=c_1, \quad B_3=2c_1, \quad c_1(1-q^2)=q.$$

The sextic is now, in Cartesian coordinates,

$$\text{(V)} \quad C_6 \equiv c_1(xu-y^3)^2-(xu-y^3)(u+c_1uy+xy^2)-(d_0+c_1)u^3=0.$$

Returning to analysis, the transformations

$$\text{(v)} \quad (xy-y)/x, \qquad \text{(vi)} \quad xy/x, \qquad \text{(vii)} \quad (xy-\tfrac{1}{2}c_1y)/x,$$

followed each time by division by y^2, give

$$c_1 = 2q, \quad q^2 = \tfrac{1}{2}.$$

The sextic is finally

$$\text{(VI)} \quad C_6 \equiv 2q(xu-y^3)^2-(xu-y^3)(u+2quy+xy^2)+\tfrac{1}{2}u^3 = 0,$$

with the condition $q^2 = \frac{1}{2}$.

Equation (VI) gives precisely two sextics; all others of the type considered are projectively equivalent to one or other of these two.

4. Since the sextic (VI) of § 3 is rational, the coordinates of any point on it may be expressed rationally as functions of a parameter.

The transformation

$$x = X+Z, \quad y = Y, \quad z = \{Y^3-(X+Z)^2 Z\}/YZ,$$

transforms the sextic into the quartic

$$\text{(A)} \quad (XZ-qY^2)^2+X^2Z(X+2qY)-XY^3 = 0,$$

which has all three of its double points coincident at (0, 0, 1).

The transformation

$$X = \xi, \quad Y = \eta, \quad Z = \zeta+q\eta^2/\xi$$

further transforms the quartic (A) into the conic

$$\text{(B)} \quad q(\eta+\zeta)^2 = -\zeta[\xi+\zeta(1-q)].$$

The parametral coordinates of any point on (B) are

$$\xi = -(t^2+2tq+q), \quad \eta = t, \quad \zeta = q,$$

whence any point on the quartic is

$$X = -(t^2+2tq+q)^2, \quad Y = t(t^2+2tq+q), \quad Z = t+\tfrac{1}{2},$$

and any point on the sextic (VI) is

$$x = -(t^3+2t+1)-2tq(2t+1),$$

$$y = t^2+2tq+q,$$

$$z = \frac{t(t^2+2tq+q)^2}{t+\frac{1}{2}} - \frac{\{(t^2+2t+1)+2tq(2t+1)\}^2}{t^2+2tq+q}.$$

The three values of the parameter at the triple point are $t = -\frac{1}{2}$, $t^2+2tq+q = 0$. The sixth intersection of $y = 0$ with the sextic is obtained by putting $t = \infty$, and is $(1,\ 0,\ \frac{1}{2}+2q)$.

The line $y = 2x$ meets the sextic four times at O, and again at the two points whose parameters are $t = -2q \pm \{-q\}^{\frac{1}{2}}$. Hence the remaining two intersections of the tangent at O to the third branch of the sextic are real if $q = -1/\sqrt{2}$, unreal if $q = +1/\sqrt{2}$.

5. The sextic (2), (ii) of § 1 has been already discussed.* We will therefore proceed at once to (2), (iii).

We may take $u^2y = 0$ as the approximation to the shape of the curve at the origin ; then, if the equation of the general sextic is written down in the form of equation (I) of § 3, Newton's diagram gives

$$a_0, a_1, a_2, a_3\,; \quad A_0, A_1, A_2\,; \quad b_0, b_1, b_2\,; \quad B_0, B_1\,; \quad c_0 = 0, \quad C_0 = 0\,; \quad d_0 = 1-c_1.$$

Now every conic which has 4-point contact at O with each of two of the branches of the sextic considered must have also 3-point contact with the third branch at O, so that, for every value of λ, eleven of the intersections of $u = \lambda y^2$ with the sextic are coincident at O. We obtain

$$a_4, a_5\,; \quad A_3, A_4\,; \quad B_2, b_3, c_1 = 0.$$

The equation of the sextic can therefore be written

$$\text{(VII)} \quad (a_6y^6+b_4uy^4+c_2u^2y^2+u^3)+xy(A_5y^4+B_3uy^2+C_1u^2) = 0.$$

The conics of closest contact at O are $u = \lambda_1y^2$ and $u = \lambda_2y^2$, where λ_1, λ_2 are the roots of the equation

$$A_5+B_3\lambda+C_1\lambda^2 = 0.$$

We can verify equation (VII) by analysis, using in succession the transformations

(i) yx/y (and then dividing by x^3),

(ii) $(xy-y)/x$, (iii) xy/x (dividing each time by y^3).

6. The sextic (1) of § 1 has been considered by Sharpe and Craig.†

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* H. Hilton, "On certain Types of Plane Algebraic Curves," *Rendiconti del Circolo Matematico di Palermo*, Vol. 44 (1920).

† "Plane Curves with Consecutive Double Points," *Annals of Mathematics*, Ser. 2, Vol. 16.

They give, as the equation of a sextic having all ten double points consecutive at the origin

$$c_3^2-(5T^2+4T^5)\,c_3c_3'+(5T+6T^4)(c_3')^2+(2-3T^3)\,c_2c_4=0,$$

with the condition $$T^6+T^3-1=0,$$

where
$$\begin{aligned} c_2 &\equiv y-\tfrac{1}{2}x^2, \\ c_3 &\equiv y-\tfrac{1}{2}x^2-2Ty(y-\tfrac{1}{2}x^2)-4xy^2, \\ c_3' &\equiv x(y-\tfrac{1}{2}x^2)-8y^3, \\ c_4 &\equiv T(y-\tfrac{1}{2}x^2)^2-2xy(y-\tfrac{1}{2}x^2)+16y^4. \end{aligned}$$

This equation can also be obtained by the preceding methods, and the coordinates of any point on the curve can be expressed rationally in terms of a parameter.

If we start, as in § 3, with the general sextic (I), and take $u^2=0$ as the approximation to the shape of the curve at the origin, Newton's diagram gives

$$a_0,\ a_1,\ a_2;\quad A_0,\ A_1;\quad b_0,\ b_1;\quad B_0=0;\quad c_0=1.$$

Of the twelve intersections of $u=0$ with the sextic, ten are coincident at O. Hence
$$a_3,\ a_4;\quad A_2,\ A_3,\ A_4=0.$$

The transformations

(i) yx/y (followed by division by x^2),

(ii) $(xy-y)/x$, (iii) xy/x, (iv) xy/x (followed each time by division by y^2),

give in addition $$B_1=0,\quad b_2=0,\quad a_5=-\tfrac{1}{4}B_2^2.$$

Of the eighteen intersections of $xu=y^3$ with the sextic, sixteen are coincident at O. We obtain, on eliminating z between this equation and the equation of the sextic (expressed in homogeneous coordinates), and equating coefficients to zero,

$$(\beta_1)\quad \begin{cases} B_2=2 \text{ (and therefore } a_5=-1), \\ A_5-C_0-b_3=a_6+B_3-c_1=b_4+C_1-1=0. \end{cases}$$

Returning to Cartesian coordinates, the next transformation

(v) xy/x

gives

$$(\beta_2)\quad \begin{cases} b_3 = -2C_0 \text{ (and therefore } A_5 = -C_0), \\ (c_1+a_6)^2 = 4(c_2-C_0). \end{cases}$$

Forming now, by the usual process, the equation giving the ratios y/x at the points of intersection of the cubic $c_3 = 0$ of § 3 with the sextic, and remembering that all but two of these ratios are zeros, we obtain the further relations

$$(\beta_3)\quad \begin{cases} q^2-q(c_1+a_6)+(c_2-C_0) = 0, \\ d_0-a_6 = -q(C_1-C_0q). \end{cases}$$

Combining (β_2) and (β_3), we have

$$c_1 = 2q-a_6 \text{ [and therefore } B_3 = 2(q-a_6)], \quad c_2 = q^2+C_0.$$

The sextic can now be put into the form

$$\text{(IX)}\quad (u+uqy+xy^2)^2+a_6(xu-y^3)^2+C_0(xu-y^3)(u+uqy+xy^2)$$
$$= (C_1-C_0q)\,u(qu^2-uxy+y^4).$$

This has deficiency two, as is apparent from analysis.

If we put $\quad C_1-C_0q \equiv A, \quad 2+C_0q-\tfrac{1}{2}C_1 \equiv B,$

the transformation

(vi) $(xy-qy)/x$

gives the relation

$$(\beta_4)\quad a_0q^2 = -A(1+C_0q)+\tfrac{1}{4}C_1^2.$$

The next transformation

(vii) $(xy-By)/x,$

combined with (β_4), gives

$$B(1-B) = q^3, \qquad A = 4(1-B)^3/(1-2B)^2.$$

The final transformation

(viii) $[xy - \{q^2 + (4 - A - 6B + 2B^2 + 2AB)/2q\}\, y]/x$

gives $$A = 2(1-B)^2.$$

From the above relations, B and q are easily determined. They are found to satisfy respectively the equations

$$4B^2 - 2B - 1 = 0, \quad 16q^2 + 4q^3 - 1 = 0,$$

whence $$B = 2q^3 + \tfrac{1}{2}.$$

The values of a_6, C_0, A follow at once in terms of q.

If we replace q by $\frac{1}{2}T$, and express all the coefficients in terms of T, the sextic becomes

(X) $$c_3^2 - \tfrac{1}{2}(5T^2 + 2T^5)c_3c_3' + \tfrac{1}{2}(5T + 3T^4)(c_3')^2 = \tfrac{1}{8}u(4 - 3T^3)(Tu^2 - 2uxy + 2y^4),$$

where $$c_3 \equiv u + \tfrac{1}{2}Tuy + xy^2,$$

$$c_3' \equiv xu - y^3,$$

$$T^6 + 2T^3 - 4 = 0.$$

It is evident that there are only two distinct rational sextics with real equations, having ten consecutive double points, if we consider all curves which are projectively equivalent to be essentially the same.

Note.—On replacing T by $2^{\frac{1}{3}}T$, y by $-2^{\frac{4}{3}}.y$, z by $1/2^{\frac{1}{3}}.z$, equation (X) becomes identical with (VIII).

7. To find the parametral coordinates of any point on the sextic of the preceding section, let us consider the intersections of (X) with the cubic

(3′) $$c_3 + kc_3' = 0.$$

If we put

$$-\tfrac{1}{2}(5T^2 + 2T^5) = C_0, \quad \tfrac{1}{2}(5T + 3T^4) = a_6, \quad \tfrac{1}{4}(4 - 3T^3) = A,$$

and then $$k^2 - C_0k + a_6 = P,$$

and eliminate z between equations (X) and (3′) (after making them both homogeneous), or, which is equivalent to the same thing, eliminate u between

(C) $$u^2 - u(x^2 - kxy - \tfrac{1}{2}Ty^2) + y^3(x - ky) = 0,$$

and

$$\text{(D)}\quad u^2\{Px^2-\tfrac{1}{2}AT(x^2-kxy-\tfrac{1}{2}Ty^2)+Axy\}$$
$$+uy^3\{-2Px+\tfrac{1}{2}AT(x-ky)-Ay\}+Py^6=0,$$

we obtain

$$\text{(i)}\quad [Py^6-y^3(x-ky)\{Px^2-\tfrac{1}{2}AT(x^2-kxy-\tfrac{1}{2}Ty^2)+Axy\}]^2$$
$$=[-Py^6(x^2-kxy-\tfrac{1}{2}Ty^2)+y^6(x-ky)\{2Px-\tfrac{1}{2}AT(x-ky)+Ay\}]$$
$$\times[y^3\{-2Px+\tfrac{1}{2}AT(x-ky)-Ay\}$$
$$+(x^2-kxy-\tfrac{1}{2}Ty^2)\{Px^2-\tfrac{1}{2}AT(x^2-kxy-\tfrac{1}{2}Ty^2)+Axy\}],$$

which reduces, after division by y^{10}, to the quadratic

$$\text{(i}'\text{)}\quad x^2[\tfrac{1}{4}T^2K^2+AKH]+xy[TKL-AMH]$$
$$+y^2[L^2-AH(Ak+\tfrac{1}{2}ATk^2-\tfrac{1}{2}PT)]=0,$$

where

$$H\equiv\tfrac{1}{2}Tk+1+\tfrac{1}{8}T^3,$$
$$K\equiv P-\tfrac{1}{2}AT,$$
$$L\equiv P+\tfrac{1}{4}AT^2k,$$
$$M=Pk-ATk-A.$$

The roots of (i′), considered as a quadratic in x/y, are

$$\text{(ii)}\quad \frac{x}{y}=\frac{-TKL+AMH\pm H[A\{A(Pk+A)^2-4KP^2\}]^{\frac{1}{2}}}{K\{\frac{1}{2}T^2K+2AH\}}.$$

Now the coefficient of x^2 in (i′) becomes, on making use of the relation $T^6+2T^3-4=0$,

$$\tfrac{1}{4}T^2K\{k+\tfrac{1}{4}T^2(4+3T^3)\}^2.$$

The coefficient of xy is also found to involve the expression

$$k+\tfrac{1}{4}T^2(4+3T^3)$$

as a factor (in the first degree only). The cubic of closest contact with the sextic, having all eighteen intersections coincident at O, is therefore

$$uz+\tfrac{1}{2}Tuy+xy^2=\tfrac{1}{4}T^2(4+3T^3)(xu-y^3).$$

If we replace k by $t-\frac{1}{4}T^2(4+3T^3)$, we have

$$\begin{aligned}
P &= t^2+\tfrac{1}{2}T^2t(1-T^3)+\tfrac{1}{2}AT^4,\\
H &= \tfrac{1}{2}Tt+\tfrac{1}{8}T^3(1-T^3),\\
K &= t^2+\tfrac{1}{2}T^2t(1-T^3)+\tfrac{1}{16}AT^{10},\\
L &= t^2+\tfrac{1}{4}T^2t\,\{A+2(1-T^3)\}+\tfrac{1}{4}A^2T^4,\\
M &= t^3-\tfrac{1}{4}T^2t^2(2+5T^3)+\tfrac{1}{4}T^4t(7-2T^3)+\tfrac{1}{4}AT^6(1-T^3),
\end{aligned}$$

$$T^2K+4AH = T^2t^2,$$

$$\begin{aligned}
A\,\{A(Pk+A)^2-4P^2K\} &= At^4\,\{-\tfrac{3}{4}t^2(4+T^3)+\tfrac{1}{2}T^2t(1+T^3)-\tfrac{3}{16}T^4(2-T^3)\}\\
&= -\tfrac{3}{16}T^6t^4\,\{t^2+\tfrac{1}{6}T^2t(2-3T^3)+\tfrac{1}{80}T^4(2-3T^3)^2\}.
\end{aligned}$$

Equation (ii) becomes

$$\text{(ii}'\text{)} \qquad \frac{x}{y} = \frac{\phi_3(t) \pm T^3Ht\sqrt{\{\phi_2(t)\}}}{T^2Kt},$$

where $\phi_3(t)$ is a function of the third degree in t, and

$$\phi_2(t) = -\tfrac{3}{4}[\{t+\tfrac{1}{12}T^2(2-3T^3)\}^2+\tfrac{1}{180}].$$

It follows that, since $\phi_2(t)$ is a quadratic function of t, the ratio x/y can be expressed as a rational function of a parameter, and from this the ratio z/y can be deduced.

We see also that since $\phi_2(t)$ is negative for every value of t except $t = 0$, the curves of equation (X) are unreal, except for the one isolated point O. Hence there can exist no real unicursal sextic having ten consecutive double points.

The following values are of some interest. $K = 0$ has unreal roots

$$t = -\tfrac{1}{4}T^2\,(1-T^3) \pm \tfrac{1}{8}iT^5(2A)^{\frac{1}{2}},$$

thus verifying that the remaining two intersections of $y = 0$ with the sextic are unreal.

$$H = 0\ [t = \tfrac{1}{4}T^2(1-T^3)],$$

makes both values of x/y equal to $\frac{1}{2}T^2$.

The cubic corresponding to this value of t is

$$uz+\tfrac{1}{2}Tuy+xy^2 = \tfrac{1}{4}T^2(5+2T^3)(xu-y^3).$$

Replacing x by $\frac{1}{2}T^2y$ in this equation, we obtain unreal values for the ratio y/z, showing that there is one real cubic of the pencil which touches the sextic again, but the point of contact is unreal.

There are besides two unreal cubics touching the sextic again, which are obtained from the roots of $\phi_2(t) = 0$.

Note.—We can deduce that O is an isolated point on the sextic from Craig and Sharpe's investigation. For they obtain

$$y = \tfrac{1}{2}x^2 + x^5 + Tx^7 + Ux^8 + Vx^9 + Wx^{10} + \ldots,$$

as the expansion at the origin, where

$$U = 7 - T^3,$$

$$V = 2T^2 + T^5,$$

$$W = 42T - 7T^4 \pm \sqrt{\{-39T^2 + 63T^5\}},$$

$$T^3 = -\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{5}.$$

The expansion is imaginary, since W is unreal for both values of T^3.

ON LEAST PRIMITIVE ROOTS

By ALLAN CUNNINGHAM, H. J. WOODALL, *and the late* T. G. CREAK.

[Read March 9th, 1922.]

1. *Scope of Paper.*—The object of this paper is to introduce a table of *Least Primitive Roots.*

2. *Fermat's Theorem.*—By this theorem

$$y^{p-1} \equiv +1 \text{ (mod } p\text{), } \textit{always} \text{ [} p \text{ an odd prime, } y \text{ prime to } p\text{].} \qquad (1)$$

Also, if
$$p-1 = \nu\xi, \qquad (2)$$

then, for certain values of y (prime to p),

$$y^{(p-1)\div\nu} = y^{\xi} \equiv +1 \text{ (mod } p\text{).} \qquad (3)$$

When ξ is the *minimum* exponent possible for y in this last congruence, and (consequently) ν the *maximum* possible, ξ is styled the *Haupt-Exponent* of y (to mod p), and ν is styled the *Residue-Index* of y, whilst y itself is styled a *Root* of the congruence.

3. *Primitive Roots* (G, g).—When y has such a value ($< p$) that $\xi = p-1$, whereby $\nu = 1$, then g is styled a *Primitive Root* of the prime p, and is here denoted by G or g.

Using the symbol $\tau(n)$ to denote the *Totient* of n, it is known that every prime p has exactly $\tau(p-1)$ *different* primitive roots (G); it follows that the *number* of such roots (G) cannot exceed $\frac{1}{2}(p-1)$ for any prime (p).

The whole set of $\tau(p-1)$ primitive roots of any prime (p) may be derived from any given one of them (say G_1), as being the Least Residues modulo p of the powers of G_1, viz.

$$G_1^{\alpha},\ G_1^{\beta},\ G_1^{\gamma},\ \ldots,\ G_1^{\zeta}\ \ [\alpha,\ \beta,\ \gamma,\ \ldots,\ \zeta \text{ all prime to } (p-1)]. \qquad (4)$$

4. *Least Primitive Root* (g).—The *least* of the set of $\tau(p-1)$ primitive roots (G) of a prime (p) will be denoted by g.

As the practical use of primitive roots (G) in computation involves *frequent multiplication* of successive Residues by that primitive root (G), it is of great importance—for ease of calculation—to have the Least Primitive Root known and available for such work.

5. *Table of Least Primitive Roots* ($\pm g$).—The table at end of this paper gives the Least Primitive Roots, both + and − (*i.e.* $\pm g$) of all odd primes $p < 25410$.

6. *Distribution of* $\pm g$.—The Table on p. 345 shows the *number* of $\pm g$ which $= 2, 3, 4, 5$, &c. in the three Ranges of odd primes—

167 primes < 1000, 1228 primes < 10000, 2800 primes < 25410,

arranged in three classes of $\pm g$, viz.

(1) Only $+g$ when $p = 4\varpi-1$.

(2) Equal $\pm g$ when $p = 4\varpi+1$.

(3) Only $-g$ when $p = 4\varpi-1$.

As the $\pm g$ are always unequal when $p = 4\varpi-1$, they have to be shown in different classes (1), (3).

7. *Chief points of note in Table.*

(1) Remarkable smallness of the highest $\pm g$, as shown below:—

Max. $\pm g$.	In primes.
± 21, once	in 167 primes < 1000
± 31, once	in 1228 primes < 10000
$+31$, twice; -37, -39 once	in 2800 primes < 25410

(2) Remarkable efficiency of small numbers $\not> 12$, in supplying $\pm g$

Number of each $\pm g$.

	Limits of p	Values of g																									Total
		2	3	4	5	6	7	9	10	11	12	13	14	15	17	18	19	20	21	22	23	26	29	31	37	39	
Only $+g$	1000	31	19	.	16	6	7	.	.	3	.	1	.	1	2	.	1	.	.	.	.	.	.	.	.	.	87
	10000	238	138	.	101	31	47	.	7	23	1	13	3	2	8	.	6	.	.	1	.	.	.	.	.	.	619
	25410	519	306	.	243	81	101	.	13	58	7	33	6	3	13	.	13	.	1	1	4	1	3	.	.	.	1406
Both $\pm g$	1000	36	21	.	10	5	3	.	2	1	.	1	.	.	.	.	.	.	1	.	.	.	.	.	.	.	80
	10000	232	154	.	70	37	41	.	20	18	1	13	4	2	7	.	3	.	2	2	2	.	.	1	.	.	609
	25410	522	355	.	155	81	100	.	47	47	2	31	13	3	12	.	10	.	4	4	5	.	1	2	.	.	1394
Only $-g$	1000	33	19	14	9	3	4	4	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	87
	10000	233	154	111	44	23	17	23	5	5	.	2	.	.	1	.	.	.	.	1	.	.	.	.	.	.	619
	25410	521	344	238	109	55	35	50	14	12	1	12	4	4	1	1	1	1	.	1	.	.	.	.	1	1	1406

as shown below: —

Number of		In primes.
$+g \not> 12$	$-g \not> 12$	
160	165	in 167 primes < 1000
1159	1188	in 1228 primes < 10000
2637	2688	in 2800 primes < 25410

(3) Composite values of $\pm g$, especially $g = 4\gamma$, are much rarer than prime values.

Results (1) and (2) are of great importance for practical use of primitive roots in computation, as rendering it probable that quite small primitive roots generally exist for all primes. The advantage of small primitive roots is obvious.

8. *Origin of present Tables.*—The origin of these tables (of $\pm g$) was a Table* of the Haupt-Exponents (ξ) and Residue-Indices (ν) of the eight small Bases

$$y = 2, 3, 5, 6, 7, 10, 11, 12, \quad [\text{all taken } +],$$

for all odd primes $p < 25410$ prepared by the present joint authors independently. In the course of this work it was seen that among these eight small Bases are found the Least Primitive Roots ($\pm g$) of by far the greater number (2637) of the 2800 odd primes < 25410, leaving only 163 primes for which no primitive root was so far known. This made it seem worth while to undertake the research of the Least Primitive Roots of these 163 primes also.

9. *Relation of ± primitive roots.*—As the tables of ξ, ν just quoted give—(as is usual in such tables)—the ξ, ν of *positive Bases* $(+y)$ only, the derivation of the ξ, ν for *negative Bases* $(-y)$ will now be given. This includes the connexion of $\pm G$, and also of $\pm g$.

Let ξ, ν refer to $+y$, and ξ', ν' to $-y$ [same y in both]. The rules are different for the two forms of $p = 4\varpi \pm 1$,

(1) When $p = 4\varpi + 1$, then $\xi = \xi', \ \nu = \nu'$. *(5a)*

Hence $\nu = 1$ gives both $\pm y = G$ [may be g]. *(5b)*

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* These tables are now printed off, and ready for publication.

(2) When $p = 4\varpi - 1$, then $\frac{1}{2}(p-1)$ is *odd*, and

$$\xi' = 2\xi, \quad \nu' = \tfrac{1}{2}\nu. \tag{6a}$$

Hence $\nu = 1$ gives $+y =$ some G [may be g]

and $-y^2 =$ some G [may be g], (6*b*)

$\nu = 2$ gives $\nu' = 1$, whence $-y =$ some G [may be g]. (6*c*)

10. *Method of research.*—This consists of *two* Steps: (1) selection of "trial roots," (2) testing the "trial roots."

STEP (1). *Selection of "trial roots"* (z).—Here z is known to be > 12. Selection is then made from the successive small Bases $z = 13, 14, 15$, &c. of those only for which

$(z/p)_2 = -1$, for each prime required;

$(z/p)_3 \neq 1$, for each prime required of form $= 6\varpi + 1$.

This can readily be done by known* Rules. Considerable help in excluding *composite* numbers (z) from this Test is given by the above tables of ξ, ν; thus:—

If a, b be any pair of the eight Bases 2, 3, ..., 12 above named.

If ν_a, ν_b be the (known) Residue-Indices of a, b, modulo p.

If ν_z ,, (unknown) ,, of z ,,

And taking $z = a^\alpha . b^\beta$, or a^α / b^β, [α, β both odd, may $= 1$].

Then (1) if ν_a, ν_b be *both odd*, or *both even*, this gives $(z/p)_2 = +1$.

Also (2) if ν_a, ν_b have a common factor (> 1), then ν_z also has that factor.

In both the above cases z *cannot be a primitive root.* But, if ν_a be *prime to* ν_b, and *not both odd*, then ν_z will be *odd*, and will be *free from all factors* of ν_a, ν_b; so that in such cases z *may be* a primitive root (G or g).

The above considerations often greatly limit the number of (composite) numbers (z) requiring detailed testing in search for a primitive root; and they have the merit of being easily applied.

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* The 2-ic test is of course well known. The 3-bic test can be applied by aid of the Table of 3-bic Residuacity criteria on pp. 28, 29 of the paper on "4-tic and 3-bic Residuacity Tables," by Col. Cunningham and Mr. Th. Gosset, in *Messenger of Mathematics*, Vol. 50 (1920); which gives the criteria of $(z/p)_3 = 1$ and $\neq 1$ for all prime Bases < 50.

The numbers (z) thus selected are termed *trial roots.* With primes of form $p = 2^\kappa . 3^\lambda + 1$ this step is sufficient: giving z a primitive root.

STEP (2). *Testing the trial roots* (z).—The process is practically that of determining the Haupt-Exponent (ξ) of each "trial root" (z) until one is found giving $\xi = p-1$.

Let $p = K.Q+1$, where $K = 2^\kappa . 3^\lambda$, Q prime to K.

„ $Q = a^\alpha . b^\beta . c^\gamma \ldots$ [a, b, c different primes > 3].

Since z is such that $(z/p)_2 = -1$ and $(z/p)_3 \neq 1$, it follows that ξ_z must contain $K = 2^\kappa . 3^\lambda$.

The process consists in finding the Residues of $z^{\frac{1}{2}Kq}$ modulo p for *every possible factor* q (prime or composite) of Q—[including $q = 1$]. If at any step it is found that $z^{\frac{1}{2}Kq} \equiv -1$ (mod p) with $Kq < (p-1)$, then z is *not a primitive root,* but has $\xi_z = Kq$, $\nu_z = Q \div q$. If, finally, $z^{\frac{1}{2}Kq} \equiv -1$ (mod p), with $Kq = p-1$ (and for no less value of Kq), then z is a *primitive root.*

The *number* of different factors (q) of Q is $(\alpha+1)(\beta+1)(\gamma+1)\ldots$, and the one-half of that number is the full number of trials of $z^{\frac{1}{2}Kq}$ required to prove that $z^{\frac{1}{2}Kq} \equiv -1$ (mod p) only when $Kq = p-1$ (*i.e.* for no smaller value of Kq); when $Kq = \xi$ is $< (p-1)$ fewer trials will be required.

For primes of a given magnitude (say $p \not> M$), that number is a maximum when $K = 2$, giving $p = 2Q+1$, along with Q composed of *small* primes.

Ex.—The maximum number of trials required in the present tables is given by

$$p = 2 . 5^3 . 7 . 13 + 1 = 22751;$$

requiring 16 trials: of course, the number of trials is usually much less.

[The above process is *very much shorter* than that given in Mathew's *Theory of Numbers,* Art. 20 (there said to be due to Gauss): the process involves finding the Residue of *every power* (z^X) of each "trial root" up to the limit $X = \frac{1}{2}\xi$ if $\xi < p-1$, until a root z is found giving $\xi = p-1$: this is a very lengthy operation.]

11. *Previous tables of primitive roots.*—The existence in general of quite small primitive roots—at any rate of all primes $\not> 25410$—and the great importance (for practical numerical calculations) of having quite small primitive roots available seems to have been overlooked in the past, as will be evident from an examination of some of the more important existing tables. The tables examined below are:

(1) *Jacobi's Canon Arithmeticus,* Berlin, 1839.

(2) Desmarest's table of G in his *Traité de l'Analyse Indéterminée du second degré, &c.,* pp. 298–300, Paris, 1852.

(3) Reuschle's *Neue Zahlentheoretische Tabellen, &c.,* Stuttgart, 1856.

In the accounts given below of those tables the values of $\pm G$ here quoted are all $< \frac{1}{2}p$: whenever the tabulated value of $+G$ is $> \frac{1}{2}p$ its equivalent negative value, *i.e.* $-G' = -(p-G)$ is always quoted and this is always $< \frac{1}{2}p$.

(1) *Jacobi's Canon Arithmeticus.*—This gives the complete set of Residues of all $(p-1)$ powers (G^x) of the primitive root (G) used as the *Base* for each prime $p > 3$ up to $p < 1000$.

The following short table is an abstract of the *number of times* each of the small primitive roots $\pm G \ngtr 10$, and also the larger roots taken in groups $\pm G > 10$ but < 100, and $\pm G > 100$ but < 400, were actually used as the Bases of the tables.

$\pm G =$	2,	3,	5,	6,	9,	10;	> 10, < 100;	> 100, < 400	Total
Number of $+G$	5,	3,	2,	3,	.	59,	14	12	166
Number of $-G$	2,	.	.	.	2,	26,	21	17	

It will be seen that—with the exception of $\pm G = 10$ which was used 85 times—the very small roots were seldom used, whilst roots > 100 were used 29 times.

(2) *Desmarest's tables.*—In this work (pp. 298–300) is a table giving *one* primitive root $(\pm G)$ for nearly every odd prime < 10000. By far the greater number of these roots are small numbers < 30; but there is still quite a number (57) of large roots: an abstract is given below.

	$p < 1000$		$p > 1000$		
$\pm G =$	>30, <100;	>100, <500	>30, <100;	>100, <1000;	>1000, <5000
Number of $+G$	2	4	11	7	9
Number of $-G$	4	.	.	10	10

(3) *Reuschle's table.*—In this work (pp. 42–53) is a table giving the least primitive root of all odd primes $\ngtr 1000$; and one or two primitive roots (nearly always ± 2 or ± 10) for all odd primes > 1000 up to $\ngtr 5000$. In a few cases only the roots given are inconveniently large.

$$p = 2689,\quad 3001,\quad 3181,\quad 4733,$$

$$G = 130,\quad 640,\quad 294,\quad 140.$$

LEAST PRIMITIVE ROOTS (g, $-g$).

p	g, $-g$	p	g, $-g$	p	g, $-g$	p	g, $-g$	p	g, $-g$
3	2, 4	311	17, 2	701	2, 2	1123	2, 4	1583	5, 2
5	2, 2	313	10, 10	709	2, 2	1129	11, 11	1597	11, 11
7	3, 2	317	2, 2	719	11, 2	1151	17, 2	1601	3, 3
11	2, 3	331	3, 5	727	5, 7	1153	5, 5	1607	5, 2
13	2, 2	337	10, 10	733	6, 6	1163	5, 3	1609	7, 7
17	3, 3	347	2, 3	739	3, 6	1171	2, 4	1613	3, 3
19	2, 4	349	2, 2	743	5, 2	1181	7, 7	1619	2, 3
23	5, 2	353	3, 3	751	3, 2	1187	2, 3	1621	2, 2
29	2, 2	359	7, 2	757	2, 2	1193	3, 3	1627	3, 6
31	3, 7	367	6, 2	761	6, 6	1201	11, 11	1637	2, 2
37	2, 2	373	2, 2	769	11, 11	1213	2, 2	1657	11, 11
41	6, 6	379	2, 4	773	2, 2	1217	3, 3	1663	3, 2
43	3, 9	383	5, 2	787	2, 4	1223	5, 2	1667	2, 3
47	5, 2	389	2, 2	797	2, 2	1229	2, 2	1669	2, 2
53	2, 2	397	5, 5	809	3, 3	1231	3, 2	1693	2, 2
59	2, 3	401	3, 3	811	3, 5	1237	2, 2	1697	3, 3
61	2, 2	409	21, 21	821	2, 2	1249	7, 7	1699	3, 6
67	2, 4	419	2, 3	823	3, 2	1259	2, 3	1709	3, 3
71	7, 2	421	2, 2	827	2, 3	1277	2, 2	1721	3, 3
73	5, 5	431	7, 5	829	2, 2	1279	3, 2	1723	3, 6
79	3, 2	433	5, 5	839	11, 2	1283	2, 3	1733	2, 2
83	2, 3	439	15, 5	853	2, 2	1289	6, 6	1741	2, 2
89	3, 3	443	2, 3	857	3, 3	1291	2, 4	1747	2, 4
97	5, 5	449	3, 3	859	2, 4	1297	10, 10	1753	7, 7
101	2, 2	457	13, 13	863	5, 2	1301	2, 2	1759	6, 2
103	5, 2	461	2, 2	877	2, 2	1303	6, 2	1777	5, 5
107	2, 3	463	3, 2	881	3, 3	1307	2, 3	1783	10, 2
109	6, 6	467	2, 3	883	2, 4	1319	13, 2	1787	2, 3
113	3, 3	479	13, 2	887	5, 2	1321	13, 13	1789	6, 6
127	3, 9	487	3, 2	907	2, 4	1327	3, 9	1801	11, 11
131	2, 3	491	2, 5	911	17, 3	1361	3, 3	1811	6, 3
137	3, 3	499	7, 5	919	7, 5	1367	5, 2	1823	5, 2
139	2, 4	503	5, 2	929	3, 3	1373	2, 2	1831	3, 9
149	2, 2	509	2, 2	937	5, 5	1381	2, 2	1847	5, 2
151	6, 5	521	3, 3	941	2, 2	1399	13, 5	1861	2, 2
157	5, 5	523	2, 4	947	2, 3	1409	3, 3	1867	2, 4
163	2, 4	541	2, 2	953	3, 3	1423	3, 9	1871	14, 2
167	5, 2	547	2, 4	967	5, 2	1427	2, 3	1873	10, 10
173	2, 2	557	2, 2	971	6, 3	1429	6, 6	1877	2, 2
179	2, 3	563	2, 3	977	3, 3	1433	3, 3	1879	6, 2
181	2, 2	569	3, 3	983	5, 2	1439	7, 2	1889	3, 3
191	19, 2	571	3, 5	991	6, 2	1447	3, 2	1901	2, 2
193	5, 5	577	5, 5	997	7, 7	1451	2, 3	1907	2, 3
197	2, 2	587	2, 3	1009	11, 11	1453	2, 2	1913	3, 3
199	3, 2	593	3, 3	1013	3, 3	1459	3, 6	1931	2, 3
211	2, 4	599	7, 2	1019	2, 3	1471	6, 5	1933	5, 5
223	3, 9	601	7, 7	1021	10, 10	1481	3, 3	1949	2, 2
227	2, 3	607	3, 2	1031	14, 2	1483	2, 4	1951	3, 2
229	6, 6	613	2, 2	1033	5, 5	1487	5, 2	1973	2, 2
233	3, 3	617	3, 3	1039	3, 2	1489	14, 14	1979	2, 3
239	7, 2	619	2, 4	1049	3, 3	1493	2, 2	1987	2, 4
241	7, 7	631	3, 9	1051	7, 5	1499	2, 3	1993	5, 5
251	6, 3	641	3, 3	1061	2, 2	1511	11, 2	1997	2, 2
257	3, 3	643	11, 7	1063	3, 2	1523	2, 3	1999	3, 5
263	5, 2	647	5, 2	1069	6, 6	1531	2, 4	2003	5, 3
269	2, 2	653	2, 2	1087	3, 2	1543	5, 2	2011	3, 5
271	6, 2	659	2, 3	1091	2, 4	1549	2, 2	2017	5, 5
277	5, 5	661	2, 2	1093	5, 5	1553	3, 3	2027	2, 3
281	3, 3	673	5, 5	1097	3, 3	1559	19, 2	2029	2, 2
283	3, 6	677	2, 2	1103	5, 3	1567	3, 2	2039	7, 2
293	2, 2	683	5, 10	1109	2, 2	1571	2, 3	2053	2, 2
307	5, 7	691	3, 6	1117	2, 2	1579	3, 5	2063	5, 2

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p	$g, -g$	p	$g, -g$	p	$g, -g$	p	$g, -g$	p	$g, -g$
2069	2, 2	2579	2, 3	3079	6, 2	3607	5, 11	4129	13, 13
2081	3, 3	2591	7, 2	3083	2, 3	3613	2, 2	4133	2, 2
2083	2, 4	2593	7, 7	3089	3, 3	3617	3, 3	4139	2, 3
2087	5, 2	2609	3, 3	3109	6, 6	3623	5, 2	4153	5, 5
2089	7, 7	2617	5, 5	3119	7, 2	3631	15, 10	4157	2, 2
2099	2, 3	2621	2, 2	3121	7, 7	3637	2, 2	4159	3, 2
2111	7, 2	2633	3, 3	3137	3, 3	3643	2, 4	4177	5, 5
2113	5, 5	2647	3, 2	3163	3, 6	3659	2, 3	4201	11, 11
2129	3, 3	2657	3, 3	3167	5, 2	3671	13, 2	4211	6, 3
2131	2, 4	2659	2, 4	3169	7, 7	3673	5, 5	4217	3, 3
2137	10, 10	2663	5, 2	3181	7, 7	3677	2, 2	4219	2, 4
2141	2, 2	2671	7, 5	3187	2, 4	3691	2, 4	4229	2, 2
2143	3, 9	2677	2, 2	3191	11, 5	3697	5, 5	4231	3, 2
2153	3, 3	2683	2, 4	3203	2, 3	3701	2, 2	4241	3, 3
2161	23, 23	2687	5, 3	3209	3, 3	3709	2, 2	4243	2, 4
2179	7, 5	2689	19, 19	3217	5, 5	3719	7, 2	4253	2, 2
2203	5, 7	2693	2, 2	3221	10, 10	3727	3, 2	4259	2, 3
2207	5, 2	2699	2, 3	3229	6, 6	3733	2, 2	4261	2, 2
2213	2, 2	2707	2, 4	3251	6, 3	3739	7, 5	4271	7, 3
2221	2, 2	2711	7, 2	3253	2, 2	3761	3, 3	4273	5, 5
2237	2, 2	2713	5, 5	3257	3, 3	3767	5, 2	4283	2, 3
2239	3, 2	2719	3 2	3259	3, 5	3769	7, 7	4289	3, 3
2243	2, 3	2729	3, 3	3271	3, 5	3779	2, 3	4297	5, 5
2251	7, 5	2731	3 5	3299	2, 3	3793	5, 5	4327	3, 2
2267	2, 3	2741	2, 2	3301	6, 6	3797	2, 2	4337	3, 3
2269	2, 2	2749	6, 6	3307	2, 4	3803	2, 3	4339	10, 5
2273	3, 3	2753	3, 3	3313	10, 10	3821	3, 3	4349	2, 2
2281	7, 7	2767	3, 9	3319	6, 2	3823	3, 9	4357	2, 2
2287	19, 7	2777	3, 3	3323	2, 3	3833	3, 3	4363	2, 4
2293	2, 2	2789	2, 2	3329	3, 3	3847	5, 2	4373	2, 2
2297	5, 5	2791	6, 7	3331	3, 5	3851	2, 4	4391	14, 2
2309	2, 2	2797	2, 2	3343	5, 11	3853	2, 2	4397	2, 2
2311	3, 2	2801	3, 3	3347	2, 3	3863	5, 2	4409	3, 3
2333	2, 2	2803	2, 4	3359	11, 2	3877	2, 2	4421	3, 3
2339	2, 3	2819	2, 3	3361	22, 22	3881	13, 13	4423	3, 7
2341	7, 7	2833	5, 5	3371	2, 3	3889	11, 11	4441	21, 21
2347	3, 6	2837	2, 2	3373	5, 5	3907	2, 4	4447	3, 2
2351	13, 3	2843	2, 4	3389	3, 3	3911	13, 2	4451	2, 3
2357	2, 2	2851	2, 4	3391	3, 5	3917	2, 2	4457	3, 3
2371	2, 4	2857	11, 11	3407	5, 2	3919	3, 2	4463	5, 2
2377	5, 5	2861	2, 2	3413	2, 2	3923	2, 3	4481	3, 3
2381	3, 3	2879	7, 2	3433	5, 5	3929	3, 3	4483	2, 4
2383	5, 13	2887	5, 2	3449	3, 3	3931	2, 4	4493	2, 2
2389	2, 2	2897	3, 3	3457	7, 7	3943	3, 9	4507	2, 4
2393	3, 3	2903	5, 2	3461	2, 2	3947	2, 3	4513	7, 7
2399	11, 2	2909	2, 2	3463	3, 9	3967	6, 2	4517	2, 2
2411	6, 3	2917	5, 5	3467	2, 3	3989	2, 2	4519	3, 9
2417	3, 3	2927	5, 2	3469	2, 2	4001	3, 3	4523	5, 3
2423	5, 2	2939	2, 3	3491	2, 3	4003	2, 4	4547	2, 3
2437	2, 2	2953	13, 13	3499	2, 4	4007	5, 2	4549	6, 6
2441	6, 6	2957	2, 2	3511	7, 2	4013	2, 2	4561	11. 11
2447	5, 2	2963	2, 3	3517	2, 2	4019	2, 4	4567	3, 7
2459	2, 3	2969	3, 3	3527	5, 2	4021	2, 2	4583	5, 2
2467	2, 4	2971	10, 5	3529	17, 17	4027	3, 6	4591	11, 2
2473	5, 5	2999	17, 2	3533	2, 2	4049	3, 3	4597	5, 5
2477	2, 2	3001	14, 14	3539	2, 3	4051	10, 5	4603	2, 4
2503	3, 2	3011	2, 3	3541	7, 7	4057	5, 5	4621	2, 2
2521	17, 17	3019	2, 4	3547	2, 4	4073	3, 3	4637	2, 2
2531	2, 3	3023	5, 2	3557	2, 2	4079	11, 2	4639	3, 2
2539	2, 4	3037	2, 2	3559	3, 2	4091	2, 3	4643	5, 3
2543	5, 2	3041	3, 3	3571	2, 4	4093	2, 2	4649	3, 3
2549	2, 2	3049	11, 11	3581	2, 2	4099	2, 4	4651	3, 5
2551	6, 2	3061	6, 6	3583	3, 2	4111	12, 2	4657	15, 15
2557	2, 2	3067	2, 4	3593	3, 3	4127	5, 2	4663	3, 9

p	g, $-g$	p	g, $-g$	p	g, $-g$	p	g, $-g$	p	g, $-g$
4673	3, 3	5233	10, 10	5791	6, 2	6329	3, 3	6907	2, 4
4679	11, 2	5237	3, 3	5801	3, 3	6337	10, 10	6911	7, 2
4691	2, 3	5261	2, 2	5807	5, 2	6343	3, 2	6917	2, 2
4703	5, 2	5273	3, 3	5813	2, 2	6353	3, 3	6947	2, 3
4721	6, 6	5279	7, 3	5821	6, 6	6359	13, 2	6949	2, 2
4723	2, 4	5281	7, 7	5827	2, 4	6361	19, 19	6959	7, 3
4729	17, 17	5297	3, 3	5839	6, 2	6367	3, 2	6961	13, 13
4733	5, 5	5303	5, 2	5843	2, 4	6373	2, 2	6967	5, 13
4751	19, 3	5309	2, 2	5849	3, 3	6379	2, 4	6971	2, 4
4759	3, 5	5323	5, 10	5851	2, 4	6389	2, 2	6977	3, 3
4783	6, 2	5333	2, 2	5857	7, 7	6397	2, 2	6983	5, 2
4787	2, 3	5347	3, 6	5861	3, 3	6421	6, 6	6991	6, 2
4789	2, 2	5351	11, 2	5867	5, 3	6427	3, 6	6997	5, 5
4793	3, 3	5381	3, 3	5869	2, 2	6449	3, 3	7001	3, 3
4799	7, 2	5387	2, 3	5879	11, 2	6451	3, 6	7013	2, 2
4801	7, 7	5393	3, 3	5881	31, 31	6469	2, 2	7019	2, 3
4813	2, 2	5399	7, 2	5897	3, 3	6473	3, 3	7027	2, 4
4817	3, 3	5407	3, 2	5903	5, 2	6481	7, 7	7039	3, 2
4831	3, 2	5413	5, 5	5923	2, 4	6491	2, 3	7043	2, 4
4861	11, 11	5417	3, 3	5927	5, 2	6521	6, 6	7057	5, 5
4871	11, 3	5419	3, 5	5939	2, 3	6529	7, 7	7069	2, 2
4877	2, 2	5431	3, 2	5953	7, 7	6547	2, 4	7079	7, 2
4889	3, 3	5437	5, 5	5981	3, 3	6551	17, 2	7103	5, 2
4903	3, 2	5441	3, 3	5987	2, 3	6553	10, 10	7109	2, 2
4909	6, 6	5443	2, 4	6007	3, 9	6563	5, 10	7121	3, 3
4919	13, 2	5449	7, 7	6011	2, 4	6569	3, 3	7127	5, 2
4931	6, 3	5471	7, 3	6029	2, 2	6571	3, 7	7129	7, 7
4933	2, 2	5477	2, 2	6037	5, 5	6577	5, 5	7151	7, 3
4937	3, 3	5479	3, 2	6043	5, 6	6581	14, 14	7159	3, 2
4943	7, 2	5483	2, 3	6047	5, 2	6599	13, 2	7177	10, 10
4951	6, 2	5501	2, 2	6053	2, 2	6607	3, 2	7187	2, 3
4957	2, 2	5503	3, 9	6067	2, 4	6619	2, 4	7193	3, 3
4967	5, 2	5507	2, 3	6073	10, 10	6637	2, 2	7207	3, 2
4969	11, 11	5519	13, 2	6079	17, 7	6653	2, 2	7211	2, 3
4973	2, 2	5521	11, 11	6089	3, 3	6659	2, 3	7213	5, 5
4987	2, 4	5527	5, 2	6091	7, 11	6661	6, 6	7219	2, 4
4993	5, 5	5531	10, 5	6101	2, 2	6673	5, 5	7229	2, 2
4999	3, 9	5557	2, 2	6113	3, 3	6679	7, 5	7237	2, 2
5003	2, 3	5563	2, 4	6121	7, 7	6689	3, 3	7243	2, 4
5009	3, 3	5569	13, 13	6131	2, 3	6691	2, 4	7247	5, 2
5011	2, 4	5573	2, 2	6133	5, 5	6701	2, 2	7253	2, 2
5021	3, 3	5581	6, 6	6143	5, 2	6703	5, 2	7283	2, 3
5023	3, 2	5591	11, 2	6151	3, 7	6709	2, 2	7297	5, 5
5039	11, 2	5623	5, 2	6163	3, 6	6719	11, 2	7307	2, 3
5051	2, 3	5639	7, 2	6173	2, 2	6733	2, 2	7309	6, 6
5059	2, 4	5641	14, 14	6197	2, 2	6737	3, 3	7321	7, 7
5077	2, 2	5647	3, 2	6199	3, 2	6761	3, 3	7331	2, 4
5081	3, 3	5651	2, 3	6203	2, 3	6763	2, 4	7333	6, 6
5087	5, 2	5653	5, 5	6211	2, 4	6779	2, 3	7349	2, 2
5099	2, 3	5657	3, 3	6217	5, 5	6781	2, 2	7351	6, 5
5101	6, 6	5659	2, 4	6221	3, 3	6791	7, 3	7369	7, 7
5107	2, 4	5669	3, 3	6229	2, 2	6793	10, 10	7393	5, 5
5113	19, 19	5683	2, 4	6247	5, 2	6803	2, 3	7411	2, 4
5119	3, 2	5689	11, 11	6257	3, 3	6823	3, 2	7417	5, 5
5147	2, 3	5693	2, 2	6263	5, 2	6827	2, 3	7433	3, 3
5153	5, 5	5701	2, 2	6269	2, 2	6829	2, 2	7451	2, 4
5167	6, 11	5711	19, 3	6271	11, 17	6833	3, 3	7457	3, 3
5171	2, 4	5717	2, 2	6277	2, 2	6841	22, 22	7459	2, 4
5179	2, 4	5737	5, 5	6287	7, 2	6857	3, 3	7477	2, 2
5189	2, 2	5741	2, 2	6299	2, 3	6863	5, 2	7481	6, 6
5197	7, 7	5743	10, 2	6301	10, 10	6869	2, 2	7487	5, 3
5209	17, 17	5749	2, 2	6311	7, 2	6871	3, 9	7489	7, 7
5227	2, 4	5779	2, 4	6317	2, 2	6883	2, 4	7499	2, 3
5231	7, 2	5783	7, 2	6323	2, 3	6899	2, 3	7507	2, 4

p	$g, -g$	p	$g, -g$	p	$g, -g$	p	$g, -g$	p	$g, -g$
7517	2, 2	8087	5, 2	8677	2, 2	9227	2, 3	9791	11, 2
7523	2, 3	8089	17, 17	8681	15, 15	9239	19, 2	9803	2, 3
7529	3, 3	8093	2, 2	8689	13, 13	9241	13, 13	9811	3, 5
7537	7, 7	8101	6, 6	8693	2, 2	9257	3, 3	9817	5, 5
7541	2, 2	8111	11, 2	8699	2, 3	9277	5, 5	9829	10, 10
7547	2, 3	8117	2, 2	8707	5, 7	9281	3, 3	9833	3, 3
7549	2, 2	8123	2, 3	8713	5, 5	9283	2, 4	9839	7, 2
7559	13, 2	8147	2, 3	8719	3, 5	9293	2, 2	9851	2, 4
7561	13, 13	8161	7, 7	8731	2, 4	9311	7, 2	9857	5, 5
7573	2, 2	8167	3, 9	8737	5, 5	9319	3, 2	9859	2, 5
7577	3, 3	8171	2, 3	8741	2, 2	9323	2, 3	9871	3, 2
7583	5, 2	8179	2, 4	8747	2, 3	9337	5, 5	9883	2, 4
7589	2, 2	8191	17, 11	8753	3, 3	9341	2, 2	9887	5, 2
7591	6, 2	8209	7, 7	8761	23, 23	9343	5, 2	9901	2, 2
7603	2, 4	8219	2, 3	8779	11, 22	9349	2, 2	9907	2, 4
7607	5, 2	8221	2, 2	8783	5, 2	9371	2, 3	9923	2, 3
7621	2, 2	8231	11, 2	8803	2, 4	9377	3, 3	9929	3, 3
7639	7, 5	8233	10, 10	8807	5, 2	9391	3, 2	9931	10, 5
7643	2, 3	8237	2, 2	8819	2, 3	9397	2, 2	9941	2, 2
7649	3, 3	8243	2, 3	8821	2, 2	9403	3, 6	9949	2, 2
7669	2, 2	8263	3, 2	8831	7, 5	9413	3, 3	9967	3, 2
7673	3, 3	8269	2, 2	8837	2, 2	9419	2, 3	9973	11, 11
7681	17, 17	8273	3, 3	8839	3, 2	9421	2, 2	10007	5, 2
7687	6, 2	8287	3, 7	8849	3, 3	9431	7, 3	10009	11, 11
7691	2, 3	8291	2, 3	8861	2, 2	9433	5, 5	10037	2, 2
7699	3, 5	8293	2, 2	8863	3, 9	9437	2, 2	10039	3, 5
7703	5, 2	8297	3, 3	8867	2, 3	9439	22, 7	10061	3, 3
7717	2, 2	8311	3, 2	8887	3, 2	9461	3, 3	10067	2, 3
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17077	2, 2	17707	3, 6	18301	6, 6	19013	2, 2	19609	13, 13
17093	2, 2	17713	7, 7	18307	11, 13	19031	11, 5	19661	2, 2
17099	2, 3	17729	3, 3	18311	13, 2	19037	2, 2	19681	11, 11
17107	3, 6	17737	7, 7	18313	10, 10	19051	2, 4	19687	5, 7
17117	3, 3	17747	2, 3	18329	3, 3	19069	2, 2	19697	3, 3
17123	2, 3	17749	2, 2	18341	3, 3	19073	3, 3	19699	7, 5
17137	5, 5	17761	19, 19	18353	3, 3	19079	7, 2	19709	2, 2

p	$g, -g$	p	$g, -g$	p	$g, -g$	p	$g, -g$	p	$g, -g$
19717	2, 2	20333	3, 3	20983	3, 9	21589	2, 2	22189	2, 2
19727	5, 2	20341	2, 2	21001	11, 11	21599	7, 2	22193	3, 3
19739	2, 3	20347	3, 6	21011	2, 3	21601	7, 7	22229	2, 2
19751	7, 2	20353	5, 5	21013	2, 2	21611	2, 3	22247	5, 2
19753	5, 5	20357	2, 2	21017	3, 3	21613	2, 2	22259	2, 3
19759	3, 5	20359	11, 5	21019	2, 4	21617	3, 3	22271	7, 2
19763	2, 3	20369	3, 3	21023	5, 2	21647	5, 2	22273	5, 5
19777	11, 11	20389	6, 6	21031	12, 2	21649	14, 14	22277	2, 2
19793	3, 3	20393	3, 3	21059	2, 3	21661	2, 2	22279	3, 2
19801	13, 13	20399	7, 5	21061	7, 7	21673	10, 10	22283	2, 3
19813	2, 2	20407	5, 13	21067	2, 4	21683	2, 3	22291	3, 5
19819	3, 5	20411	6, 3	21089	3, 3	21701	2, 2	22303	6, 11
19841	3, 3	20431	3, 2	21101	2, 2	21713	3, 3	22307	2, 3
19843	19, 11	20441	3, 3	21107	2, 3	21727	3, 7	22343	5, 2
19853	2, 2	20443	2, 4	21121	19, 19	21737	6, 6	22349	2, 2
19861	11, 11	20477	2, 2	21139	2, 4	21739	2, 4	22367	5, 2
19867	2, 4	20479	3. 2	21143	10, 2	21751	3, 2	22369	11, 11
19889	3, 3	20483	5, 10	21149	3, 3	21757	5, 5	22381	10, 10
19891	2, 4	20507	2, 3	21157	2, 2	21767	5, 2	22391	13, 2
19913	3, 3	20509	2, 2	21163	2, 4	21773	2, 2	22397	2, 2
19919	7, 2	20521	11, 11	21169	13. 13	21787	23, 13	22409	3, 3
19927	6, 2	20533	2, 2	21179	2, 3	21799	7, 5	22433	3, 3
19937	3, 3	20543	5, 2	21187	2, 4	21803	2, 3	22441	14, 14
19949	2, 2	20549	2, 2	21191	7, 2	21817	7, 7	22447	3, 2
19961	6, 6	20551	3, 2	21193	11, 11	21821	2, 2	22453	5, 5
19963	2, 4	20563	3, 6	21211	2, 4	21839	11, 2	22469	2, 2
19973	2, 2	20593	3, 5	21221	2, 2	21841	11, 11	22481	3, 3
19979	2, 3	20599	3, 2	21227	2, 3	21851	6, 12	22483	2, 4
19991	11, 2	20611	2, 4	21247	3, 2	21859	2, 4	22501	2, 2
19993	10, 10	20627	2, 3	21269	2, 2	21863	5, 2	22511	11, 2
19997	2, 2	20639	11, 2	21277	6, 6	21871	6, 2	22531	2, 4
20011	12, 6	20641	7, 7	21283	11, 7	21881	3, 3	22541	3, 3
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20023	3, 2	20681	3, 3	21317	2, 2	21911	13, 2	22549	2, 2
20029	2, 2	20693	2, 2	21319	14, 2	21929	3, 3	22567	3, 2
20047	3, 2	20707	5, 14	21323	5, 3	21937	7, 7	22571	6, 3
20051	2, 3	20717	2, 2	21341	2. 2	21943	5, 2	22573	6, 6
20063	5, 3	20719	3, 7	21347	2, 4	21961	17, 17	22613	2, 2
20071	3, 2	20731	2, 4	21377	3, 3	21977	3, 3	22619	2, 3
20089	7, 7	20743	5, 11	21379	2, 4	21991	3, 9	22621	2, 2
20101	6, 6	20747	5, 3	21383	5, 2	21997	7, 7	22637	2, 2
20107	2, 4	20749	2, 2	21391	6, 2	22003	2, 4	22639	6, 5
20113	10, 10	20753	3, 3	21397	2, 2	22013	2, 2	22643	2, 3
20117	2, 2	20759	7, 2	21401	3, 3	22027	3, 6	22651	3, 6
20123	2, 3	20771	2, 3	21407	5, 2	22031	7, 2	22669	2, 2
20129	3, 3	20773	2, 2	21419	2, 3	22037	2, 2	22679	13, 2
20143	5, 11	20789	2, 2	21433	5, 5	22039	6, 2	22691	2, 3
20147	6, 3	20807	5, 2	21467	2, 3	22051	3, 5	22697	3, 3
20149	2, 2	20809	7, 7	21481	13, 13	22063	5, 2	22699	12, 5
20161	13, 13	20849	3, 3	21487	3, 9	22067	2, 3	22709	2, 2
20173	2, 2	20857	10, 10	21491	2, 3	22073	3, 3	22717	2, 2
20177	3, 3	20873	3, 3	21493	2, 2	22079	7, 2	22721	3, 3
20183	5, 2	20879	11, 2	21499	3, 6	22091	2, 3	22727	5, 2
20201	6, 6	20887	3, 2	21503	5, 2	22093	6, 6	22739	2, 3
20219	2, 3	20897	3, 3	21517	5, 5	22109	2, 2	22741	7, 7
20231	29, 5	20899	2, 4	21521	3, 3	22111	6, 10	22751	21, 2
20233	5, 5	20903	5, 2	21523	2, 4	22123	2, 4	22769	3, 3
20249	3, 3	20921	3, 3	21529	11, 11	22129	19, 19	22777	7, 7
20261	2, 2	20929	7, 7	21557	2, 2	22133	2, 2	22783	3, 2
20269	2, 2	20939	2, 3	21559	15, 5	22147	5, 7	22787	2, 3
20287	5, 2	20947	3, 6	21563	2, 3	22153	5, 5	22807	3, 9
20297	3, 3	20959	7, 13	21569	3, 3	22157	2, 2	22811	6, 3
20323	2, 4	20963	2, 3	21577	5, 5	22159	6, 2	22817	3, 3
20327	5, 2	20981	2, 2	21587	2, 3	22171	2, 4	22853	2, 2

p	$g, -g$	p	$g, -g$	p	$g, -g$	p	$g, -g$	p	$g, -g$
22859	2, 3	23339	2, 3	23869	2, 2	24371	2, 4	24943	5, 2
22861	2, 2	23357	2, 2	23873	3, 3	24373	7, 7	24953	3, 3
22871	7, 2	23369	3, 3	23879	7, 2	24379	2, 4	24967	3, 2
22877	2, 2	23371	2, 4	23887	3, 9	24391	3, 9	24971	2, 4
22901	2, 2	23399	17, 2	23893	5, 5	24407	5, 2	24977	3, 3
22907	5, 3	23417	3, 3	23899	2, 4	24413	2, 2	24979	2, 4
22921	7, 7	23431	3, 5	23909	2, 2	24419	2, 3	24989	2, 2
22937	3, 3	23447	5, 2	23911	6, 2	24421	7, 7	25013	2, 2
22943	5, 2	23459	2, 3	23917	2, 2	24439	6, 2	25031	13, 2
22961	6, 6	23473	5, 5	23929	7, 7	24443	2, 3	25033	5, 5
22963	3, 6	23497	5, 5	23957	2, 2	24469	14, 14	25037	2, 2
22973	2, 2	23509	2, 2	23971	10, 6	24473	3, 3	25057	5, 5
22993	5, 5	23531	2, 3	23977	5, 5	24481	11, 11	25073	3, 3
23003	2, 3	23537	3, 3	23981	3, 3	24499	2, 4	25087	3, 2
23011	7, 5	23539	2, 4	23993	3, 3	24509	2, 2	25097	3, 3
23017	5, 5	23549	2, 2	24001	14, 14	24517	5, 5	25111	3, 9
23021	2, 2	23557	5, 5	24007	17, 13	24527	5, 2	25117	5, 5
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23029	2, 2	23563	2, 4	24023	5, 2	24547	2, 4	25127	5, 2
23039	7, 2	23567	5, 2	24029	2, 2	24551	7, 2	25147	2, 4
23041	11, 11	23581	6, 6	24043	2, 4	24571	7, 37	25153	10, 10
23053	2, 2	23593	5, 5	24049	19, 19	24593	3, 3	25163	2, 3
23057	5, 5	23599	3, 9	24061	10, 10	24611	13, 3	25169	3, 3
23059	3, 5	23603	2, 3	24071	11, 2	24623	5, 2	25171	3, 5
23063	5, 2	23609	6, 6	24077	2, 2	24631	3, 5	25183	5, 2
23071	3, 2	23623	6, 2	24083	2, 3	24659	2, 3	25189	2, 2
23081	3, 3	23627	2, 3	24091	7, 5	24671	11, 2	25219	2, 4
23087	5, 2	23629	2, 2	24097	5, 5	24677	2, 2	25229	2, 2
23099	2, 3	23633	5, 5	24103	5, 2	24683	2, 3	25237	2, 2
23117	2, 2	23663	5, 2	24107	2, 3	24691	2, 4	25243	2, 4
23131	3, 5	23669	2, 2	24109	2, 2	24697	5, 5	25247	5, 2
23143	5, 13	23671	3, 5	24113	3, 3	24709	2, 2	25253	2, 2
23159	11, 2	23677	5, 5	24121	13, 13	24733	2, 2	25261	7, 7
23167	3, 2	23687	5, 2	24133	6, 6	24749	2, 2	25301	3, 3
23173	5, 5	23689	11, 11	24137	3, 3	24763	2, 4	25303	3, 2
23189	2, 2	23719	6, 2	24151	6, 5	24767	5, 2	25307	2, 3
23197	2, 2	23741	2, 2	24169	11, 11	24781	2, 2	25309	13, 13
23201	3, 3	23743	3, 9	24179	2, 3	24793	5, 5	25321	19, 19
23203	3, 6	23747	2, 4	24181	17, 17	24799	6, 2	25339	3, 6
23209	31, 31	23753	3, 3	24197	2, 2	24809	6, 6	25343	5, 2
23227	3, 9	23761	7, 7	24203	2, 3	24821	2, 2	25349	2, 2
23251	2, 4	23767	3, 2	24223	3, 2	24841	14, 14	25357	2, 2
23269	6, 6	23773	5, 5	24229	2, 2	24847	3, 2	25367	5, 2
23279	7, 3	23789	2, 2	24239	13, 2	24851	2, 3	25373	2, 2
23291	6, 3	23801	3, 3	24247	3, 2	24859	2, 4	25391	7, 3
23293	5, 5	23813	2, 2	24251	6, 3	24877	5, 5	25409	3, 3
23297	3, 3	23819	2, 3	24281	3, 3	24889	11, 11		
23311	3, 9	23827	2, 4	24317	2, 2	24907	2, 4		
23321	3, 3	23831	11, 2	24329	3, 3	24917	2, 2		
23327	5, 2	23833	5, 5	24337	5, 5	24919	3, 2		
23333	2, 2	23857	5, 5	24359	11, 2	24923	2, 3		

ON WATER WAVES DUE TO DISTURBANCE BENEATH THE SURFACE

By HORACE LAMB.

[Received April 10th, 1922.—Read April 20th, 1922.]

THE waves on deep water caused by a disturbing agency beneath the surface have been discussed in one or two cases of steady motion.* The present paper deals with some problems where the disturbance is due to a simple source whose strength varies with the time according to a prescribed law. These problems were suggested by a study of the phenomena of a submarine explosion, but it must be acknowledged that the results have only a restricted application in this connection. In the case of an explosion of more than a very moderate intensity (in relation to the depth) the free surface is disintegrated on the first arrival of the condensational wave. This modifies the subsequent development, whilst the cloud of spray produced masks whatever of a more orderly character follows. In this paper no further reference is made to this phenomenon; compressibility is in fact neglected throughout. It is hoped that in spite of this limitation the investigations may have some hydrodynamical interest.

1. As a preliminary we take the case of a periodic point-source at a depth f. To avoid indeterminateness due to the implicit inclusion of free waves of the prescribed period $(2\pi/\sigma)$, we have recourse to the usual hypothesis of a small frictional force varying as the velocity. The formula for the pressure is then†

$$\frac{p}{\rho}=\frac{\partial\phi}{\partial t}+\mu\phi-gz, \tag{1}$$

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* The waves generated by the uniform translation of a submerged cylinder were discussed by the author in *Ann. di Mat.* (3), Vol. 21 (1913), p. 237. The much more difficult case of the sphere was worked out by Havelock in *Proc. Roy. Soc.*, Vol. 93 (1919), p. 520.

† The square of the velocity (near the surface) is here neglected.

where the axis of z is supposed drawn vertically upwards through the source, and the origin is in the free surface. Hence if the pressure at a distance r due to the source alone is

$$p_1 = Ce^{i\sigma t}/r, \tag{2}$$

the corresponding velocity potential is

$$\phi_1 = \frac{Ce^{i\sigma t}}{(i\sigma + \mu)\,\rho r}. \tag{3}$$

By a known formula we have, at points above the level of the source,

$$\frac{1}{r} = \int_0^\infty e^{-\xi(f+z)} J_0(\xi\varpi)\, d\xi, \tag{4}$$

where ϖ denotes distance from the axis of z. We therefore assume, for the motion reflected from the free surface,

$$\phi_2 = e^{i\sigma t} \int_0^\infty e^{\xi z} J_0(\xi\varpi) f(\xi)\, d\xi. \tag{5}$$

The form of the function $f(\xi)$ is determined by the surface-condition $Dp/Dt = 0$, or

$$\frac{\partial^2 \phi}{\partial t^2} + \mu \frac{\partial \phi}{\partial t} + g \frac{\partial \phi}{\partial z} = 0 \quad [z = 0]. \tag{6}$$

Writing $\phi = \phi_1 + \phi_2$, and substituting, we find

$$f(\xi) = \frac{g\xi + \sigma^2 - i\mu\sigma}{g\xi - \sigma^2 + i\mu\sigma} \, \frac{Ce^{-\xi f}}{(i\sigma + \mu)\rho}. \tag{7}$$

The surface-elevation ζ is therefore given by

$$g\rho\zeta = (i\sigma + \mu)\,\rho\phi = 2Ce^{i\sigma t} \int_0^\infty \frac{g\xi e^{-\xi f} J_0(\xi\varpi)\, d\xi}{g\xi - \sigma^2 + i\mu\sigma}$$

$$= 2Ce^{i\sigma t} \left\{ \frac{1}{r} + \kappa \int_0^\infty \frac{e^{-\xi f} J_0(\xi\varpi)\, d\xi}{\xi - (\kappa - i\mu_1)} \right\}, \tag{8*}$$

where

$$\kappa = \sigma^2/g, \qquad \mu_1 = \mu/g. \tag{9}$$

The interpretation is simplest for the point vertically above the source.

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* Here, and elsewhere in the sequel, we put $\mu_1 = 0$ wherever this can be done without introducing indeterminateness.

Putting $\varpi = 0$, we have there

$$g\rho\zeta = 2Ce^{i\sigma t}\left\{\frac{1}{f}+\kappa\int_0^\infty \frac{e^{-\xi f}\,d\xi}{\xi-(\kappa-i\mu_1)}\right\}. \tag{10}$$

By a contour integration round the first quadrant of the plane of a complex variable $\xi+i\eta$, we find

$$\int_0^\infty \frac{e^{-\xi f}\,d\xi}{\xi-(\kappa-i\mu_1)} = \int_0^\infty \frac{e^{-i\eta f}\,d\eta}{\eta+i\kappa}, \tag{11}$$

where μ_1 has been finally put $= 0$. The latter integral can be expressed in terms of a known function. If we take the integral

$$\int \frac{e^{zf}\,dz}{z+\kappa}, \tag{12}$$

where $z = \xi+i\eta$, round the *third* quadrant of the aforesaid plane, excluding the singular point $(-\kappa, 0)$ by a small semicircle below the axis of ξ, we find

$$P\int_0^\infty \frac{e^{-\xi f}\,d\xi}{\xi-\kappa} - i\pi e^{-\kappa f} - \int_0^\infty \frac{e^{-i\eta f}\,d\eta}{\eta+i\kappa} = 0, \tag{13}$$

where the symbol P stands for "principal value of." Since

$$P\int_0^\infty \frac{e^{-\xi f}\,d\xi}{\xi-\kappa} = e^{-\kappa f}P\int_{-\kappa f}^\infty \frac{e^{-u}\,du}{u} = -e^{-\kappa f}\,\mathrm{Ei}(\kappa f), \tag{14}$$

in the usual notation of the exponential integral, we have, finally,

$$g\rho\zeta = 2\kappa Ce^{i\sigma t}\left\{\frac{1}{\kappa f} - e^{-\kappa f}\mathrm{Ei}(\kappa f) - i\pi e^{-\kappa f}\right\}. \tag{15}$$

Hence if the disturbing pressure be

$$p_1 = \frac{C}{r}\cos\sigma t, \tag{16}$$

the surface oscillation is

$$\zeta = \frac{2C}{g\rho f}(A\cos\sigma t + B\sin\sigma t), \tag{17}$$

where

$$A = 1-\kappa f e^{-\kappa f}\mathrm{Ei}(\kappa f), \qquad B = \pi\kappa f e^{-\kappa f}. \tag{18}$$

The following values of the coefficients A, B, and of the combination $\sqrt{(A^2+B^2)}$ which determines the amplitude of ζ, have been calculated with the help of Glaisher's tables* of $\mathrm{Ei}(x)$. It will be remembered that κ is proportional to the *square* of the frequency (σ).

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* *Phil. Trans.*, Vol. 160 (1870), p. 367.

TABLE I.

κf	A	B	$\surd(A^2+B^2)$
0	+1·0000	0	1·000
·1	1·1468	·2843	1·181
·2	1·1346	·5144	1·246
·3	1·0673	·6982	1·275
·4	·9719	·8423	1·286
·45	·9183	·9014	1·287
·5	·8623	·9527	1·285
·6	·7465	1·0345	1·276
·7	·6298	1·0920	1·261
·8	·5157	1·1293	1·241
·9	·4062	1·1495	1·219
1·0	·3028	1·1557	1·195
1·2	+ ·1173	1·1355	1·142
1·4	− ·0382	1·0846	1·085
1·6	·1646	1·0148	1·028
1·8	·2645	·9347	·971
2·0	·3379	·8503	·915
3·0	·4837	·4692	·674
4·0	·4382	·2302	·495
5·0	·3538	·1058	·369
6·0	·2789	·0467	·283
7·0	·2523	·0201	·253
8·0	·1818	·0084	·182
9·0	·1528	·0035	·153
10·0	·1315	·0014	·132
15·0	− ·0781	·0000	·078

The maximum amplitude is 1·287 corresponding to $\kappa f = {\cdot}455$ approximately, and makes $A = B$, so that the phase of the elevation follows that of the disturbing pressure p_1 by $\frac{1}{4}\pi$. The corresponding wave-length is $\lambda = 1{\cdot}38 f$.

For large values of the variable there is the asymptotic expansion

$$e^{-x}\,\mathrm{Ei}(x) = \frac{1}{x} + \frac{1}{x^2} + \frac{1\,.\,2}{x^3} + \frac{1\,.\,2\,.\,3}{x^4} + \dots. \tag{19}$$

Hence for rapid oscillations, or great depth of the source,

$$\zeta = -\frac{2C\cos\sigma t}{\rho f^2 \sigma^2}, \tag{20}$$

approximately, gravity having, as was to be expected, little influence. For small values of x on the other hand

$$\mathrm{Ei}(x) = \gamma + \log x + x + \frac{1}{2}\,\frac{x^2}{1\,.\,2} + \dots, \tag{21}$$

and therefore for slow oscillations, or small depth of the source,

$$\zeta = \frac{2C \cos \sigma t}{g\rho f}. \tag{22}$$

If we denote the amplitude of the source by Q, the constant C in (2) is equivalent to $i\sigma Q\rho/4\pi$. Hence, taking the real part, the surface-elevation due to a primary source

$$\phi_1 = \frac{Q}{4\pi r} \cos \sigma t \tag{23}$$

is given by

$$\zeta = \frac{Q}{2\pi g^{\frac{1}{2}} f^{\frac{3}{2}}} (A' \cos \sigma t - B' \sin \sigma t), \tag{24}$$

where $A' = \pi(\kappa f)^{\frac{3}{2}} e^{-\kappa f} = (\kappa f)^{\frac{1}{2}} B, \quad B' = (\kappa f)^{\frac{1}{2}} (1 - \kappa f e^{-\kappa f}) = (\kappa f)^{\frac{1}{2}} A.$ (25)

In the annexed Fig. 1, the value of $\sqrt{(A'^2 + B'^2)}$ is plotted against $\sqrt{(\kappa f)}$. This shows how the amplitude varies as a function of the frequency (σ).

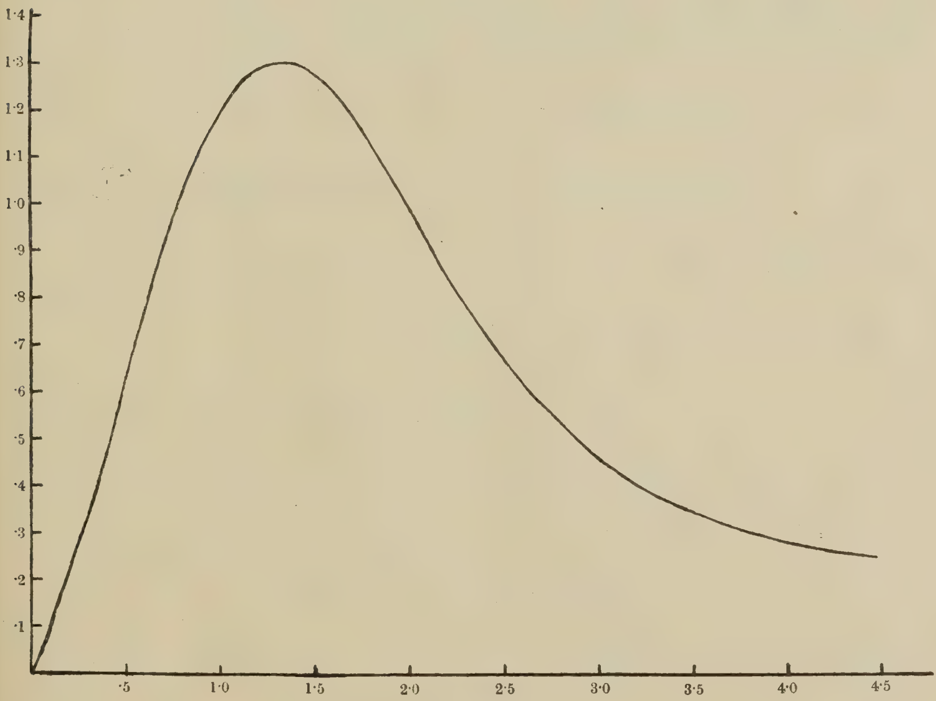

FIG. 1.

It is a matter of less interest to trace the annular waves propagated to a distance. Recalling that

$$J_0(x) = \frac{2}{\pi} \int_0^\infty \sin\,(x \cosh u)\, du,$$

and referring to (8), we have to consider the double integral

$$\int_0^\infty \left\{ \int_0^\infty (e^{im\xi} - e^{-im\xi})\, du \right\} \frac{e^{-\xi f}\, d\xi}{\xi - (\kappa - i\mu_1)}, \tag{26}$$

where $$m = \varpi \cosh u.$$

We may invert the order of integration. By integration round the contour of the first quadrant of the plane of $\xi + i\eta$, we have

$$\int_0^\infty \frac{e^{-\xi f + im\xi}\, d\xi}{\xi - (\kappa - i\mu_1)} = \int_0^\infty \frac{e^{-m\eta - i\eta f}\, d\eta}{\eta + i\kappa}, \tag{27}$$

where μ_1 has been put $= 0$ in the end. The fourth quadrant contains the singular point $\kappa - i\mu_1$. Excluding this by a small circle, and integrating round the boundary of the region which remains, we find

$$\int_0^\infty \frac{e^{-\xi f - im\xi}\, d\xi}{\xi - (\kappa - i\mu_1)} = -2\pi i e^{-\kappa f - im\kappa} + \int_0^\infty \frac{e^{-m\eta + i\eta f}\, d\eta}{\eta - i\kappa}. \tag{28}$$

Hence $$\int_0^\infty \frac{e^{-\xi f} \sin m\xi\, d\xi}{\xi - (\kappa - i\mu_1)} = \pi e^{-\kappa f - im\kappa} - \int_0^\infty \frac{(\eta \sin \eta f + \kappa \cos \eta f) e^{-m\eta}\, d\eta}{\eta^2 + \kappa^2}. \tag{29}$$

Thus (8) becomes

$$g\rho\xi = 2C e^{i\sigma t} \left\{ \frac{1}{r} + 2\kappa e^{-\kappa f} \int_0^\infty e^{-i\kappa\varpi \cosh u}\, du \right.$$
$$\left. - \frac{2\kappa}{\pi} \int_0^\infty \int_0^\infty \frac{(\eta \sin \eta f + \kappa \cos \eta f)\, e^{-\eta\varpi \cosh u}}{\eta^2 + \kappa^2}\, d\eta\, du \right\}. \tag{30}$$

The definite integral in the second term is a Bessel's function of the "second kind." Writing

$$D_0(x) = \frac{2}{\pi} \int_0^\infty e^{-ix \cosh u}\, du,$$

we have, for large values of x,

$$D_0(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{-i(x + \frac{1}{4}\pi)}, \tag{31}$$

approximately. For sufficiently large values of $\kappa\varpi$ the term in question therefore preponderates over the rest, and we have

$$g\rho\zeta = 2\kappa C\, e^{-\kappa f} \left(\frac{2\pi}{\kappa\varpi}\right)^{\frac{1}{2}} e^{i(\sigma t - \kappa\varpi - \frac{1}{4}\pi)}. \tag{32}$$

The wave-velocity has the usual value

$$\sigma/\kappa = \surd(g/\kappa) = \surd(g\lambda/2\pi),$$

where λ is the wave-length.

2. We proceed to the problems more especially in view. Let us suppose in the first instance that the pressure due to the source is zero up to the time $t = 0$, and then follows the law

$$p_1 = A\, e^{-\lambda t}/r. \tag{33}$$

This gives an abrupt rise followed by a gradual fall, and so resembles in a general way the case of an explosion. The formula (33) is adopted, however, for mathematical convenience, rather than as a very exact representation of the facts. The proposed law of variation is equivalent to

$$p_1 = \frac{A}{2\pi i r} \int_{-\infty}^{\infty} \frac{e^{i\sigma t}\, d\sigma}{\sigma - i\lambda} \tag{34}$$

for all values of t.*

We confine our attention to the part of the surface vertically above the source. Referring to (8), and putting $\varpi = 0$, we have

$$g\rho\zeta = \frac{A}{\pi i} \int_{-\infty}^{\infty} \left\{ \int_0^{\infty} \frac{g\xi e^{-\xi f} d\xi}{g\xi - \sigma^2 + i\mu\sigma} \right\} \frac{e^{i\sigma t}}{\sigma - i\lambda}\, d\sigma. \tag{35}$$

This makes

$$g\rho\left(\frac{d\zeta}{dt} + \lambda\zeta\right) = \frac{A}{\pi} \int_{-\infty}^{\infty} \left\{ \int_0^{\infty} \frac{g\xi\, e^{-\xi f}\, d\xi}{g\xi - \sigma^2 + i\mu\,\sigma} \right\} e^{i\sigma t} d\sigma. \tag{36}$$

To integrate first with respect to σ, we write

$$\sigma^2 - i\mu\sigma - g\xi = (\sigma - \sigma_1)(\sigma - \sigma_2), \qquad 37)$$

where

$$\left.\begin{matrix}\sigma_1 \\ \sigma_2\end{matrix}\right\} = \tfrac{1}{2} i\mu \pm \alpha. \tag{38}$$

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* *Hydrodynamics*, Art. 248.

Here
$$a = \sqrt{(g\xi)} \tag{39}$$
with sufficient approximation. We take the integral
$$\int \frac{e^{izt}\,dz}{(z-\sigma_1)(z-\sigma_2)}, \tag{40}$$
where $z = \sigma + i\tau$, round the boundary of the region lying above the axis of σ, the singular points σ_1, σ_2 being excluded by small circles described about them. We find
$$\int_{-\infty}^{\infty} \frac{e^{i\sigma t}\,d\sigma}{(\sigma-\sigma_1)(\sigma-\sigma_2)} = \frac{\pi i}{a}\,(e^{iat} - e^{-iat}) = -\,2\pi\frac{\sin at}{a}, \tag{41}$$
putting $\mu = 0$ in the end. Since $\xi = a^2/g$, by (39), the formula (36) becomes
$$g\rho\left(\frac{d\zeta}{dt} + \lambda\zeta\right) = \frac{4A}{g}\int_0^{\infty} a^2 e^{-a^2 f/g}\sin at\,da. \tag{42}$$

To connect the second member with a known form, we put
$$\omega = \sqrt{(a^2f/g)}, \qquad \beta = \sqrt{(gt^2/4f)}, \qquad k = \sqrt{(4\lambda^2 f/g)}. \tag{43}$$

Thus
$$\begin{aligned} g\rho\left(\frac{d\zeta}{d\beta} + k\zeta\right) &= \frac{8A}{f}\int_0^{\infty} \omega^2 e^{-\omega^2}\sin 2\beta\omega\,d\omega \\ &= -\frac{2A}{f}\,\frac{d^2}{d\beta^2}\int_0^{\infty} e^{-\omega^2}\sin 2\beta\omega\,d\omega. \end{aligned} \tag{44}$$

Now
$$\int_0^{\infty} e^{-\omega^2}\sin 2\beta\omega\,d\omega = e^{-\beta^2}\int_0^{\beta} e^{x^2}\,dx, \tag{45}$$
and therefore
$$\frac{d\zeta}{d\beta} + k\zeta = \frac{4A}{g\rho f}\,F(\beta), \tag{46}$$
where
$$F(\beta) = \beta + (1-2\beta^2)\,e^{-\beta^2}\int_0^{\beta} e^{x^2}\,dx. \tag{47}$$

We have to solve (46) subject to the condition that $\zeta = 0$ for $\beta = 0$.

The integral
$$\int_0^{\beta} e^{x^2}\,dx \tag{48}$$
has been tabulated by Dawson* for values of β ranging from 0 to 2 at intervals of ·01. The following table gives a sufficient range of values of $F(\beta)$ and $F'(\beta)$ for our purpose.

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* *Proc. London Math. Soc.*, Ser. 1, Vol. 29 (1898), p. 521.

TABLE II.

β	$F(\beta)$	$F'(\beta)$	β	$F(\beta)$	$F'(\beta)$
0	0	+2·000			
·1	+ ·197	1·921	1·1	+ ·353	−1·079
·2	·379	1·693	1·2	·246	1·026
·3	·531	1·342	1·3	·150	·912
·4	·649	·908	1·4	·067	·744
·5	·712	+ ·439	1·5	+ ·001	·573
·6	·733	− ·019	1·6	− ·048	·407
·7	·710	·425	1·7	·081	·259
·8	·651	·744	1·8	·100	·136
·9	·565	·963	1·9	·109	− ·041
1·0	+ ·462	−1·076	2·0	− ·110	+ ·027

The asymptotic value of the integral (48) is*

$$e^{\beta^2}\left(\frac{1}{2\beta}+\frac{1}{2^2\beta^3}+\frac{1\,.\,3}{2^3\beta^5}+\frac{1\,.\,3\,.\,5}{2^4\beta^7}+\dots\right), \tag{49}$$

so that $F(\beta)=-1/2\beta^3$, ultimately, for large values of β.

In most cases of any interest the quantity k defined in (43) is a large number. Hence, if we identify β with the time (to which it is, in fact, proportional), and for a moment regard ζ as a velocity, (46) is the equation of motion of a particle subject to a certain law of force increasing from zero, and to a great viscous resistance. In this analogy the inertia of the particle hardly counts, except in the very beginning of the motion, and the term $d\zeta/d\beta$ may be neglected. Hence, except for very small values of β, the solution of (46) under the condition stated is

$$\zeta=\frac{2A}{g\rho fk}F(\beta), \tag{50}$$

approximately. The same thing appears from the formal solution

$$\zeta=\frac{4A\,e^{-k\beta}}{g\rho f}\int_0^\beta F(\beta)\,e^{k\beta}\,d\beta. \tag{51}$$

By a partial integration

$$e^{-k\beta}\int_0^\beta F(\beta)\,e^{k\beta}\,d\beta=\frac{F(\beta)}{k}-\frac{e^{-k\beta}}{k}\int_0^\beta F'(\beta)\,e^{k\beta}\,d\beta, \tag{52}$$

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* Stokes, *Camb. Trans.*, Vol. 10, p. 106.

where
$$F'(\beta) = 2-2\beta^2+(4\beta^3-6\beta)\,e^{-\beta^2}\int_0^\beta e^{x^2}\,dx. \tag{53}$$

It appears from Table II that $F'(\beta)$ never exceeds 2. The second term on the right-hand side of (52) is therefore less than $2/k^2$, and for the most part negligible. This verifies (50).

For very minute values of β, however, the first and second terms may be comparable, though both are very small. We have then $F(\beta) = 2\beta$, approximately, and
$$e^{-k\beta}\int_0^\beta F(\beta)\,e^{k\beta}\,d\beta = \frac{2}{k^2}(k\beta-1+e^{-k\beta}). \tag{54}$$

It is to be noted that, though β may be small, $k\beta$ is not necessarily so. When, however, $k\beta$ is also small the expression in brackets is of the second order in $k\beta$. It was, in fact, to be expected that ζ would vary initially as t^2.

The annexed Fig. 2 shows the relation between ζ and t. The abscissa is β, or $\sqrt{(gt^2/4f)}$, and the time-scale therefore varies as $\sqrt{f}$. The ordinate of the curve, except close to the origin, is $F(\beta)$, to which ζ is proportional, by (43).

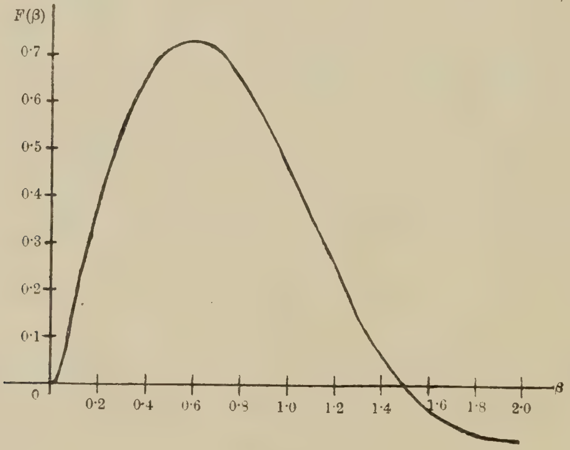

FIG. 2.

3. I take next a case where the strength of the source rises gradually from zero to a maximum and then falls gradually to zero again, the velocity-potential of the source alone being
$$\phi_1 = \frac{Q}{4\pi^2 r}\,\frac{\tau}{t^2+\tau^2}, \tag{55}$$

where Q is the total output. If we make τ approach the limit zero, we can pass to the case of an instantaneous expansion of finite amount. Omitting a constant factor, we may write in place of (55),

$$\phi_1 = \frac{1}{(\tau - it)r} = \frac{1}{r}\int_0^\infty e^{-\sigma\tau + i\sigma t}\, d\sigma, \tag{56}$$

provided we retain in the end only the real part.

Proceeding, as in § 1, we should find that the complete velocity-potential resulting from a source $e^{i\sigma t}/r$ is, at the free surface,

$$2\int_0^\infty \frac{g\xi e^{-\xi f} J_0(\xi\varpi)\, d\xi}{g\xi - \sigma^2 + i\mu\sigma}. \tag{57}$$

Hence corresponding to (56), we have

$$\phi = 2\int_0^\infty \left\{ \int_0^\infty \frac{g\xi e^{-\xi f} J_0(\xi\varpi)\, d\xi}{g\xi - \sigma^2 + i\mu\sigma} \right\} e^{-\sigma\tau + i\sigma t}\, d\sigma, \tag{58}$$

for $z = 0$.

Inverting the order of integration, we have to evaluate

$$\int_0^\infty \frac{e^{-\sigma\tau + i\sigma t}\, d\sigma}{(\sigma - \sigma_1)(\sigma - \sigma_2)}, \tag{59}$$

where σ_1, σ_2 are defined by (38). We transform this by contour integration, taking separately the cases of $t > 0$ and $t < 0$. For $t > 0$ we use the contour of the first quadrant in the plane of the complex variable $\sigma + im$. Since this region includes the singular point $\sigma_1 = a + \frac{1}{2}i\mu$, we obtain

$$\frac{i\pi}{a} e^{-a\tau + iat} - i\int_0^\infty \frac{e^{-mt - im\tau}\, dm}{m^2 + a^2}, \tag{60}$$

putting $\mu = 0$ in the result. For $t < 0$ we use the contour of the fourth quadrant. Since there is here no singularity the result is

$$i\int_0^\infty \frac{e^{mt + im\tau}\, dm}{m^2 + a^2}. \tag{61}$$

It is easily verified that (60) and (61) coincide for $t = 0$. Confining our attention as before to the part of the surface directly above the source, and therefore putting $\varpi = 0$ in (58), and rejecting the imaginary part, we find, for $t > 0$,

$$\phi/g = 2\pi\int_0^\infty \xi e^{-a\tau - \xi f} \frac{\sin at}{a}\, d\xi + 2\int_0^\infty \left\{ \int_0^\infty \frac{e^{-mt} \sin m\tau\, dm}{m^2 + a^2} \right\} e^{-\xi f} \xi\, d\xi, \tag{62}$$

and, for $t < 0$,

$$\phi/g = 2\int_0^\infty \left\{\int_0^\infty \frac{e^{mt}\sin m\tau\, dm}{m^2+a^2}\right\} e^{-\xi f}\xi\, d\xi. \tag{63}$$

It is to be remembered here that $a^2 = g\xi$. Hence, for $t > 0$,

$$\zeta = \frac{4\pi}{g^2}\int_0^\infty e^{-a^2f/g-a\tau}\cos at\, a^3 da - \frac{4}{g^2}\int_0^\infty \left\{\int_0^\infty \frac{e^{-mt}m\sin m\tau\, dm}{m^2+a^2}\right\} e^{-a^2f/g}a^3 da, \tag{64}$$

and, for $t < 0$,

$$\zeta = \frac{4}{g^2}\int_0^\infty \left\{\int_0^\infty \frac{e^{mt}\, m\sin m\tau\, dm}{m^2+a^2}\right\} e^{-a^2f/g}a^3 da. \tag{65}$$

The case of an instantaneous expansion ($\tau \to 0$) is specially simple. We have then $\zeta = 0$ for $t < 0$, and

$$\zeta = \frac{4\pi}{g^2}\int_0^\infty e^{-a^2f/g}\cos at\, a^3 da, \tag{66}$$

for $t > 0$. If we put

$$\omega = \sqrt{(a^2f/g)}, \qquad \beta = \sqrt{(gt^2/4f)}, \tag{67}$$

the latter expression becomes

$$\zeta = \frac{4\pi}{f^2}\int_0^\infty e^{-\omega^2}\cos 2\beta\omega\,\omega^3 d\omega = -\frac{\pi}{2f^2}\frac{d^3}{d\beta^3}\int_0^\omega e^{-\omega^2}\sin 2\beta\omega\, d\omega$$

$$= -\frac{\pi}{2f^2}\frac{d^3}{d\beta^3}e^{-\beta^2}\int_0^\beta e^{x^2}dx = \frac{\pi}{f^2}F'(\beta), \tag{68}$$

where $F'(\beta)$ is defined by (53). Restoring the factors omitted in (53), we have

$$\zeta = \frac{Q}{4\pi f^2}F'(\beta). \tag{69}$$

The function $F'(\beta)$ has been tabulated on p. 367. The lower curve of Fig. 3 shows the graph of $\frac{1}{10}F'(\beta)$, or $4\pi f^2\zeta/10Q$, the factor $\frac{1}{10}$ being introduced in order to bring the curve within the range of the diagram. The asymptotic value of $F'(\beta)$ is $3/2\beta^4$, whence ultimately

$$\zeta = \frac{6Q}{\pi g^2 t^4}. \tag{70}$$

For values of τ other than zero the interpretation is more difficult. The functions involved in (64) and (65) can be reduced to simple quadratures, but the numerical work for any assigned value of τ is considerable. I was tempted to work out, by way of illustration, the case of $\tau = \frac{1}{4}\pi\sqrt{(f/g)}$.

this apparently fanciful value being chosen with a view to diminish the labour of interpolation with the tables employed. It is not worth while to reproduce the rather intricate details of the numerical calculations, but I give the resulting graph of $4\pi f^2 \zeta / Q$ in Fig. 3. The comparison with the case of $\tau = 0$, shown (with diminished ordinates) on the same diagram, is interesting.

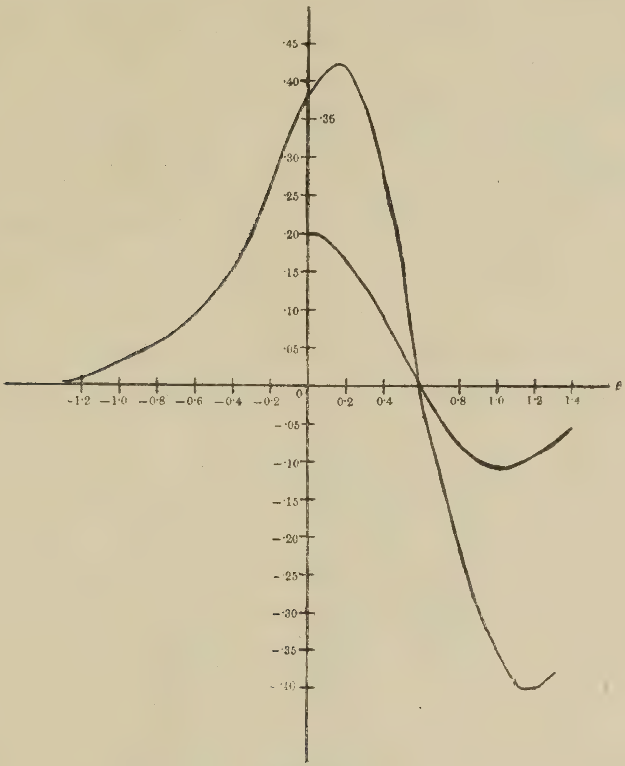

FIG. 3.

It may be noticed that the value of ζ at the instant $t = 0$ can be reduced to known functions. It is true that this is of minor interest, since the maximum value is then not yet attained, as is obvious dynamically, and is further illustrated by the diagram. Since

$$\int_0^\infty \frac{m \sin m\tau \, dm}{m^2 + a^2} = \tfrac{1}{2}\pi e^{-a\tau}, \tag{71}$$

we have, putting $t = 0$ in (64) or (65),

$$\zeta = \frac{2\pi}{g^2} \frac{\partial^3}{\partial \tau^3} \int_0^\infty e^{-a^2 f/g - a\tau} \, da. \tag{72}$$

Writing $$\alpha+\frac{g\tau}{2f}=\left(\frac{g}{f}\right)^{\frac{1}{2}}x, \quad \frac{g\tau^2}{4f}=\omega^2, \tag{73}$$

we have $$\int_0^\infty e^{-\alpha^2 f/g-\alpha\tau}\,d\alpha=\left(\frac{g}{f}\right)^{\frac{1}{2}}e^{\omega^2}\int_\omega^\infty e^{-x^2}\,dx, \tag{74}$$

and therefore

$$\zeta=-\frac{\pi}{4f^2}\frac{d^3}{d\omega^3}e^{\omega^2}\int_\omega^\infty e^{-x^2}\,dx=\frac{\pi}{f^2}\left\{1+\omega^2-(3\omega+2\omega^3)\,e^{\omega^2}\int_\omega^\infty e^{-x^2}\,dx\right\}. \tag{75}$$

This is easily calculated with the help of the tables of the "error-function"

$$\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}\,dx. \tag{76}$$

The following brief table shows how the elevation at the middle instant $t=0$ of the original disturbance diminishes as τ increases.

TABLE III.

ω	$\zeta f^2/\pi$	ω	$\zeta f^2/\pi$
0	1·000		
·05	·877	·8	·156
·1	·770	·9	·128
·2	·598	1·0	·105
·3	·469	1·1	·088
·4	·371	1·2	·073
·5	·295	1·3	·061
·6	·237	1·4	·052
·7	·192	1·5	·044

SEXTACTIC CONES AND TRITANGENT PLANES OF THE SAME SYSTEM OF A QUADRI-CUBIC CURVE

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[Received April 26th, 1922.—Read May 11th, 1922.]

1. *Introduction.*

In 1864, Clebsch investigated by means of the periods of Abelian Functions the fundamental properties of the Quadri-Cubic Curve (*Crelle*, Vol. 63, "Ueber die Anwendung der Abelschen Functionen in der Geometrie"). This curve is the complete intersection of the quadric and cubic surfaces and is of genus $p = 4$. Many properties are obtained in the above memoir analogous to those of the bitangents, quadritangent conics, sextactic cubics, etc. in the case of the plane quartic curve of genus $p = 3$. In particular Clebsch enunciates the following statement:—

> "*There are* 255 *systems of surfaces of the second order, which touch the curve of the sixth order in* 6 *different points; the points of contact of every two surfaces of the same system lie on a surface of the second order.*
>
> "*In each of these systems exist* 28 *surfaces, which break up into pairs of triple-tangent planes. The* 12 *points of contact of every two such pairs lie on a pencil of surfaces of the second order, i.e. on a space-curve of the fourth order.*"

The problem of the tritangent planes of the quadri-cubic curve was again attacked in 1893 by E. Pascal in four papers published in the *Atti della Reale Accademia dei Lincei, Rendiconti*, Vol. 2, 1° Semestre:—

(i) (Page 65) "Su di una estensione della configurazione delle 10 rette della superficie di 5° ordine a quintica doppia."

(ii) (Page 120) "Osservazioni sui gruppi di sostituzioni fra le caratteristiche dispari di genere 3 e di genere 4."

(iii) (Page 204) "Sulla configurazione dei 120 piani tritangenti della storta di genere 4."

(iv) (Page 239) "Sui piani tritangenti della sestica storta di genere 4."

Reference ought to be made also to a set of three papers by Ed. Weyr in Vol. 76 (1873) of the *Comptes Rendus*, "Classification des courbes du sixième ordre dans l'espace." Weyr's three papers, however, deal with the different possible types of sextic curves in space, and not with the properties of quadri-cubics in particular.

In none of these memoirs, however, is the problem attacked by the processes of synthetic geometry, and I propose in the present communication to establish by direct methods the above-mentioned properties given by Clebsch in so far as they appertain to sextactic quadrics and tritangent planes *of the same system.* No attempt will be made in this paper to discuss the mutual properties of the 255 systems. By this application of synthetic methods, however, many new properties are revealed, and the whole configuration is considerably simplified and rendered easily capable of visualization.

2. *The Sextactic Quadrics and Cones.*

Let the quadri-cubic curve be regarded as the complete intersection of the surfaces of the second and third degrees Γ_2 and Γ_3. Let S_2 be a surface of the second degree touching the curve $\Gamma_2\Gamma_3$ at six distinct points. We wish to obtain in simple form the complete system (S_2).

Since S_2 touches $\Gamma_2\Gamma_3$ at six distinct points, the quadri-quadric $S_2\Gamma_2$ must touch the cubic surface Γ_3 at six distinct points. Regarding $S_2\Gamma_2$ as an elliptic curve of genus $p = 1$, we see that the six elliptic parameters (u) of the points of contact of $S_2\Gamma_2$ with Γ_3 must satisfy the relation $\sum_1^6 u_r = \frac{1}{2}$ (period). Hence any other quadric Q_2 through the six points of contact will cut $S_2\Gamma_2$ in two points whose elliptic parameters v, w satisfy the relation $v+w = \frac{1}{2}$ (period). The chord vw must therefore pass through U, one of the vertices of the four cones containing $S_2\Gamma_2$. Consider now the twisted cubic T through the above six points of contact (u). Since the points v, w, together with the six points (u), constitute the eight points of intersection of the three quadrics S_2, Γ_2, Q_2, we know that the line vw is a chord of the twisted cubic T. But it has been shown that the line vw generates a cone of the second degree. Hence this cone must be one of the quadrics containing T, since it is generated by a chord of the curve, and being a cone its vertex U must lie on T. Hence we have the following result:—

If S_2 *be a sextactic quadric of the quadri-cubic* $\Gamma_2\Gamma_3$, *one of the vertices of the cones of the pencil* $S_2\Gamma_2$ *lies on the twisted cubic through the six points of contact.*

Denoting this sextactic cone by U_2, its vertex by U, and the system of such sextactic cones by (U_2), it will be found convenient to regard the system of sextactic quadrics as those belonging to the pencils (U_2, Γ_2). It must be noted also that of the pencil of sextactic quadrics containing the curve $U_2\Gamma_2$ four are cones, and only one of these sextactic cones has its vertex on the twisted cubic through the six points of contact.

Now all the cubic surfaces of the linear system

$$\Gamma_3' \equiv \Gamma_3 + (lx+my+nz+pt)\Gamma_2 = 0$$

contain the quadri-cubic curve $\Gamma_2\Gamma_3$, the coefficients l, m, n, p being entirely at our disposal. Let us so choose these coefficients that Γ_3' shall pass through four points on T other than the six points (u); there will thus lie $6+4 \equiv 10$ points of T on Γ_3', this number being one more than the maximum number of intersections of a general cubic surface with a twisted cubic. Hence T must lie completely on Γ_3', which is thus completely determined.

We shall now use the property that there are on every cubic surface *two* systems of twisted cubics corresponding to each double-six on the surface, such that two twisted cubics of the same system intersect in *one* point, while two twisted cubics of opposite systems intersect in *five* points. Also a quadric through a twisted cubic of one system intersects the cubic surface in a twisted cubic of the opposite system, and has, in fact, five points of contact with the cubic surface where these two twisted cubics intersect. Finally, if the cubic surface be such that these two systems of twisted cubics coalesce, the surface has four nodes through which passes every twisted cubic curve on the surface. Let, therefore, P be any point on T, and let UP meet Γ_3' again in P_1. Plainly, therefore, the locus of P_1 will be a twisted cubic T_1 of the opposite system to T. But the line UP has been shown to be a chord vw of $U_2\Gamma_2$. Hence, if P approach indefinitely close to the point of intersection u of T and $U_2\Gamma_2$, let v also approach indefinitely close to u. Now the quadri-quadric $U_2\Gamma_2$ is known to touch the cubic surface Γ_3' at u. Hence v must be the third point of intersection P_1 of the chord UP with Γ_3'. Thus P and P_1 will ultimately coincide at u. The twisted cubics T and T_1 will therefore intersect at the *six* points (u), and hence the cone U_2 containing both T and T_1 must touch Γ_3' at the six points (u). But a quadric containing two twisted cubics of a cubic surface can touch that surface at *five* points at most. Hence T and T_1 must coincide and the cone U_2 will touch Γ_3' at all points of T. Now the enveloping cone from U to Γ_3' is of the fourth degree and part of this enveloping cone has been shown to be U_2. Hence another cone $\bar{U}_2$ of the second degree can be drawn having U as vertex and such that every generator touches Γ_3'. The surface Γ_3' will therefore be quadrinodal having its four nodes at the points where it is met by the four common rays of U_2 and $\bar{U}_2$. Consider now any other point U' lying on Γ_3', and let U_2' be one of the two cones of the second degree that can be drawn from U' enveloping Γ_3', and let its curve of contact be the twisted cubic T'. Then the quadri-cubic $\Gamma_2\Gamma_3'$ will plainly touch U_2' at the six points in which T' intersects Γ_2. Also the twisted cubics T and T' are known to lie on the same quadric. The complete configuration of sextactic quadrics of the same system as described by Clebsch in the above memoir is now plain:—

If one of the quadrinodal cubic surfaces that contain a quadri-cubic curve be taken, the enveloping cones of the second degree that can be drawn to the surface from points upon it constitute the complete set of sextactic cones (U_2) *of the same system that can be drawn to the quadri-cubic curve; and the quadrics* (Q_2) *that contain the points of contact of any two sextactic cones (of the same system) with the quadri-cubic curve are the pencils of quadrics determined by* Γ_2 *and the quadrics that contain any two twisted cubics lying wholly on the quadrinodal cubic surface.*

Many interesting properties spring at once from the above result:—

(1) *All twisted cubics through the sets of six points of contact of sextactic cones of the same system, pass through the same four points, namely, the four nodes of the quadrinodal cubic surface. Hence, since six points uniquely determine a twisted cubic, any two points of contact of a sextactic cone will determine (along with the four nodal points) a twisted cubic of the system, and therefore the other four points of contact of the set of six. Also, these twisted cubics all meet each of three fixed coplanar lines, namely, the three lines lying wholly on the quadrinodal surface (other than those lines joining nodes).*

(2) *All sextactic cones* (U_2) *of the same system pass through the same four points, namely, the four nodal points; and touch the same three coplanar lines, namely, the three lines of the quadrinodal surface.*

(3) *All quadrics on which lie pairs of twisted cubics through the points of contact of two sextactic cones* (U_2) *of the same system pass through the same four points, namely, the four nodal points.*

(4) *There are three coplanar chords of the same system of the quadri-cubic through each of which can be drawn two bitangent planes of the curve, and the chords joining the points of contact of each such bitangent plane form the edges of a tetrahedron.*

Let p, q, r be the three coplanar lines lying completely in Γ_3', and let the nodes be A, B, C, D. Let p, q, r meet the pairs of lines BC, AD; CA, BD; AB, CD respectively. Then of the two enveloping cones that can be drawn from any point of the line p, one consists of the two planes pBC, pAD [Townsend, *Quarterly Journal*, Vol. 10 (1870)]. Hence the planes pBC, pAD will touch the quadri-cubic at the points where BC, AD meet Γ_2 respectively, thus accounting for four of the points of contact of the degenerate sextactic cone (pBC ; pAD). But as the line p lies on each plane, the points of intersection of p with Γ_2 must be counted twice over amongst the intersections of the sextactic cone (pBC ; pAD) with the quadri-cubic. Hence the set of six "points of contact" of the degenerate sextactic cone (pBC ; pAD) are accounted for. The line joining the points of contact of the bitangent plane pBC plainly passes through the nodal points B, C; and similarly with regard to the other such planes. Hence the lines joining the points of contact of the bitangent planes pBC, pAD; qCA, qBD; rAB, rCD form respectively the opposite edges of the tetrahedron whose vertices are the nodal points A, B, C, D.

3. *The Tritangent Planes.*

Consider now a cubic surface C_3 and a plane γ such that Γ_3' is the Polo-Cubic Surface of γ with respect to C_3; that is, Γ_3' is the envelope of the polar planes with respect to C_3 of points on γ, or Γ_3' is the locus of points whose polar-quadrics with respect to C_3 touch γ (Reye, *Geometrie der Lage*, Vol. 3, Chap. 15). Also, the locus of all points whose polar planes with respect to C_3 touch Γ_2 is a quartic surface K_4. Hence the locus of all points whose polar-planes with respect to C_3 touch both Γ_3' and Γ_2 is the plane quartic curve k_4, which is the curve of intersection of K_4 and γ. Let now l be any straight line lying in γ. Then the polar-planes with respect to C_3 of all points on l is a quadric cone U_2 having its vertex U on Γ_3' and enveloping the surface. It is therefore evident that the four common tangent planes of U_2 and the enveloping cone from U to Γ_2 are the polar-planes with respect to C_3 of the four points in which l intersects k_4. If, in particular, l be a bitangent of k_4, the above two cones will have double contact with each other, and therefore U_2 will touch Γ_2 at two distinct points. Hence the curve of intersection of U_2 and Γ_2 will consist of two conics σ_1 and σ_2 lying in two planes π_1 and π_2 respectively. The conics σ_1 and σ_2 will meet each other at the two points in which they are each met by the line $\pi_1\pi_2$. Also each conic σ and each plane π touch the quadri-cubic curve $\Gamma_2\Gamma_3'$ at three distinct points, and therefore constitute a system of tritangent conics and tritangent planes. We have therefore the following result:—

There are 28 *positions of* U *on* Γ_3' *corresponding to the* 28 *bitangents of* k_4, *such that* U_2 *touches* Γ_2 *at two distinct points. The curve of intersection of* U_2 *and* Γ_2 *consists of two conics lying in planes which are each tritangent planes of the quadri-cubic curve* $\Gamma_2\Gamma_3'$.

We shall, hereafter, refer to π_1, π_2 as a "tritangent plane-pair" of the system, corresponding to the vertex U. Many properties of these plane-pairs can be deduced at once from the properties of the bitangents of a plane quartic curve. It will, therefore, be found useful to state those properties of a quadrinodal cubic surface which will be required in what follows. A quadrinodal cubic surface is the reciprocal of Steiner's Quartic Surface, which is discussed by Sturm (*Math. Annalen*, Vol. 3, "Ueber die Römische Fläche von Steiner"). On Steiner's Surface lie pairs of Rational Twisted Quartics which constitute the complete intersection of the surface with a quadric. These pairs of Twisted Quartics intersect in four points, and hence the quadric containing such a pair of Twisted Quartics touches the surface at four points. Reciprocally, the quadrinodal cubic surface is inscribed in pairs of Rational Developables of class 4, the generating planes of which all touch the same quadric, which therefore touches the surface at four distinct points, and the tangent planes at these four points are the common tangent planes of the two developables. Such surfaces of the second degree we shall refer to as "Quadritangent Quadrics" of Γ_3'. Each rational quartic on Steiner's Surface is uniquely defined by five points on the curve, and reciprocally each rational developable of class 4 circumscribed to the quadrinodal cubic surface is uniquely defined by five tangent planes to the surface.

Let, therefore, S_1 be a conic lying in γ, and let the polar-planes of points on S_1 with respect to C_3 envelop the rational developable Δ_1 of class 4, circumscribed to Γ_3'. Then, to the eight points in which S_1 intersects k_4 will correspond the eight generating planes of Δ_1 that touch Γ_2. In particular, if S_1 touch k at four distinct points, Δ_1 will touch Γ_2 at four distinct points. Now we know that if S_1 touch k_4 at four distinct points, any conic S_{12} through these four points of contact will meet k_4 again in other four points at each of which another conic S_2 will touch k_4. We have therefore the following result :—

If Δ_1 *be a rational developable of class 4 circumscribed to* Γ_3' *and touching* Γ_2 *at four distinct points, and if* Δ_{12} *be another such rational developable touching the four tangent planes common to* Δ_1 *and* Γ_2, *the remaining four tangent planes common to* Δ_{12} *and* Γ_2 *will be such that a rational developable* Δ_2 *of* Γ_3' *will touch* Γ_2 *at four points and have these four planes as the common tangent planes at the points of contact.*

This may be otherwise put in a form which brings into prominence the quadritangent quadrics of Γ_3'. Let Ψ be the unique quadritangent quadric of Γ_3' inscribed in Δ, a rational developable of class 4, circumscribed to Γ_3'.

If the developable surface $\Psi_1\Gamma_3'$ *touch* Γ_2 *at four distinct points, and if another developable surface* $\Psi_{12}\Gamma_3'$ *touch these four common tangent planes, then the remaining four tangent planes common to* $\Psi_{12}\Gamma_3'$ *and* Γ_2 *will be such that a quadritangent quadric* Ψ_2 *of* Γ_3' *can be found such that the developable* $\Psi_2\Gamma_3'$ *touches* Γ_2 *at four points and has these four planes as the common tangent planes at the points of contact.*

In particular, for six positions of S_{12}, the conic S_2 will degenerate into pairs of bitangents of k_4, and therefore for six positions of Δ_{12} (or Ψ_{12}), Δ_2 will degenerate into a pair of cones U_2, V_2 of the second degree, each enveloping Γ_3' and intersecting Γ_2 respectively in the tritangent conic-pairs σ_1, σ_2 ; τ_1, τ_2 which lie in the respective tritangent plane-pairs π_1, π_2 ; ρ_1, ρ_2. We shall say that such a configuration of six cone-pairs (U_2, V_2) belong to the same "set." We may take S_1 to consist of two bitangents of k_4, and hence Δ_1 will consist of a cone-pair. Δ_{12} will therefore be defined as the pencil of rational developables touching the tangent planes to Γ_2 at the points where Γ_2 is touched by the cone-pair. The following result is therefore immediately evident in virtue of what has gone before :—

If $U_2^{(r)}$; $V_2^{(r)}$ *and* $U_2^{(s)}$; $V_2^{(s)}$ *be two cone-pairs of the same set, so that their conics of section with* Γ_2 *define the tritangent plane-pairs* $\pi_1^{(r)}, \pi_2^{(r)}$; $\rho_1^{(r)}, \rho_2^{(r)}$ *and* $\pi_1^{(s)}, \pi_2^{(s)}$; $\rho_1^{(s)}, \rho_2^{(s)}$ *respectively, then the eight tangent planes to* Γ_2 *at the points where it is met by the lines* $\pi_1^{(r)}\pi_2^{(r)}$, $\rho_1^{(r)}\rho_2^{(r)}$, $\pi_1^{(s)}\pi_2^{(s)}$, $\rho_1^{(s)}\rho_2^{(s)}$ *all touch the same quadritangent quadric* Ψ_{rs} *of* Γ_3'

Finally, we know that the six points of intersection of the six pairs of bitangents of k_4 belonging to the same set (S_2) lie on a conic. But these six points of intersection correspond to the fourth common tangent plane of the two cones U_2, V_2 (excluding the three common tangent planes joining the vertices U, V to the lines p, q, r lying wholly on Γ_3'). The following result appears immediately:—

The six common tangent planes of the six cone-pairs (U_2, V_2) *of the same set (excluding those common tangent planes that contain* p, q, r) *all touch the same quadritangent quadric of* Γ_3'.

Thus, knowing the properties of the 28 bitangents of a plane quartic curve, the complete configuration of the 28 tritangent plane-pairs of the same system can be easily deduced.

ON THE GENERAL INVARIANT THEORY OF QUADRICS

By H. W. TURNBULL.

[Received May 26th, 1922.—Read June 8th, 1922.]

The Prepared System for n Quadrics.

The success with which Gordan* and Ciamberlini† dealt with systems of quadratics in ternary forms by symbolic methods was largely due to a quality of equivalence which seems to be inherent in quadratics of all dimensions—ternary and higher. This may be stated as follows :—

Let
$$\begin{aligned} f &= a_x^2 = a_x'^2 = \dots, \\ f_1 &= b_x^2 = b_x'^2 = \dots, \\ f_2 &= c_x^2 = c_x'^2 = \dots, \\ &\dots \quad \dots \quad \dots \quad \dots \end{aligned}$$

be a system of quadratics in m variables. Then any concomitant P of these quadratics may be expressed as a sum of terms ΣT, each term being an aggregate of factors $(\alpha_r \beta_s \dots \iota_k)$, where

$\alpha_r = aa'a'' \dots,$ a convolution of r symbols, of *currency* r,

$\beta_s = bb' \dots$,, s ,, ,, s,

$\upsilon_k =$ a convolution of k cogredient variables u, v, w,

and
$$r+s+\dots+k = m, \quad 0 \leqslant r, s, \dots, k.$$

This is the fundamental theorem, which as it stands is too general to be of practical use. For although every symbol convolved in α_r must occur again elsewhere in the same term of T, the chance that the whole set $aa'a'' \dots$ in α_r should be convolved twice is small. But by adding new bracket types, here symbolized by square brackets

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* Gordan, *Math. Ann.*, Vol. 56.

† Ciamberlini, *Battaglini*, Vol. 24.

$$[a_r \beta_s \dots \nu_k];$$

where the sum of the suffixes may exceed m, and where the elements composing the bracket do not necessarily obey the law of determinants, we may explicitly pair off every symbol a_r, β_s, ... which occurs in the original form of T. When this is done, in the event of T containing more than one symbol a_r belonging to f, all such symbols a_r, a'_r, a''_r may be interchanged indifferently, so that the distinguishing marks $'$ $''$ may be suppressed. The form P so expressed is said to be *prepared*, and the set of brackets including the generalized brackets $[a_r \beta_s \dots \nu_k]$ is called the *prepared system.*

In general the prepared system has not been found, beyond the case of three quadrics* (where $m = 4$). I propose to determine the system for any number of quadrics, showing that the most general type of bracket required to express any concomitant of any number of quadrics involves nine quadrics each to the second degree.

Gordan's Theorem.

The existence of a finite prepared system for any number of quadratics $f, f_1, f_2, \dots, f_p$ not only proves Gordan's theorem, that the number of irreducible concomitants of $f, f_1, f_2, \dots, f_p$ is finite, but it also leads to an explicit statement of the system. This may be shown as follows: let N, a finite number, denote the number of brackets in the prepared system. Let G_i typify one such bracket. Then any concomitant P may be expressed as a series of terms such as $G_{i_1} G_{i_2} G_{i_3} \dots G_{i_k}$. If these suffixes do not all differ the form is reducible, for it contains a square factor $G_{i_1}^2$ say, which is an actual concomitant of the system. Thus the irreducible forms can only occur when all the factors G differ. So the number of such forms is not greater than the number of combinations of N things taken one, two, three, ... at a time without repetition; and this number is finite.

Types.

It is convenient to use this term for quadratics in a slightly different sense from the usual.† Two brackets $[a_r \beta_s \dots]$, $[a_{r_1} \beta_{s_1} \dots]$ are of the same type if by an interchange of all the symbols of one quadratic f with those of another quadratic f_1, or by a series of like interchanges, one bracket is transformed into the other.

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* Turnbull, *Proc. London Math. Soc.*, Ser. 2, Vol. 20, p. 465.

† Cf. Grace and Young, *Algebra of Invariants*, p. 319.

It follows that the prepared system for n quadrics is known if all the necessary types are known.

Order and Degree.

Let $[a_r \beta_s \dots \kappa_t \nu_k]$ be a bracket factor, where $a, \beta, \dots, \kappa$ refer to i different quadratics $f, f_1, \dots$, then the total degree of the factor is $(r+s+\dots+t)$, or, what is more important, the distributed degree in the i ground forms is $(r, s, \dots, t)$, while the order is i. The symbol F_i denotes any such factor. The same definition applies when perhaps a, β refer to the same form f, but $r \neq s$.

With quaternary and higher forms a complexity arises because the factor may contain two equivalent symbols as well as the i different symbols. Such pairs of included symbols increase the degree by half the total suffix of the pairs, but have no effect on the order F_i. For example, $(AB\gamma\gamma p)$ is an F_2 bracket for quaternary forms, and its degree is (2, 2, 3).

Characteristic.

The original brackets contain variables

$$u, \ \overline{uu_1} = p, \ \overline{uu_1u_2}, \ \dots, \ \overline{uu_1u_2\dots u_{m-2}} = x$$

of currencies, let us say, 1, 2, 3, ..., $m-1$, where all the symbols u are cogredient, and x denotes, geometrically, point coordinates. This leads us to consider the *characteristic* of a bracket, which is best shown by taking particular cases. Thus in quinary forms, the brackets $(abcdu)$, $(abcp)$, $(ab\overline{uu_1u_2})$, a_x have characteristics (11111), (1112), (113), (14), where the numerals indicate the currency of the convolved elements. And, further, all brackets containing symbols of the same three currencies as those in $(A\beta\gamma)$ for quaternary forms have the same characteristic, namely, (233). For example, $(pa\beta)$, $(\overline{ab}\,\gamma\delta)$, $(A\beta x)$ all have this characteristic.

Principle of Duality.

Two characteristics $(i_1 i_2 i_3 \dots i_k)$ and $(i'_1 i'_2 i'_3 \dots i'_k)$ are said to be reciprocal if $i_r + i'_r = m$ $(r = 1, 2, 3, \dots, k)$. A prepared system must involve reciprocal characteristics, for it admits the symbols a_{m-1}, β_{m-1} on exactly the same footing as a and b; in fact, for quaternary forms, the existence of the fundamental bracket type (abp) implies the existence of the reciprocal type $(\alpha\beta p)$. We can indeed forecast a considerable portion of a prepared system by forming brackets reciprocal to the original bracket types.

If m is even, as in binary and quaternary forms, there are self-reciprocal elements of currency $m/2$, and self-reciprocal bracket factors entirely made up of these elements. They form as it were a backbone to the system with symmetrical groupings on either side, towards x and towards u respectively. When $m = 4$, the symbols A, B and the line coordinates p occupy this central position; so, when $m = 6$ for space of five dimensions, *plane* coordinates, of currency three, occupy the central position.

If m is odd, as in ternary forms, there is not the same striking symmetry, for no symbol is self-reciprocal.

Although the discussion of the algebra of quadratic forms opens with a reference to the point coordinate forms a_x^2, b_x^2, c_x^2, ..., this view, so to say, enfilades the field of operations from one extreme position. To proceed to the prepared system is to take up the central position which gives the true perspective.

Quaternary Forms.

The first two steps in seeking the prepared system for any number of quadrics are the same as those for three quadrics. In fact, if A is (aa') of currency two, and α is $(aa'a'')$ of currency three, then (1) a bracket $(aa'a''a''')$ or a'''_α implies a duplicate factor, and the invariant a_α^2 emerges; and (2) a bracket $(aa'a''i)$, where i does not refer to the quadric $f = a_x^2$, implies a second convolution $aa'a''$ $(= \alpha)$, provided that we introduce the new bracket types

$$(\alpha\beta p), \quad (\alpha\beta C), \quad (\alpha\beta\gamma x), \quad (\alpha\beta\gamma\delta),$$

involving two, three, or four quadrics.

Suppose all the α, β, γ, ... symbols of currency three to be paired, then the possible types of brackets now arising are

$$a_\beta, \ (abC), \ (abcd), \ (\alpha\beta C), \ (\alpha\beta\gamma\delta), \tag{1}$$

and brackets with u, p, x taking the place of currency 1, 2, 3 symbols in this same group of brackets.

Next suppose $A (= aa')$ to exist: then, as in the case of three quadrics the complements of a, a' may also be paired, provided that we introduce the following series of new brackets :

$$\begin{array}{ll}
(abb'a), & (iaa'b)\,b'_\lambda, \\
(abb'c)(c'dd'a'), & (iaa'b)(b'cc'j), \\
(abb'c)(c'dd'e)(e'ff'a'), & (\lambda a)(a'bb'c)(c'\mu), \\
\dots\quad\dots\quad\dots\quad\dots & (iaa'b)(bcc'd)(d'\lambda), \\
 & \dots\quad\dots\quad\dots
\end{array}$$

where each of aa', bb', cc', ... stand for p or the complements of a currency two symbol A, B, C ; and where the currencies of i, j, λ, μ are 1, 1, 3, 3 respectively.

Let the operator Ω_b denote the determinantal permutation of b, b' : let Ω_{ac} denote the same for a, a' and c, c' permuted independently ; then the above series of brackets may be conveniently written

$$\left.\begin{array}{ll}
(AB), & (iAB\lambda) \;\;= \Omega_b(iAb)\,b'_\lambda, \\
(ABCD) \;\;\;= \Omega_{ac}(abb'c)(c'dd'a'), & (iABCj) \;= \Omega_b(iAb)(b'Cj), \\
(ABCDEF) = \Omega_{ace}(abb'c)(c'dd'e)(e'ff'a'), & (\lambda ABC\mu) = \Omega_{ac}(\lambda a)(a'Bc)(c'\mu), \\
\dots\quad\dots\quad\dots & (iABCD\lambda). \\
 & \dots\quad\dots\quad\dots
\end{array}\right\} \tag{2}$$

This set (2) together with (1) gives rise to the prepared system ; for every symbol u, p, x is explicit, and each symbol a, A, α is explicitly paired. Moreover, all symbols of one currency and one quadric are now equivalent, since the interchange of any two surviving symbols a'', a''' would lead to a new A, and the interchange of A, A' to a new α, and the interchange of α, α' to a reducing factor a_α^2.

It is best to classify these brackets in four types—

(I) a_β, $(abcd)$, $(\alpha\beta\gamma\delta)$, a_x, u_α, $(abcu)$, $(\alpha\beta\gamma x)$.

(II) $(iAB\,\dots\,Kj)$ and its dual $(\lambda AB\,\dots\,K\mu)$.

(III) $(iAB\,\dots\,K\lambda)$.

(IV) $(AB\,\dots\,K)$.

Here type (I) has no symbol A or p : (II) has an odd number of symbols

A, p; III and IV have each an even number of symbols A, p. The number of symbols $A, B, \ldots, K$ in the bracket is called its *grade*: so the grade may be zero, one, two, three and upwards.

Properties of these Brackets.

1. If the grade exceeds two, the symbols of currency two in a bracket may be interchanged as in a determinant.

In fact if P denote the bracket $(\ldots DEF \ldots)$, it may be written

$$\Omega_{df}(\rho d)(d'Ef)f'_\sigma M \text{ or else } \Omega_e(lDe)(e'Fm)M.$$

The first of these is

$$= \Omega_{ef}(\rho e)(d'de'f)f'_\sigma M + \Omega_f(\rho f)(d'Ed)f'_\sigma M$$

$$= -(\ldots EDF \ldots) - (DE)(\ldots F \ldots),$$

and similarly for the second. Hence the bracket $(\ldots DEF \ldots)$ differs from $-(\ldots EDF \ldots)$ by a reducible term made of two factors of lower grade. As any two consecutive symbols D, E may thus be interchanged, with change of sign, the whole bracket behaves as a determinant, and the symbols $A, B, \ldots, K$ can be written in any order. It follows that $A, B, \ldots, K$ must all differ, else the bracket reduces.

2. If two of $A, B, \ldots, K$ are equivalent, A and A' say, the bracket reduces. For AA' may be brought into adjacent positions, and a currency three symbol emerges.

3. For the same reason the end symbols i, j, λ, μ must not belong to the same quadrics as any of $A, B, \ldots, K$; and if p is among $A, B, \ldots, K$, then u and x are excluded from i, j, λ, μ.

4. The end symbols may or may not be alike.

5. The upper limit to the grade is ten, the number of coefficients of x in a_x^2. For any concomitant of more than nine quadrics may, by a general theorem of Peano,* be expressed in terms of the nine quadrics, with the possible exception of the invariant of degree one in the coefficients of ten quadrics. This exceptional invariant must be a product of factors of type

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* Cf. Grace and Young, *Ibid.*, p. 359.

$(abcd)$ only. So we have ten different symbols $A, B, \ldots, p$ at our disposal for one bracket, limiting the grade to ten.

6. Type (IV) reduces if the grade is a multiple of 4. For let it contain $2n$ symbols $A, B, \ldots$ and therefore n original brackets, being in fact

$$P = \Omega_n (aBc)(c'De) \ldots (k'La'),$$

where Ω_n implies the n separate interchanges in the n pairs $aa', cc', \ldots, kk'$.

Since $(k'La') = -(a'Lk')$ and there are n such brackets, P may be written backwards as $(-)^n \Omega_n (a'Lk') \ldots (eDc')(cBa)$.

But $$\Omega_a a_\rho a'_\sigma = -\Omega_a a'_\rho a_\sigma,$$

and the same for $k, \ldots, e, c$. Hence

$$P = (-)^{2n} \Omega_n (aLk) \ldots (e'Dc)(c'Ba') = (-)^{2n} (ALK \ldots DCB),$$

which is $\overline{n-1}$ interchanges from the original form, so that

$$P \equiv (-)^{2n+n-1} P.$$

This reduces P if n is even, but not if n is odd; hence it reduces if $2n$, the grade, is 0 mod 4.

7. In type (II) if the end symbols are equal, $i = j$ or $\kappa = \lambda$, the same reduction applies for grades 1, 5, 9. Also since when $i \neq j$ or $\kappa \neq \lambda$ there would be eleven symbols in a grade 9 bracket, it follows that all grade 9 brackets are reducible.

8. If the grade is greater than unity a form reduces which contains besides this bracket the factor of type $(abcd)$ or $(\alpha\beta\gamma\delta)$. For a reduction in grade can be made by introducing one A symbol from the chain bracket to this other bracket.

These considerations show that the prepared system contains brackets of all grades not exceeding eight, together with

$$(ABCDEFGHIp)$$

of grade ten, involving p and nine quadrics. We may thus exhibit the types of the prepared system in ascending grade as follows:

Grade.				
0	a_x, u_a, a_β	$(abcu)$, $(\alpha\beta\gamma x)$	$(abcd)$, $(\alpha\beta\gamma\delta)$	
1	(iAj)	(kpj)	$(\lambda A\mu)$	$(\nu p\mu)$
2	$(iAB\lambda)$	$(kAp\mu)$	(AB)	(Ap)
3	$(iABCj)$	$(kABpj)$	$(\lambda ABC\mu)$	$(\nu ABp\mu)$
4	$(iABCD\lambda)$	$(kABCp\mu)$		
5	$(iABCDEj)$	$(kABCDpj)$	$(\lambda ABCDE\mu)$	$(\nu ABCDp\mu)$
6	$(iABCDEF\lambda)$	$(kABCDEp\mu)$	$(ABCDEF)$	$(ABCDEp)$
7	$(iABCDEFGj)$	$(kABCDEFpj)$	$(\lambda ABCDEFG\mu)$	$(\nu ABCDEFp\mu)$
8	$(iABCDEFGH\lambda)$	$(kABCDEFGp\mu)$		
10			$(ABCDEFGHIp)$	

where $$i = u, a, b, c, \ldots, \qquad \lambda = x, a, \beta, \gamma, \ldots,$$

$$j, k = \quad a, b, c, \ldots, \qquad \mu, \nu = \quad a, \beta, \gamma, \ldots,$$

and no two elements in the same bracket belong to the same quadric, or are both variables, excepting the case of the end elements $i, j, k, \lambda, \mu, \nu$ for grades 2, 3, 4, 6, 7, 8.

ON THE MANNER OF DIVERGENCE OF THE LEGENDRE SERIES OF AN INTEGRABLE FUNCTION

By B. M. WILSON.

[Received April 27th, 1922.—Read May 11th, 1922.]

1. It is well known that the Legendre series of a function $f(x)$ which has a Lebesgue integral in the interval $(-1, 1)$ is not necessarily convergent, and may, in fact, fail to converge for every value of x in the interval.* It is therefore natural to enquire how pronounced this failure to converge may be; an analogous question presents itself in the theory of Fourier series, and there receives its most complete answer in a theorem due to Hardy, who has shown† that, *if* a_n, b_n *are the Fourier coefficients of an integrable function, the series*

$$\sum \frac{a_n \cos nx + b_n \sin nx}{\log n}$$

converges for almost all values of x *in* $(0, 2\pi)$. The object of the present note is to establish, for Legendre series, a theorem analogous to Hardy's theorem for Fourier series; the required theorem is Theorem 4 below. Theorem 1 would seem to be the natural analogue for Legendre series of the well known theorem of Riemann-Lebesgue in the theory of Fourier series.‡ In Theorem 2 we obtain an estimate of the order of the partial sums of a Legendre series; the result is interesting, as, on comparison

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* Thus Stieltjes noted that the Legendre series of $(1-x)^{-\omega}$ does not converge for any value of x if $\omega \geqslant \frac{3}{4}$. See *Correspondance d'Hermite et de Stieltjes*, Vol. 2, p. 46 (letter of May 1890).

† *Proc. London Math. Soc.*, Ser. 2, Vol. 12 (1913), pp. 365–372. I take pleasure in expressing my thanks to Prof. Hardy for having called my attention to his paper, and suggested to me the discussion, from a similar point of view, of Legendre series.

‡ By the theorem of Riemann-Lebesgue we understand the theorem that, if $F(x)$ is integrable in (α, β),

$$\int_\alpha^\beta F(x) \frac{\sin}{\cos} nx\, dx = o(1)$$

as $n \to \infty$. Appeal is made to this theorem several times in the course of this paper.

with Theorem 1, it shows that, when the series diverges, the order of the nth partial sum is, in general, not greater than that of the nth term.

Since the results obtained apply to the Legendre series of any integrable function, it is hoped that they will retain some interest in spite of the recent important memoirs of A. Haar and of W. H. Young, wherein more far-reaching theorems are proved, but under the restriction that (1) $[f(x)]^2$ is integrable,[*] or (2) the general term of the Legendre series tends to zero.[†]

2. Throughout this note the following notation will be adhered to: $f(x)$ denotes a function which has a Lebesgue integral in $(-1, 1)$, and k denotes a number between 0 and $\frac{1}{2}$ (inclusive) which is such that the integral

$$\text{(A)} \qquad \int_{-1}^{1} \frac{f(x)}{(1-x^2)^{\frac{1}{4}-\frac{1}{2}k}}\, dx = \int_0^{\pi} f(\cos\theta)(\sin\theta)^{\frac{1}{2}+k}\, d\theta \quad (0 \leqslant k \leqslant \tfrac{1}{2}),$$

exists.[‡] We then have the following theorem :

THEOREM 1.—*If* $0 \leqslant \alpha < \beta \leqslant \pi$,

$$\int_{\alpha}^{\beta} f(\cos\theta)\, P_n(\cos\theta) \sin\theta\, d\theta = o(n^{k-\frac{1}{2}}),$$

as $n \to \infty$.

On account of the relation

$$(2.1) \qquad P_n(-x) = (-1)^n P_n(x),$$

it is sufficient to prove the equation when

$$0 \leqslant \alpha < \beta \leqslant \tfrac{1}{2}\pi;$$

we may also clearly suppose that $\alpha = 0$. Taking n sufficiently large, and denoting by η a suitably chosen constant, we write

$$\int_0^{\beta} f(\cos\theta)\, P_n(\cos\theta) \sin\theta\, d\theta = \int_0^{n^{-1}} + \int_{n^{-1}}^{n^{-\frac{1}{8}}} + \int_{n^{-}}^{\eta} + \int_{\eta}^{\beta}$$

$$= J_1 + J_2 + J_3 + J_4,$$

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[*] A. Haar, *Math. Annalen*, Vol. 78 (1918), pp. 121–136.

[†] W. H. Young, *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1919), pp. 141–162.

[‡] The case when $k = 0$ is treated by Hobson : "On the Representation of a Function by a Series of Legendre's Functions," *Proc. London Math. Soc.*, Ser. 2, Vol. 7 (1909), pp. 24–39. In this case the Legendre series of $f(\cos\theta)$ behaves, as to convergence at any interior point of the interval, precisely like the Fourier series of $f(\cos\theta)\sqrt{\sin\theta}$. See, in particular, W. H. Young's memoir already quoted; and compare Theorems 3 and 4 below with pp. 369–371 of Hardy's paper.

say. Since

$$(2.15) \qquad |P_n(\cos\theta)| \leqslant 1,$$

$$(2.2) \qquad |J_1| \leqslant \int_0^{n^{-1}} |f(\cos\theta)(\sin\theta)^{\frac{1}{2}+k}|\,(\sin\theta)^{\frac{1}{2}-k}\,d\theta = o(n^{k-\frac{1}{2}}),$$

by a fundamental property of the Lebesgue integral.

Again, Hobson has established the existence of an absolute constant K, such that

$$(2.25) \qquad |P_n(\cos\theta)| < \frac{K}{\sqrt{(n\sin\theta)}} \quad (0<\theta<\pi);^*$$

hence

$$(2.3) \qquad |J_2| < \frac{K}{\sqrt{n}} \int_{n^{-1}}^{n^{-\frac{1}{8}}} |f(\cos\theta)(\sin\theta)^{\frac{1}{2}+k}|\,(\sin\theta)^{-k}\,d\theta = o\,(n^{k-\frac{1}{2}}).$$

In estimating J_3 and J_4 we use the asymptotic equation for $P_n(\cos\theta)$:

$$(2.4) \quad P_n(\cos\theta)$$

$$= \sqrt{\left(\frac{2}{n\pi\sin\theta}\right)} \left[\cos\{(n+\tfrac{1}{2})\theta-\tfrac{1}{4}\pi\} + \frac{1}{8n}\cot\theta\sin\{(n+\tfrac{1}{2})\theta-\tfrac{1}{4}\pi\} + O(n^{-1})\right],$$

if

$$n^{-\frac{1}{8}} \leqslant \theta \leqslant \tfrac{1}{2}\pi.\dagger$$

We have therefore, writing

$$(n+\tfrac{1}{2})\theta-\tfrac{1}{4}\pi = \omega_n,$$

$$(2.5) \quad J_3 = \sqrt{\left(\frac{2}{n\pi}\right)} \int_{n^{-\frac{1}{8}}}^{\eta} f(\cos\theta)(\sin\theta)^{\frac{1}{2}+k}\cos\omega_n(\sin\theta)^{-k}\,d\theta$$

$$+O(n^{-\frac{11}{8}}) \int_{n^{-\frac{1}{8}}}^{\eta} |f(\cos\theta)(\sin\theta)^{\frac{1}{2}+k}|\,(\sin\theta)^{-k}\,d\theta.$$

Since $(\sin\theta)^{-k}$ is monotone decreasing, an application of the Second Mean Value Theorem to the first integral on the right of (2.5) now gives

$$(2.6) \quad J_3 = O(n^{\frac{1}{8}k-\frac{1}{2}}) \int_{n^{-\frac{1}{8}}}^{\eta_1} f(\cos\theta)(\sin\theta)^{\frac{1}{2}+k}\cos\omega_n\,d\theta + O(n^{\frac{1}{8}k-\frac{11}{8}})$$

$$= o(n^{\frac{1}{8}k-\frac{1}{2}}), \qquad (n^{-\frac{1}{8}} \leqslant \eta_1 \leqslant \eta)$$

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* Hobson, *loc. cit.*, pp. 25–29.

† Heine, *Handbuch der Kugelfunktionen*, Vol. 1, p. 178. See also E. Hilb, *Math. Zeitschrift*, Vol. 5 (1919), pp. 17–25 (p. 18). For $\eta \leqslant \theta \leqslant \pi$ the second term of the formula is, of course, absorbed into the error-term.

by the theorem of Riemann-Lebesgue, since $f(\cos\theta)(\sin\theta)^{\frac{1}{2}+k}$ is integrable in $(0, \eta)$.

Finally, from (2 . 4),

$$J_4 = \sqrt{\left(\frac{2}{n\pi}\right)} \int_\eta^\beta f(\cos\theta)(\sin\theta)^{\frac{1}{2}} \cos\omega_n \, d\theta + O(n^{-\frac{3}{2}}) = o(n^{-\frac{1}{2}}), \tag{2 . 7}$$

by the theorem of Riemann-Lebesgue. Combining these results, the theorem is proved.

3. We follow the notation customary in the theory of Legendre series

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n(x), \tag{3 . 1}$$

$$a_n = (n+\tfrac{1}{2}) \int_{-1}^{1} f(x)\, P_n(x)\, dx = (n+\tfrac{1}{2}) \int_0^\pi f(\cos\theta) P_n(\cos\theta) \sin\theta \, d\theta; \tag{3 . 15}$$

and we write

$$s_n(\cos\theta) = \sum_{\nu=0}^{n} a_\nu P_\nu(\cos\theta) \quad (\cos\theta = x). \tag{3 . 2}$$

We now have :—

Theorem 2.—*With the notation of this note*

$$s_n(\cos\theta) = o(n^k) \quad (0 < k \leqslant \tfrac{1}{2}),$$

$$= o(\log n), \quad (k = 0),$$

for almost all values of θ *in* $(0, \pi)$.

We suppose, as we clearly may,

$$\theta \neq 0, \quad \theta \neq \tfrac{1}{2}\pi, \quad \theta \neq \pi.^*$$

By the well known formula of Christoffel, if we write

$$F(n, \theta, \phi) = \frac{P_{n+1}(\cos\theta) P_n(\cos\phi) - P_n(\cos\theta) P_{n+1}(\cos\phi)}{\cos\theta - \cos\phi}, \tag{3 . 3}$$

$$\frac{2 s_n(\cos\theta)}{n+1} = \int_0^\pi F(n, \theta, \phi) f(\cos\phi) \sin\phi \, d\phi \tag{3 . 31}$$

$$= \int_0^{\theta-\mu} + \int_{\theta-\mu}^{\theta+\mu} + \int_{\theta+\mu}^{\pi} = I_1 + I_2 + I_3,$$

---

[*] The exclusion of $\frac{1}{2}\pi$ is merely for brevity and convenience; that of 0 and π is essential.

say. Here, if we write

$$h(\cos\phi) = \frac{f(\cos\phi)}{\cos\theta - \cos\phi},$$

so that $h(\cos\phi)(\sin\phi)^{\frac{1}{2}+k}$ is integrable in $(0, \theta-\mu)$,

$$(3.35) \quad I_1 = P_{n+1}(\cos\theta)\int_0^{\theta-\mu} h(\phi)\, P_n(\cos\phi)\sin\phi\, d\phi$$
$$-P_n(\cos\theta)\int_0^{\theta-\mu} h(\phi)\, P_{n+1}(\cos\phi)\sin\phi\, d\phi.$$

Thus, noting that, since θ is fixed and unequal to 0 or π,

$$(3.36) \qquad P_n(\cos\theta) = O(n^{-\frac{1}{2}}),$$

it follows, by applying Theorem 1 to each of the integrals on the right of (3.35), that

$$(3.4) \qquad I_1 = o(n^{k-1}) \quad (0 \leqslant k \leqslant \tfrac{1}{2}).$$

Similarly we prove that $\qquad I_3 = o(n^{k-1})$.

Theorem 2 will obviously be proved if we prove that

$$I_2 = o(n^{-1}\log n)$$

for almost all values of θ in $(0, \pi)$. We have

$$I_2 = \int_{-\mu}^{\mu} F(n, \theta, \theta+t) f[\cos(\theta+t)] \sin(\theta+t)\, dt.$$

Further, since $\displaystyle\int_{-\mu}^{\mu} F(n, \theta, \theta+t)\sin(\theta+t)\, dt \to \frac{2}{n+1}$,*

the theorem will be proved if it is shown that

$$I_2' = \int_{-\mu}^{\mu} F(n, \theta, \theta+t)\, \psi(t)\, dt = o(n^{-1}\log n)$$

almost everywhere; we have here written

$$(3.6) \qquad \psi(t) = \{f[\cos(\theta+t)] - f(\cos\theta)\} \sin(\theta+t).$$

We will suppose that $\qquad \theta+\mu \leqslant \tfrac{1}{2}\pi$;

this is, of course, no real restriction in virtue of (2.1). Write

$$I_2' = \left(\int_{-\mu}^{-n^{-1}} + \int_{-n^{-1}}^{n^{-1}} + \int_{n^{-1}}^{\mu}\right) F(n, \theta, \theta+t)\, \psi(t)\, dt = L_1 + L_2 + L_3,$$

---

* This is readily seen by taking, in Christoffel's formula (3.31), $f(\cos\ \ ) \equiv 1$.

say; we may suppose μ to be so chosen that

$$\theta-2\mu>0, \qquad \theta+2\mu<\pi. \tag{3.65}$$

The integral L_2 may be estimated very easily; for Lebesgue has shown that the equation

$$\int_0^\lambda |\psi(t)|\,dt = o(\lambda) \tag{3.7}$$

holds almost everywhere if only $f(x)$ is integrable.* Also, in virtue of (3.65) and (3.36),

$$(n+1)\,F(n,\,\theta,\,\theta+t) = \sum_{\nu=0}^{n} (2\nu+1)\,P_\nu(\cos\theta)\,P_\nu[\cos(\theta+t)]$$

$$= \sum_{\nu=0}^{n} O(1) = O(n);$$

so that

$$|L_2| = O\int_{-n^{-1}}^{n^{-1}} |\psi(t)|\,dt = o(n^{-1})$$

almost everywhere.

Again, using the equation

$$P_n(\cos\theta) = \sqrt{\left(\frac{2}{n\pi\sin\theta}\right)}\cos\left[(n+\tfrac{1}{2})\theta-\tfrac{1}{4}\pi\right]+O(n^{-\frac{3}{2}}), \dagger$$

and the corresponding formula for $P_n[\cos(\theta+t)]$, we find, by a simple reduction, that for

$$n^{-1} \leqslant |t| \leqslant \mu,$$

$$F(n,\theta,\theta+t)-\frac{\sin(n+1)t\operatorname{cosec}\frac{1}{2}t-\cos(n+1)(2\theta+t)\operatorname{cosec}(\theta+\frac{1}{2}t)}{\pi\sqrt{\{n(n+1)\sin\theta\sin(\theta+t)\}}}$$

$$= \operatorname{cosec}(\tfrac{1}{2}t)\,O(n^{-2}) = O(n^{-1}).$$

We have, therefore,

$$L_2 = \pi^{-1}n^{-\frac{1}{2}}(n+1)^{-\frac{1}{2}}(\sin\theta)^{-\frac{1}{2}}\left\{\int_{n^{-1}}^{\mu}\frac{\sin(n+1)t}{\sin\frac{1}{2}t}\,\frac{\psi(t)\,dt}{\sqrt{\{\sin(\theta+t)\}}}\right. \tag{3.8}$$

$$\left.-\int_{n^{-1}}^{\mu}\frac{\cos(n+1)(2\theta+t)}{\sin(\theta+\frac{1}{2}t)\sqrt{\{\sin(\theta+t)\}}}\,\psi(t)\,dt\right\}+O(n^{-1})$$

$$= Kn^{-\frac{1}{2}}(n+1)^{-\frac{1}{2}}(L_3^{(1)}-L_3^{(2)})+O(n^{-1}),$$

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* Lebesgue, *Leçons sur les séries trigonométriques*, pp. 15–16.

† This equation is, of course, a degenerate form of (2.4).

say. The theorem of Riemann-Lebesgue shows at once that

$$L_3^{(2)} = o(1);$$

for the multiplier of $\cos(n+1)(2\theta+t)$ is integrable in $(0, \mu)$. The integral $L_3^{(1)}$, on the other hand, is a Dirichlet's integral

$$\int_{n^{-1}}^{\mu} \frac{\sin(n+1)t}{\sin\frac{1}{2}t}\chi(t)\,dt,$$

where $$\chi(t) = \psi(t)[\sin(\theta+t)]^{-\frac{1}{2}}$$

is integrable in $(0, \mu)$. Hence, in virtue of the theorem by Hardy already referred to,

$$L_3^{(1)} = o(\log n),$$

almost everywhere;* and thus, from equation (3.8),

$$L_3 = o(n^{-1}\log n)$$

almost everywhere. Hence the proof of the theorem, as enunciated, is completed.

4. From the fact that

$$\sum_{\nu=1}^{n} u_\nu = o(n^k) \quad \text{or} \quad o(\log n),$$

it does not, of course, follow that $\Sigma n^{-k}u_n$ or $\Sigma(\log n)^{-1}u_n$ is convergent. In the case of Fourier series Hardy was able to make the required deduction by using the additional fact that the series Σu_n is then necessarily summable $(C, 1)$ almost everywhere. The establishment of the corresponding result for Legendre series would appear to be difficult;† we are, however, able to dispense with it, using, instead of Cesàro summability, the existence of Abel's limit. The latter, although a consequence of the former, is yet sufficient for our purpose, and has the advantage of being proved much more easily. We therefore proceed to

THEOREM 3.—With the notation of the preceding pages, *if $f(x)$ is in-*

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* Hardy, *loc. cit.*, p. 369.

† See W. H. Young, "On Series of Bessel Functions," *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1920), footnote on p. 171 for some indications of the results of that writer's researches on the Cesàro summability of Legendre series. In using Abel's limit in this note, instead of a Cesàro sum, I am following a suggestion made to me by Prof. Hardy.

tegrable in $(-1, 1)$, *the limit*

$$\lim_{r \to 1-0} \sum_{n=0}^{\infty} a_n P_n(x)\, r^n$$

exists for almost all values of x.

The known expansion

$$(4.1) \qquad \frac{1-r^2}{(1-2r\cos\lambda+r^2)^{\frac{3}{2}}} = \sum_{n=0}^{\infty} (2n+1)\, P_n(\cos\lambda)\, r^n, \quad (-1 < r < 1),$$

together with the equation

$$(4.2) \qquad \int_{-\alpha}^{2\pi-\alpha} P_n(\cos\lambda)\, d\chi = 2\pi P_n(\cos\theta)\, P_n(\cos\phi),$$

wherein

$$(4.25) \qquad \cos\lambda = \cos\theta\cos\phi + \sin\theta\sin\phi\cos\chi,$$

and α is a constant which we take to lie between 0 and 2π, leads to the representation (with $x = \cos\theta$)

$$(4.3) \quad \sum_{n=0}^{\infty} a_n P_n(x) r^n = \tfrac{1}{2}\int_0^{\pi} f(\cos\phi)\sin\phi \sum_{n=0} (2n+1) P_n(\cos\theta) P_n(\cos\phi) r^n\, d\phi$$

$$= \frac{1}{4\pi}\int_0^{\pi} d\phi \int_{-\alpha}^{2\pi-\alpha} \frac{(1-r^2)\, f(\cos\phi)\sin\phi}{(1-2r\cos\lambda+r^2)^{\frac{3}{2}}}\, d\chi \quad (-1<r<1).$$

As ϕ and χ vary between their respective limits, λ varies between the limits 0 and π.* Since

$$(4.35) \qquad 1-2r\cos\lambda+r^2 \geqslant 4r\sin^2\tfrac{1}{2}\lambda,$$

it is at once seen that, if μ is a positive constant, the value of the double integral in (4.3), when extended only to that part of the rectangle of integration for which

$$\mu \leqslant \lambda \leqslant \pi$$

tends to zero with $1-r$. Let S denote the portion of the rectangle for which

$$0 \leqslant \lambda \leqslant \mu\,;$$

---

* The angle λ may be regarded as the angle between the radii vectores from the origin to the points on the unit sphere with spherical coordinates $(\theta, 0)$, (ϕ, χ).

it remains to be shown that the value of the double integral, taken over S, tends to a limit for almost all values of θ. We again suppose, as in the proof of Theorem 2, that

(4 . 4) $$0 < \theta < \pi.$$

Write now

(4 . 45) $$\phi = \theta + t.$$

Since $$\lim_{r \to 1-0} \frac{1}{4\pi} \int_0^{\pi} d\phi \int_{-\alpha}^{2\pi-\alpha} \frac{(1-r^2)\sin\phi}{(1-2r\cos\lambda+r^2)^{\frac{3}{2}}}\, d\chi = 1,$$

it is, in order to prove the theorem, sufficient to show that

(4 . 5) $$\iint\limits_S \frac{(1-r^2)\,\psi(t)}{(1-2r\cos\lambda+r^2)^{\frac{3}{2}}}\, dt\, d\chi = o(1)$$

for almost all values of θ; $\psi(t)$ is the function defined by equation (3 . 6).

In the region S, from equations (4 . 25) and (4 . 45),

(4 . 6) $$\begin{aligned} 1-\tfrac{1}{2}\lambda^2 \leqslant \cos\lambda &= \cos t - (1-\cos\chi)\sin\theta\sin\phi \\ &\leqslant 1-\tfrac{1}{2}t^2+\tfrac{1}{24}t^4-(\tfrac{1}{2}\chi^2-\tfrac{1}{24}\chi^4)\sin\theta\sin\phi\,; \end{aligned}$$

hence, when μ is sufficiently small, in consequence of (4 . 4), positive constants a^2, b^2 exist, such that, in S,

(4 . 65) $$\lambda^2 \geqslant a^2t^2+b^2\chi^2.^*$$

Thus the region S is included in the rectangle

$$|at| \leqslant \mu, \quad |b\chi| \leqslant \mu\,;$$

and, throughout S, from (4 . 35),

$$1-2r\cos\lambda+r^2 \geqslant K\lambda^2, \quad 1-2r\cos\lambda+r^2 \geqslant (1-r)^2.$$

Hence equation (4 . 5) will have been shown to hold almost everywhere if it is proved that the three equations that follow hold almost everywhere:

(4 . 70) $$R_1 = \int_0^{1-r} dt \int_0^{1-r} \frac{(1-r^2)\,|\psi(t)|}{(1-r)^3}\, d\chi = o(1),$$

(4 . 71) $$R_2 = \int_{1-r}^{\mu} dt \int_0^{1-r} \frac{(1-r^2)\,|\psi(t)|}{\lambda^3}\, d\chi = o(1),$$

(4 . 72) $$R_3 = \int_0^{\mu} dt \int_{1-r}^{\mu} \frac{(1-r^2)\,|\psi(t)|}{\lambda^3}\, d\chi = o(1).$$

---

* This also follows at once geometrically. See preceding footnote.

Of these, (4 . 70) follows immediately from (3 . 7). To prove (4 . 71) we note that, by (4 . 65),

$$R_2 < K\int_{1-r}^{\mu} dt \int_0^{1-r} \frac{(1-r^2)\,|\,\psi(t)\,|}{t^3}\,d\chi = K(1+r)(1-r)^2 \int_{1-r}^{\mu} \frac{|\,\psi(t)\,|}{t^3}\,dt;$$

and hence, integrating by parts and using equation (3 . 7)

$$R_2 = O[(1-r)^2]\left[O(1) + \frac{o(1-r)}{(1-r)^3} + \int_{1-r}^{\mu} \frac{o(t)}{t^4}\,dt\right]$$

for almost all values of θ; this reduces to (4 . 71). Finally, from (4.65),

$$R_3 < \int_0^{\mu} dt \int_{1-r}^{\mu} \frac{(1-r^2)\,|\,\psi(t)\,|}{(a^2t^2+b^2\chi^2)^{\frac{3}{2}}}\,d\chi. \tag{4 . 8}$$

But, with $\chi > 0$, we find, on integrating by parts and again using (3.7),

$$\int_0^{\mu} \frac{|\,\psi(t)\,|}{(a^2t^2+b^2\chi^2)^{\frac{3}{2}}}\,dt = O(1) + \int_0^{\mu} \frac{o(t^2)}{(a^2t^2+b^2\chi^2)^{\frac{5}{2}}}\,dt$$

$$= O(1) + \int_0^{a\mu/bx} \frac{o(1)\,z^2}{\chi^2(1+z^2)^{\frac{5}{2}}}\,dz$$

$$= O(1) + o(\chi^{-2}) \int_0^{\infty} \frac{z^2}{(1+z^2)^{\frac{5}{2}}}\,dz$$

$$= O(1) + o(\chi^{-2}),$$

all these equations being true almost everywhere. Hence, from (4 . 8),

$$R_3 = (1-r^2)\int_{1-r}^{\mu} o(\chi^{-2})\,d\chi = o(1) \tag{4 . 72 bis}$$

almost everywhere.

5. With the completion of the proof of Theorem 3 we are now in a position to prove the main theorem of this paper.

Theorem 4.—*With the same notation as before, the series*

$$\Sigma n^{-k} a_n P_n(x) \quad (0 < k \leqslant \tfrac{1}{2})$$

or $$\Sigma (\log n)^{-1} a_n P_n(x) \quad (k = 0),$$

is convergent for almost all values of x in $(-1, 1)$.

It is easily shown that the existence of the limit

$$(5\,.\,1)\qquad \lim_{r\to 1-0} f(r) = \lim_{r\to 1-0} \sum_{n=1}^{\infty} c_n r^n$$

involves that of the limit

$$(5\,.\,2)\qquad \lim_{r\to 1-0} \phi_k(r) = \lim_{r\to 1-0} \sum_{n=1}^{\infty} \lambda_k(n) c_n r^n,$$

where

$$(5\,.\,25)\qquad \lambda_k(n) = n^{-k} \quad (k > 0)\,; \qquad \lambda_k(n) = (\log n)^{-1} \quad (k = 0).\text{*}$$

Also it is known that if the limit

$$(5\,.\,3)\qquad \lim_{r\to 1-0} \Sigma h_n r^n$$

exist, and if, further

$$(5\,.\,35)\qquad \sum_{\nu=1}^{n} \nu h_\nu = o(n),\dagger$$

then the series

$$(5\,.\,4)\qquad \sum^{\infty} h_n$$

is convergent.

In our application of these theorems

$$c_n = a_n P_n(x), \qquad h_n = \lambda_k(n) a_n P_n(x),$$

and, by Theorem 2 of this note,

$$(5\,.\,5)\qquad \lambda_k(n)\, s_n(x) = o(1),$$

for almost all values of x in $(-1, 1)$. Further, condition (5 . 1), and hence also (5 . 3) have been shown to be satisfied for almost all values of x in $(-1, 1)$. For (5 . 35) we have

$$\sum_{\nu=1}^{n} \nu h_\nu = \sum_{\nu=1}^{n} \nu\lambda_k(\nu)\, a_\nu P_\nu(x) = n\lambda_k(n)\, s_n(x) - \sum_{\nu=0}^{n-1} \left[(\nu+1)\lambda_k(\nu+1) - \nu\lambda_k(\nu)\right] s_\nu(x)$$

$$= o(n) + \sum_{\nu=0}^{n-1} o(1) = o(n),$$

---

* Thus, with $k > 0$, $\phi_k(r) = \dfrac{1}{\Gamma(k)} \displaystyle\int_0^1 \frac{f(rt)}{t} \left(\log \frac{1}{t}\right)^{k-1} dt$;

and from this expression the truth of the statement in the text follows at once.

† See, for example, Bromwich, *Theory of Infinite Series*, p. 251 (Ex. 28).

for almost all values of x, by (5.5). Thus, since Abel's limit (5 . 3) exists for almost all values of x in $(-1, 1)$, and since the series (5 . 4) is convergent for almost all values of x for which (5 . 3) does exist, it follows that the series (5 . 4), namely,

$$\Sigma n^{-k} a_n P_n(x) \quad (0 < k \leqslant \tfrac{1}{2}), \qquad \Sigma (\log n)^{-1} a_n P_n(x) \quad (k = 0),$$

is necessarily convergent for almost all values of x in $(-1, 1)$.

In particular, for any integrable function $f(x)$,

$$\Sigma n^{-\frac{1}{2}} a_n P_n(x)$$

is convergent almost everywhere.

ON ANALOGUES OF WARING'S PROBLEM FOR RATIONAL NUMBERS

By H. W. RICHMOND.

[Received September 15th, 1922.—Read November 9th, 1922.]

THEOREMS on expressing numbers as a sum of powers of other numbers have been conveniently grouped together* under the title "Waring's Problem and Related Results." From Waring's original assertion about integers it is an obvious step to consider whether similar properties are possessed by Rational Numbers. Any rational number R can be reduced to the form of a fraction A/a, where A and a are integers prime to one another; and we may inquire, by a natural extension of Waring's statement, whether every such fraction can be expressed as the sum of a finite number of the squares or cubes or . . . . of other fractions.

The problem is not new. It may be said, like the problem of integers, to have been solved completely when the powers employed are squares; in fact for squares the problem of rational numbers is identical with that of integers. For powers higher than the third it would seem that nothing has been discovered; nor has the present writer anything to communicate. But for cubes† several papers have been published, and it is not a little surprising to find that here the two problems (of integral and rational numbers) are widely different, both as regards methods and results. In a sense the problem of rational cubes turns out to be simpler than that of rational squares; at any rate the number of cubes required is less than the number of squares—another curious and unexpected result.

The results which solve the problems of expressing (i) any positive integer as a sum of squares of other integers, (ii) any rational number as a sum of squares of other rational numbers, can be stated briefly. When this has been done, the corresponding problems for cubes will be considered.

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* L. E. Dickson, *History of the Theory of Numbers*, Vol. 2, Ch. xxv.

† L. E. Dickson, *loc. cit.*, pp. 726–729.

I. *Waring's Problem for Squares.*

1. *Every Positive Integer I* can be expressed as the sum of either four or a smaller number of squares of other integers. Further, rules have been discovered by which, when I is given, the minimum number of squares required can be foretold. To quote these rules is unnecessary; but they show that the minimum number of squares required for I is the same as that required for any integer obtained by multiplying or dividing I by a perfect square.

2. *Positive Rational Numbers.*—Any such number R having been reduced to the form A/a, where A and a are positive integers prime to one another, if the integer $A\times a$ is expressed as the sum of integer squares, R can be expressed as the sum of the same number of squares by dividing throughout by a^2. Every positive rational number can therefore be expressed as the sum of the squares of at most four rational numbers.

No smaller number than four will suffice in certain cases. For suppose that R can be expressed as the sum of (say) three squares:

$$R = A/a = (X/x)^2+(Y/y)^2+(Z/z)^2\,;$$

find m, the least common multiple of x, y, z, and multiply by m^2. Then

$$m^2\times A/a = \text{sum of three integer squares},$$

and must itself be an integer. Denote by U^2 and u^2 the greatest squared factors contained in A and a, the remaining factors being V and v; so that $A = U^2V$, $a = u^2v$, V and v having no squared factors. Since A is prime to a, m must contain uv as a factor; *i.e.* $m = kuv$. Thus

$$(kU)^2\times Vv = \text{sum of three integer squares}.$$

That is to say the fact that A/a can be expressed as the sum of 3 (or 2 or 4) squares implies, for the reason stated above, that $(kU)^2\times Vv$, and therefore Vv, and therefore $(Uu)^2\times Vv$ or Aa can also be expressed as the sum of 3 (or 2 or 4) squares of integers. The minimum number of squares (of rational fractions) required for the fraction A/a and the minimum number of squares (of integers) required for the integer $A\times a$ are equal. To cover all cases, *four* squares are necessary in both problems.

3. *A General Formula for R as a Sum of Squares.*—When R has been expressed in one way as the sum of a minimum number of squares, it is possible to derive a formula which includes every partition of R into that or any greater number of squares. The method, applicable in all

cases, will be clear from an example :—Suppose that

$$R = f^2 + g^2 + h^2$$

expresses R as the sum of a minimum number of squares, and that an expression for R as the sum of five squares is required. Take any five rational numbers L, M, N, P, Q, and put

$$R = (f+\theta L)^2 + (g+\theta M)^2 + (h+\theta N)^2 + (\theta P)^2 + (\theta Q)^2.$$

This quadratic in θ is satisfied by $\theta = 0$ and by one other rational value of θ which gives a solution. Every set of values of the ratios of L, M, N, P, Q to one another leads to a 5-square partition of R; moreover every partition of R into 5 squares is included, for proper values of the ratios $L : M : N : P : Q$. The solution of Waring's problem for squares of rational numbers may therefore be regarded as fairly complete.

II. *Waring's Problem for Cubes.*

1. *Positive Integers.*—In order to express *every* positive integer as a sum of cubes* of other positive integers, *nine* cubes have been proved to be both necessary and sufficient. Since Aa^2 can be expressed as the sum of nine cubes, and then divided by a^3, it follows that any rational number A/a can be expressed as a sum of nine rational cubes. But the number nine can be shown to be unnecessarily large. For it has been proved that for integers which exceed a certain value *eight* cubes are sufficient (and on the evidence of computation it cannot be doubted that the number can be further reduced, to seven, six, five, or possibly four, though no proof has yet been discovered). This number eight (probably reducible further) and not Waring's number nine must be a limit to the number of positive rational cubes required for a rational fraction A/a. For, if n is large enough, Aa^2n^3 is a sum of eight positive cubes; and on division by a^3n^3 the result follows.

2. *Integers, Positive or Negative.*—To express any integer positive or negative as a sum of cubes of integers positive or negative is a legitimate extension of Waring's problem, the exponent being an odd number. Here *five* cubes are certainly sufficient; possibly four are sufficient but no lower number.

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* See *Some Famous Problems of the Theory of Numbers*, by G. H. Hardy, especially pp. 19–23; and Dickson, *l.c.*, Chapter xxv.

3. *Rational Numbers.*—The rest of this paper deals with this case. In the problem of cubes (in contrast to that of squares) there is a formula for partitioning a number into rational cubes, not applicable to integral cubes, which makes it clear that the limits obtained above are much too high. Dickson quotes three proofs (*l.c.* pp. 726–729) that every rational number is the sum of *three* rational cubes, and two proofs that every positive rational number is the sum of *four* positive rational cubes. But even this last result can be improved and the necessary number of cubes reduced to *three* in both cases. Since it is also known (pp. 572–578) that certain numbers, for example $\frac{1}{2}$, 3, 4, 5, ..., cannot be expressed as the sum of two rational cubes, *three* is a genuine minimum for rational cubes whether restricted to positive values or not.

III. *Ryley's Formula.*

1. It is curious that on examination the three methods quoted by Dickson for resolving R into a sum of three cubes lead to identical results. The following investigation seems to be simpler than any of the three, but the formula obtained should bear the name of its discoverer, Mr. S. Ryley, of Leeds, by whom it was published in 1825. See equations (γ) below.

Suppose that $$R = x^3+y^3+z^3, \tag{α}$$

and put $$x = s-t, \quad y = t-z;$$

i.e. $$s = x+y+z, \quad t = y+z.$$

Then $$R = (s-t)^3+(t-z)^3+z^3 = s^3-3t(s^2-z^2)+3t^2(s-z). \tag{β}$$

In order to obtain a special solution assume that

$$s^3-3t(s^2-z^2) = 0,$$

and substitute $z = \theta s$. We find at once

$$s = 3t(1-\theta^2), \quad z = 3t(\theta-\theta^3),$$

$$R = 3t^2(s-z) = 9t^3(1-\theta)^2(1+\theta)$$

or $$3R = s^3(1+\theta)^{-2}(1-\theta)^{-1}.$$

If now we write $s/(1+\theta) = \lambda$, we find

$$(1+\theta)/(1-\theta) = 3R/\lambda^3, \quad \theta = (3R-\lambda^3)/(3R+\lambda^3),$$

and every rational value of λ leads us to a rational set of values of θ, s, t, z, y, x, *i.e.* to a partition of R into a sum of the cubes of three rational numbers x, y, z.

Every Rational Number can be expressed as a sum of cubes of three Rational Numbers—in a simple infinity of ways.

Ryley's formulæ may be written

$$\left.\begin{aligned} x+y+z &= s = 6R\lambda/(3R+\lambda^3),\\ y+z &= t = (3R+\lambda^3)/(6\lambda^2),\\ z &= 6R\lambda(3R-\lambda^3)/(3R+\lambda^3)^2,\\ \text{whence}\qquad x+y &= s-z = 12R\lambda^4/(3R+\lambda^3)^2,\\ \text{also}\qquad R &= x^3+y^3+z^3 = 3(x+y)(y+z)^2. \end{aligned}\right\} \qquad (\gamma)$$

2. *Positive Rational Numbers.*—It will now be shown that, if R is positive, x, y, z are also positive if θ and λ lie between certain numerical limits.

When x, y, z are positive, s and t will be positive and also s/t and z/t, *i.e.* $1-\theta^2$ and $\theta-\theta^3$. Thus θ must lie between 0 and 1.

Secondly, x is positive if $s-t$ is positive, or if

$$3(1-\theta^2) \text{ is greater than } 1$$

or

$$\theta \text{ less than } (2/3)^{\frac{1}{2}}.$$

Lastly, y is positive if t is greater than z, or if

$$3(\theta-\theta^3) \text{ is less than } 1,$$

which (for positive values of θ) is the case

(i) if θ is less than $(4/3)^{\frac{1}{2}}\times\sin 20^\circ$,

(ii) if θ is greater than $(4/3)^{\frac{1}{2}}\times\sin 40^\circ$.

Combining these conditions, we see that numerically

θ must lie either (i) between 0 and ·39493,

or (ii) between ·74223 and ·81650;

$\lambda^3/(3R)$ must lie either between 1 and ·43376,

or between ·14756 and ·10102.

Hence, again by Ryley's formulæ (γ),

Every Positive Rational Number can be expressed as a sum of cubes of three Positive Rational Numbers—in a simply infinite number of ways. Ryley's formula, rediscovered independently by Strong and others, does by means of three cubes all that the four-cube formulæ of Libri and Lebesgue accomplish (Dickson, p. 728). As an example, suppose that $R = 2/3$. We may take $\lambda = 1$, $\theta = 1/3$; then

$$x = 5/6, \quad y = 4/9, \quad z = 1/18.$$

Or, taking $R = \frac{1}{2}$ and $\theta = 1/5$, we find

$$\tfrac{1}{2} = (6/25)^3+(47/60)^3+(53/300)^3.$$

IV. *The Problem of Rational Cubes. Further Solutions.*

1. Save for Ryley's formula no resolution of R into three cubes seems to have been discovered, and it is not impossible that no other simple algebraic formula of the kind exists. Values of x, y, z given by Ryley's formula are connected by the homogeneous relation

$$x^3+y^3+z^3 = 3(x+y)(y+z)^2,$$

or

$$(x+y+z)^3 = 3(x+y)(y+z)(x+y+2z),$$

and are therefore greatly restricted. In the light of geometry of three dimensions $x^3+y^3+z^3 = R$ represents a cubic surface on which we seek points (x, y, z) whose coordinates are rational numbers. Ryley has discovered a curve lying on the surface which contains a simply infinite number of such points, each defined by a rational value of a parameter λ. This is vastly inferior to the solution attainable when R is a perfect cube, viz. expressions for x, y, z in terms of two parameters, such that every pair of rational values of the parameters leads to rational values of x, y, z, and every set of rational values of x, y, z can be so obtained.

In dealing with squares in Section I, we were able to derive from a single known partition of R into squares a formula which included all partitions. Something of the same sort is open to us also in dealing with cubes. Although it may not be possible for arbitrary values of R to devise a formula which includes *all* the partitions of R into a sum of three cubes, yet from one or more given solutions of

$$x^3+y^3+z^3 = R, \tag{a}$$

other solutions can be derived by various known methods, suggested for the most part by geometrical considerations. Other formulæ involving one or more parameters can be deduced from Ryley's, but all that I have obtained are complicated and do not really help. A fact which soon becomes evident and is better worth following up is that the rational values of x, y, z which satisfy (a) are, if plotted as coordinates of a point, generally distributed over the surface (a); they are not confined to Ryley's curve or to any particular region. [It was perhaps hardly to be expected that they would be so confined; yet it must be remembered that when $R = 0$, the surface (a) has no rational points upon it whatever.] The following theorem states the issue clearly :—

2. *Given a Positive Rational Number R, and any three positive numbers a, b, c, it is possible to find (by elementary algebra) positive rational values of x, y, z which satisfy* (a) *accurately, and at the same time make the differences*

$$x/a - y/b, \quad x/a - z/c,$$

less than any assigned small number.

Suppose that X, Y, Z are a known set of values of x, y, z which satisfy (a), and that K and F are to be determined, so that

$$(aF+XK)/(K+1), \quad (bF+YK)/(K+1), \quad (cF+ZK)/(K+1)$$

shall also satisfy (a). We must have

$$X^3+Y^3+Z^3 = R,$$

$$(aF+XK)^3+(bF+YK)^3+(cF+ZK)^3 = (K+1)^3 R.$$

On expanding, the term in K^3 disappears; that in K^2 also disappears if

$$F(aX^2+bY^2+cZ^2) = R.$$

Let F have this value, and we have a linear equation giving K, viz.

$$-3K(a^2XF^2+b^2YF^2+c^2ZF^2-R) = (a^3F^3+b^3F^3+c^3F^3-R).$$

Thus, if X, Y, Z are rational and also $a:b:c$, we have derived from a known solution X, Y, Z a second rational solution, in which the values of $x/a-y/b$ and $x/a-z/c$ are seen to be

$$\frac{K(X/a-Y/b)}{K+1} \quad \text{and} \quad \frac{K(X/a-Z/c)}{K+1}.$$

In order to make $x/a-y/b$, etc. as small as we please it is enough to show that we can find finite values of X, Y, Z from Ryley's formula which make K as small as we please. (X, Y, Z tend to infinite values only as λ or λ^3+3R tends to zero or λ to infinity, *i.e.* as θ tends to $+1$ or infinity or -1.) Now K vanishes when

$$(a^3+b^3+c^3)F^3 = R,$$

or

$$(a^3+b^3+c^3)(X^3+Y^3+Z^3)^2-(aX^2+bY^2+cZ^2)^3 = 0.$$

This equation is homogeneous; if any values of X, Y, Z given by Ryley's formula satisfy it, we can find them from the simple expressions for $X:Y:Z$ in terms of θ. Substitute for X, Y, Z respectively

$$3(1-\theta^2)-1, \quad 1-3(\theta-\theta^3), \quad 3(\theta-\theta^3),$$

and for $X^3+Y^3+Z^3$ substitute $9(1-\theta)^2(1+\theta)$. We obtain an equation of order 18 in θ; and, if we can prove that it has any real root, we can approximate through rational values to the relevant value of λ to any degree of accuracy that we desire, and thus can make $x/a-y/b$ and $x/a-z/c$ as small as we please (provided X, Y, Z are finite). Now θ is to be found from the vanishing of a polynomial in θ of order 18; it is unnecessary to write it down, as it has been stated in terms of X, Y, Z, whose values in terms of θ are also stated. If we substitute $\theta=+1$, or $\theta=-1$ in the polynomial, X, Y, Z take values -1, $+1$, 0, and the polynomial takes the value $-(a+b)^3$, which is negative.

If we substitute $\theta=0$, X, Y, Z take the values 2, 1, 0, and the polynomial takes the value

$$81(a^3+b^3+c^3)-(4a+b)^3 = 81c^3+(a-2b)^2(17a+20b).$$

For positive values of a, b, c this is positive except if $c=0$ and $a=2b$, but this difficulty would be got over by interchanging a, b, c. For all positive values of a, b, c, the equation in θ has two real roots leading to finite values of X, Y, Z, and the process of approximation indicated above can certainly be carried out. Hence the theorem is proved.

3. *Postscript.*—I find that Ryley's curve (γ) is one of a family of curves which cover the surface (α), and which give solutions of our numerical problem in a similar manner. If

$$s = x+y+z = 3\rho(1-\theta)\{(1+\theta)+3p-3p^3(1-\theta)\}, \qquad \text{(i)}$$

$$t = y+z = \rho\{(1-9p^2(1-\theta)-9p^3(1-\theta)^2(1+\theta)\}, \qquad \text{(ii)}$$

$$z = \theta(x+y+z) \qquad \text{(iii)}$$

(equations which reduce to those of § III, 1, if $p=0$), then will

$$x^3+y^3+z^3 = 3(x+y)\{(y+z)^2+3p(y+z)(x+y+z)+3p^2(x+y+z)^2$$
$$+3p^3(x+y)(x+z)\}. \qquad \text{(iv)}$$

In terms of ρ and θ,

$$x^3+y^3+z^3 = 9\rho^3(1-\theta)^2\{(1+\theta)+3p-3p^2(1-\theta)\}$$
$$\times\{1+3p(1-\theta)(1+\theta)+3p^2(1-\theta)\}^3. \qquad \text{(v)}$$

If now we write, much as in Ryley's method,

$$\lambda = 3\rho(1-\theta)\{1+3p(1-\theta)(1+\theta)+3p^2(1-\theta)\}, \tag{vi}$$

then, if $x^3+y^3+z^3 = R$,

$$3R(1-\theta) = \lambda^3\{(1+\theta)+3p-3p^2(1-\theta)\}. \tag{vii}$$

First θ, and then ρ, x, y, z, are expressed as rational functions of λ, p, and R, such that all rational values of λ and p lead to rational values of x, y, z whose cubes have the sum R. Given R, we can find values of x, y, z, satisfying (a) which are nearly proportional to any chosen numbers a, b, c, rational or irrational, positive or negative, by determining θ and p to any desired degree of approximation from equations (iii) and (iv), after replacing x, y, z by a, b, c in them; p has at least one real value, to which we approximate by standard methods.

4. The last formulæ could not be discovered by accident, as Ryley's apparently were. When the fundamental reasons for the success of Ryley's method were recognised, it became possible to deduce the existence of these formulæ from geometrical considerations, which seem to show also that no other cubic functions of θ will serve. A host of other curves of higher order are derivable which give solutions of our problem; but general theorems, not special results, are now desirable.

The simplicity of the algebraic method applicable to cubes and the utter dissimilarity between the two problems of squares and cubes both in results and treatment are very remarkable; it would be interesting to discover which of the two the higher problems of fourth and fifth powers resemble.

NOTE ON THE NODES OF THE 3-BAR SEXTIC

By Col. R. L. HIPPISLEY, C.B.

[Received February 16th, 1922.—Read March 9th, 1922.]

1. Mr. F. V. Morley has shown in his recent article on the analytical treatment of the 3-bar curve* that, using the term as defined by Dr. Bennett,† the two principal chords through a node terminate on the opposite side of the nodal triangle; and that the elliptic arguments by which the coordinates of the two ends of any principal chord can be expressed differ by a half period. This is a real half period in the case of a unipartite curve and the imaginary half period for a bipartite curve. This opposite side of the nodal triangle, therefore, cuts the curve in six known points, and, determining the infinities of the elliptic arguments in the usual way, we can write down the sum of the arguments representing these six points of intersection; and hence the sum of the arguments representing the nodes is also known. A relation is easily obtained between the two arguments for each node because, if ω_r and ω_r' are the two values of the angle which a traversing link makes with one of the rotating links in its two positions at a node, $\omega_r+\omega_r' =$ a constant.

2. A remarkable relation between the areas S_1, S_2, S_3 of the three parallelograms of the triple generation theorem at a node is obtainable by substituting the coordinates of a point on the curve in their trigonometrical form in the equation to the focal circle; the result being

$$S_1 \sin^2 A + S_2 \sin^2 B + S_3 \sin^2 C = 0.$$

3. Performing a similar operation on the equation to the circle of inflexions, we get

$$S_1 \sin^2 \omega_3 + S_2 \sin^2 \omega_1 + S_3 \sin^2 \omega_2 = 0,$$

which is true for a point of inflexion.

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* *Proc. London Math. Soc.*, Ser. 2, Vol. 21, pp. 147 and 153.

† *Ibid.*, Vol. 20, p. 62.

4. With regard to the first of these two equations, if PP' be any principal chord and perpendiculars are drawn from P to the opposite sides of the three traversing triangles, the feet of these perpendiculars form a triangle equal and homothetic to the pedal triangle of the point P' with reference to the focal triangle ABC. A glance at the triple generation diagram will show that the angles between the three perpendiculars from P are respectively ω_3-A, ω_1-B and ω_2-C. Changing ω_r into $-\omega_r$, the corresponding angles for the similar perpendiculars radiating from P' will be $-(\omega_3+A)$, $-(\omega_1+B)$ and $-(\omega_2+C)$, and the areas of the triangles formed by them will be

$$-\tfrac{1}{2}b_1 \sin A \,.\, a_3 \sin C \,.\, \sin(\omega_3+A) \quad \text{or} \quad \tfrac{1}{2}S_1 \sin^2 A,$$

$$-\tfrac{1}{2}c_2 \sin B \,.\, b_1 \sin A \,.\, \sin(\omega_1+B) \quad ,, \quad \tfrac{1}{2}S_2 \sin^2 B,$$

$$-\tfrac{1}{2}a_3 \sin C \,.\, c_2 \sin B \,.\, \sin(\omega_2+C) \quad ,, \quad \tfrac{1}{2}S_3 \sin^2 C,$$

where a_r, b_r, c_r are the sides of the traversing triangles. For a node the pedal triangle becomes a straight line; therefore the feet of the perpendiculars from the other ends of both principal chords through this node on the bases of the traversing triangles at these points lie in straight lines; hence taking account of the signs of the areas

$$S_1 \sin^2 A + S_2 \sin^2 B + S_3 \sin^2 C = 0.$$

5. If A, B, C are the foci and F, G, H are the nodes of a 3-bar curve, and 2α, 2β, 2γ, $2f$, $2g$, $2h$ are the angles with a fixed axis of the radii of the focal circle to these points, then, in virtue of the relation

$$\alpha+\beta+\gamma = f+g+h,$$

the equation to the Simson line* of F with reference to ABC may be written

$$x \sin(g+h) - y \cos(g+h)$$

$$= \tfrac{1}{2}r\left[\sin(g+h-2\alpha)+\sin(g+h-2\beta)+\sin(g+h-2\gamma)+\sin(g+h-2f)\right],$$

and that of G,

$$x \sin(h+f) - y \cos(h+f)$$

$$= \tfrac{1}{2}r\left[\sin(h+f-2\alpha)+\sin(h+f-2\beta)+\sin(h+f-2\gamma)+\sin(h+f-2g)\right],$$

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* Salmon's *Higher Plane Curves*, p. 271.

where r is the radius of the focal circle. Their intersection is at the point

$$x = \tfrac{1}{2}r\,[\Sigma \cos 2a + \Sigma \cos 2f], \qquad y = \tfrac{1}{2}r\,[\Sigma \sin 2a + \Sigma \sin 2f],$$

and the symmetry of the result shows that the Simson lines of F, G, H with reference to ABC and of A, B, C with reference to FGH all meet in a point which is midway between the orthocentres of the two triangles. This point is also the common orthopole of the six sides of the two triangles, for as will be seen from their equations the Simson lines of F, G, H, A, B, C are respectively perpendicular to GH, HF, FG, BC, CA, AB.

A RELATION BETWEEN BERTRAND'S AND KELVIN'S THEOREMS ON IMPULSES

By G. I. TAYLOR.

[Received June 2nd, 1922.—Read June 8th, 1922.]

THE theorems known as Bertrand's and Kelvin's theorems on impulses relate to the effect of constraints on the energy of a dynamical system when it is set in motion by impulses. Bertrand's theorem states that if a system is set in motion by given impulses applied to given coordinates, the effect of a constraint is to reduce the energy which the impulses give to the system. Kelvin's theorem states that if, instead of the impulses, the velocities imparted by those impulses to the corresponding coordinates are given, the effect of a constraint is to increase the energy of the motion.

The following simple connection between these two theorems does not seem to have been pointed out before. 'The reduction in energy due to the imposition of constraints in the Bertrand case is less than the increase in energy due to the imposition of the same constraints in the Kelvin case.'

The proof of this theorem is very simple. Let T_o represent the energy of a dynamical system due to a set of impulses p_r applied to coordinates q_r, another set of coordinates q_s being unconstrained, and let the velocities due to these impulses be $\dot{q}_r$. This motion will be represented by the symbol (o).

Suppose now that the system is constrained by fixing certain coordinates q_s. Two cases will be considered, (a) Bertrand's case where the system is set in motion by impulses p_r applied to coordinates q_r. The energy T_a of the constrained system is less than T_o. This is proved by showing that $T_o - T_a$ is equal to the energy of the system whose velocity is equal to the velocity of the motion which is the difference between (o) and (a). This motion, which will be denoted by $(o-a)$, is that produced by the action of a set of impulses p_s* which act on the coordinates q_s producing velocities $\dot{q}_s$,† the coordinates q_r being *unconstrained*.

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* p_s are the impulsive reactions at the constraints in case (a).

† $\dot{q}$ are the velocities of the coordinates q_s in case (o).

The next constrained motion, (b), to be considered is Kelvin's case when the constrained system is set in motion by the application of a set of impulses p'_r, say, acting on q_r so as to produce velocities $\dot{q}_r$. The energy T_b of this motion is greater than T_o. This is proved by showing that $T_b - T_o$ is equal to the energy of the motion ($o-b$) which is the difference between motions (o) and (b). This derived motion is that produced by the action of a set of impulses p'_s* which produce the same velocities $\dot{q}_s$ when they act on q_s as were produced in motion ($o-a$) by the action of the impulses p_s, but in this case the coordinates q_r are *fixed.*

It will be seen that the relation between motions ($o-a$) and ($o-b$) is exactly that contemplated in Kelvin's theorem. Hence $T_b - T_o$ is greater than $T_o - T_a$, which is the theorem that was to be proved.

The difference between $T_b - T_o$ and $T_o - T_a$, that is $T_a + T_b - 2T_o$ is, in fact, the energy of the system which is the difference between the motions (a) and (b), that is to say it is the energy of the system set in motion by the application of impulses $p'_r - p_r$ to the coordinates q_r, when the system is constrained by fixing the coordinates q_s.

The physical bearing of these results may be seen at once from the consideration of a simple case. Consider the case of a rigid rod which is set in motion by an impulse applied at some point in a direction perpendicular to its length. If the constraint consists in the fixing of a point in the rod, it is clear that the energy of the constrained motion tends to zero in the Bertrand case when the fixed point is very near to the point of application of the impulse. In the Kelvin case, however, the energy of the constrained motion tends to infinity. The reduction in energy due to the constraint in the Bertrand case can never be greater than the energy of the unconstrained motion, whereas the increase in energy due to the constraint in the Kelvin case may tend to become infinite.

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* p'_s are the impulsive reactions at the constraints in case (b).

ON THE INTEGER SOLUTIONS OF THE EQUATION $ey^2 = ax^3+bx^2+cx+d$.

By L. J. MORDELL.

[Received May 22nd, 1922.—Read June 8th, 1922.]

1. The present knowledge of general classes of indeterminate equations which have only a finite number of integer solutions is comparatively limited.[*] Apart from the results connected with Fermat's Last Theorem, the simplest and most general one is due to Thué,[†] and states that, if $f(x, y)$ a binary quantic[‡] in x and y is not the power of a linear or indefinite quadratic function of x and y, then the equation

$$f(x, y) = c \tag{1}$$

has at most a finite number of integer solutions for x and y. Extensions of this result have been given by Maillet, and very recently by Siegel.[§] The latter considers solutions in various algebraic fields, and has also just[||] given a simpler proof of Thué's theorem.

Other results follow from my paper[¶] "Indeterminate Equations of the Third and Fourth Degrees," wherein I showed that all the integer solutions of

$$z^2 = ax^3+bx^2y+cxy^2+dy^3, \tag{2}$$

with x prime to y, are given by a finite number of expressions of the form

$$x = f_1(p, q), \quad y = f_2(p, q),$$

where f_1 and f_2 are binary quartics in p and q. An immediate consequence of applying Thué's theorem to this result is that the equation

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[*] For references see Dickson's *History of the Theory of Numbers*, Vol. 2, pp. 673 *et seq.*

[†] *Crelle's Journal*, Vol. 135 (1909).

[‡] The coefficients are supposed to be rational integers, as are also all other numbers unless otherwise stated.

[§] *Mathematische Zeitschrift*, Vol. 10 (1921), and *Mathematische Annalen*, Vol. 84 (1921).

[||] *Videnskapsselskapets Skrifter* 1 *Mat. Naturv. Klasse* (Kristiania), 1921, No. 16.

[¶] *Quarterly Journal of Pure and Applied Mathematics*, Vol. 45 (1913–1914).

$$ey^2 = ax^3+bx^2+cx+d \tag{3}$$

has only a finite number of integer solutions, if the right-hand side has no squared factors in x. This I pointed out for the equation

$$y^2 = x^3+k$$

in a paper[*] "On a statement by Fermat"; a different proof is included in the similar result for

$$dy^m = ax^2+bx+c,$$

given by Thué,[†] and independently by Landau[‡] and Ostrowski. The result for the general equation (3) was noted in my booklet[§] *Three Lectures on Fermat's Last Theorem.* A very simple proof, making no use of equation (2), has just appeared in the *Messenger of Mathematics.*[‖]

It is at once suggested that a similar result holds for the equation

$$y^2 = ax^4+bx^3+cx^2+dx+e,$$

and the present method of proof for (3) was devised with this object[¶] in view.

I shall first prove a useful lemma; if ϕ_1 is a ternary quadratic and ϕ is any ternary quantic, the simultaneous equations

$$\phi_1(x, y, z) = 0,$$

$$\phi(x, y, z) = m,$$

have at most a finite number of integer solutions for x, y, z, provided that the curve represented by these equations, which is in general of non-zero genus, is not unicursal, that is that x, y and z cannot be expressed as rational functions of a parameter t. For as the second equation shows that the common factors of x, y and z are finite in number, the general solution of the first equation, by a classic result, is given by a finite number of expressions of the form

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[*] *Proc. London Math. Soc.*, Ser. 2, Vol. 18 (1919).

[†] *Archiv for Mathematik og Naturvidenskab*, Vol. 34 (1917).

[‡] *Proc. London Math. Soc.*, Ser. 2, Vol. 19 (1920).

[§] P. 3 (published by the Cambridge University Press).

[‖] Vol. 51 (1922).

[¶] Although unsuccessful, I was led in this way to the proof of the theorem that the method of infinite descent applies to all ternary homogeneous cubics $f(x, y, z) = 0$ of genus unity. See my paper in the *Proceedings of the Cambridge Philosophical Society*, Vol. 21 (1922).

$$x = f_1(p, q), \quad y = f_2(p, q), \quad z = f_3(p, q),$$

where f_1, f_2, f_3 are binary quadratics in p and q. Substituting these values of x, y and z in the second equation, the result at once follows by Thué's theorem.

2. Suppose now that the equation

$$ey^2 = ax^3+bx^2+cx+d,$$

where the right-hand side is supposed to be irreducible, has an infinite number of integer solutions in x, y. Multiplying throughout by a^2e^3, it is sufficient to consider the equation

$$y^2 = x^3+bx^2+cx+d,$$

so that θ, defined by $\theta^3+b\theta^2+c\theta+d = 0$,

is an algebraic integer. Since y^2 is divisible by $x-\theta$,

$$y^2 = (x-\theta)(x^2+px+q),$$

say. The consideration of the cubic field $K(\theta)$ shows that the common ideal factors of $x-\theta$ and x^2+px+q are finite in number for all values of x, as they are also factors of $\theta^2+p\theta+q$, and hence of the discriminant of the cubic. Hence we have the equation in ideals

$$(x-\theta) = \lambda\tau^2,$$

where τ is an ideal, and the ideal λ is one of a finite number. Since the classes of ideals are finite in number, we can write

$$\tau = \mu\nu,$$

where ν is an algebraic number given by

$$n\nu = f_1+g_1\theta+h_1\theta^2,$$

with f, g, h integers, while the integers n and the ideals μ are finite in number. We have then

$$(x-\theta) = \lambda\mu^2\nu^2,$$

so that the ideal $\lambda\mu^2$ is a principal ideal, and hence, since any unit in the field, except for a finite number of unit factors, can be absorbed in ν^2,

$$x-\theta = \sigma\nu^2/M,$$

where σ and M are a pair taken from a finite number of algebraic numbers and ordinary integers respectively. If the values of x are infinite in number, then for a particular value, say x_1,

$$x_1 - \theta = \sigma\nu_1^2/M,$$

whence

$$(x-\theta)(x_1-\theta) = \eta^2,$$

say. We put now

$$N\eta = f+g\theta+h\theta^2,$$

where N is one of a finite number of integers, and have to prove that the equation

$$N^2(x-\theta)(x_1-\theta) = (f+g\theta+h\theta^2)^2$$

has at most a finite number of integer solutions in x, f, g, h. Square the right-hand side, replace θ^4, θ^3 in terms of θ^2, θ, θ^0 from the cubic in θ and equate coefficients of θ^0, θ, θ^2 on both sides. Eliminating x from the first two of these equations, we have

$$\phi(f, g, h) = N^2, \quad \phi_1(f, g, h) = 0,$$

where ϕ, ϕ_1 are homogeneous quadratics in f, g, h. The result now follows, since by the lemma only a finite number of integer values of f, g, h can satisfy these two equations.

No exceptional cases can arise since then f, g, h would be rationally expressible in terms of a parameter t, giving us such a solution for x, y of

$$y^2 = x^3+bx^2+cx+d,$$

representing a curve of genus unity. This completes the proof for the irreducible cubics.

When the cubic is reducible it will always have a linear factor, so that it is sufficient to prove that the equations

$$fy^2 = ax^2+bx+c,$$

$$gz^2 = dx+e,$$

have only a finite number of integer solutions in x, y, z. The solution of the first equation (considering it as

$$fy^2 = ax^2+bxx_1+cx_1^2$$

with $x_1 = 1$) is given by a finite number of expressions of the form

$$x = \lambda t^2+\mu tu+\nu u^2,$$

where t and u are integer solutions of an equation

$$t^2-Du^2 = E.$$

The equation for z now gives

$$z^2 = \lambda_1 t^2+\mu_1 tu+\nu_1 u^2,$$

and the result again follows from the lemma. This completes the proof for the cubic.

It may be noted that a consideration of the equation

$$y^2 = x(ax^2+bx+e),$$

shows at once that the equation

$$y^2 = Ax^4+Bx^2+C,$$

wherein the right-hand side has no squared factors in x, has only a finite number of integer solutions. The main result of this paper shows that the same result holds for the equation

$$y^2 = ax^4+bx^3+cx^2+dx+e,$$

if the right-hand side has a rational linear factor in x.

COMMUTATIVE ORDINARY DIFFERENTIAL OPERATORS

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Received May 26th, 1922. — Read June 8th, 1922.]

1. *Preliminary.*

IT is convenient at the outset to prescribe certain conventions of notation. There is a single independent variable x and so a single differential operator $D \equiv d/dx$. Greek "lower case" letters α_0, α_2, ... denote functions of x; English "lower case" letters a_0, a_1, ... (but not, of course, x, y) denote constants. A similar distinction is, in general, attached to symbols of functional form, so that $\phi(D)$, $\psi(D)$ are operators with variable coefficients, but $f(D)$, $g(D)$ are operators with constant coefficients. In all cases we are concerned solely with direct, finite, differential operations, so that $\phi(\;)$, $\psi(\;)$, $f(\;)$, $g(\;)$ are always the functional symbols of polynomials. We shall very frequently denote an operator $\phi(D)$ by a single English capital P, Q, etc.

Now two operators $f(D)$, $g(D)$ are commutative, but two operators $\phi(D)$, $\psi(D)$ are not, as a rule, commutative. More generally, if P be any operator, the operators $f(P)$, $g(P)$ are commutative. The question may be asked: Are these the most general types of commutative operators? It is not difficult to see that the answer must be in the negative.

If P_m, Q_n are two perfectly general operators $\alpha_0 D^m + \ldots + \alpha_m$ and $\beta_0 D^n + \ldots + \beta_n$ of orders m, n respectively, their alternant $QP - PQ$ is, on expansion, an operator of order not exceeding $m+n-1$. It vanishes identically, *i.e.* its $m+n$ coefficients vanish, on condition that $m+n$ differential equations are satisfied. To satisfy these we have at our disposal $m+n+2$ arbitrary quantities $\alpha_0, \ldots, \alpha_m$, $\beta_0, \ldots, \beta_n$. Presumably then the conditions can be met. But when m, n are interprime, algebraic considerations alone show that P_m, Q_n are expressible as $f(R)$, $g(R)$, only if R is some linear operator $\lambda D + \mu$. If, for instance, writing $\delta \equiv xD$, we consider

$$(1) \qquad \begin{cases} P \equiv D^2 - 2x^{-2} \equiv x^{-2}(\delta+1)(\delta-2), \\ Q \equiv D^3 - 3x^{-2}D + 3x^{-3} \equiv x^{-3}(\delta+1)(\delta-1)(\delta-3), \end{cases}$$

then $$PQ = x^{-5}(\delta+1)(\delta-1)(\delta-2)(\delta-3)(\delta-5) = QP.$$

But P cannot be written as any $f(\lambda D+\mu)$ and so the operators (1) are commutative, but not expressible in the form $f(R)$, $g(R)$.

Our general problem is the complete determination of all commutative operators not reducible to a form $f(R)$, $g(R)$. This general determination we cannot at present achieve, but we establish in this paper certain characteristic properties of all commutative operators, and obtain a complete solution, in terms of hyperelliptic functions, of the case in which P is an operator of the second degree.

2. *The Characteristic Identity.*

Suppose P_m, Q_n commutative and take any constant h. Let $\eta_1, \ldots, \eta_m$ be a linearly distinct set of solutions of the differential equation

$$(P-h)\, y = 0.$$

Then $$(P-h)Q\eta_1 = Q(P-h)\eta_1 = 0.$$

Thus $Q\eta_1$ is a solution of the differential equation, and we have

$$Q\eta_1 = a_1\eta_1+\ldots+a_m\eta_m.$$

Similarly $$Q\eta_2 = b_1\eta_1+\ldots+b_m\eta_m,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$Q\eta_m = p_1\eta_1+\ldots+p_m\eta_m.$$

By the theory of "normal forms" there is at least one η, a linear function of $\eta_1, \ldots, \eta_m$, such that

$$Q\eta = k\eta,$$

where k is some constant.

Thus η is a common solution of

(2) $$(P-h)\, y = 0 \quad \text{and} \quad (Q-k)\, y = 0.$$

Since h was arbitrarily chosen, this result holds for any h, and k is a function of h, being given by some algebraic relation $f(h, k) = 0$.

If we form the eliminant of the differential equations (2), we have an expression of degree n in the coefficients of P and of degree m in the coefficients of Q, which, equated to zero, must reduce by disappearance of all trace of x to $f(h, k) = 0$.

The actual eliminant may be found by eliminating the $m+n$ unknowns

$$y,\ Dy,\ \ldots,\ D^{m+n-1}\, y$$

between the $m+n$ equations

$$D^r(P-h)y=0 \quad (r=0,\,1,\,\ldots,\,n-1),$$

$$D^s(Q-k)y=0 \quad (s=0,\,1,\,\ldots,\,m-1),$$

which are linear and homogeneous in them.

The two terms of highest order in h, k respectively are

$$h^n\beta_0^m-(-)^{(m+1)(n+1)}\,k^m\alpha_0^n.$$

Now the coefficient of D^{m+n-1} in the alternant $QP-PQ$ is

$$n\beta_0 D\alpha_0-m\alpha_0 D\beta_0,$$

and must vanish. Hence α_0^n is a constant multiple of β_0^m. Accordingly the eliminant, on division by α_0^n or β_0^m (which by presumption are not zero), must become a mere numerical multiple of $f(h,\,k)$, and hence $f(h,\,k)$ is of degree n in h and of degree m in k.

Consider the operator $f(P,\,Q)$ which is of order mn. If η is a common solution of the equations

$$(P-h)y=0, \qquad (Q-k)y=0,$$

then
$$f(P,\,Q)\,\eta=f(h,\,k)\,\eta=0.$$

Thus any such η is a solution of the equation

$$f(P,\,Q)=0.$$

But η depends on an arbitrary h, and so such solutions are infinitely numerous. They are linearly independent, for suppose that a linear relation

$$a_1\eta_1+\ldots+a_r\eta_r=0,$$

connects solutions of
$$Py=h_1y,\,\ldots,\,Py=h_ry,$$

where $h_1,\,\ldots,\,h_r$ are all different. Then operation with $P,\,P^2,\,\ldots,\,P^{r-1}$ on the relation adds to it the $r-1$ other linear relations

$$a_1h_1^s\eta_1+\ldots+a_rh_r^s\eta_r=0 \quad (s=1,\,\ldots,\,r-1).$$

Then either the determinant of these r equations in $a\eta$ vanishes, in which case the h's are not all different, or every η is zero.

Consequently, since the equation $f(P,\,Q)\,y=0$ has infinitely numerous solutions, linearly distinct, the operator $f(P,\,Q)$ must vanish identically. This identity of order n in P, m in Q, connecting a commutative pair, is fundamental in the theory.

Since, then, every commutative pair P_m, Q_n satisfies an identity

$f(P, Q) = 0$, it is of interest to enquire if, conversely, a pair P_m, Q_n satisfying such an identity is necessarily commutative. If we examine the form of the identity $f(P, Q)$, *i.e.* of the eliminant $f(h, k)$, we find that when m, n are interprime the terms therein of highest order in D are of the form $aQ^m - bP^n$, being of order mn, and that all other terms are of order less than mn. With this restriction on the form of the identity the converse can be established.

If, however, m, n are not interprime, but of the form $m_1 r$, $n_1 r$, we ought to contemplate the possibility that P_m, Q_n satisfy an identity

$$\begin{aligned} 0 = F(P, Q) &\equiv [aQ^{m_1} - bP^{n_1}]^s + \text{terms of lower order} \\ &\equiv \Delta^s + \ldots, \text{ say.} \end{aligned}$$

We shall prove that in this slightly more general case P, Q are necessarily commutative.

For otherwise let $QP - PQ$ be of order k in D and write it Ω. Then

$$Q^q P - P Q^q = \sum_{i=0}^{i=q-1} Q^{q-i-1} \Omega Q^i,$$

and is of order $qn - n + k$, since the terms of highest order in each element of the summation are cumulative. In like fashion

$$\begin{aligned} &(P^{p_1} Q^{q_1} \ldots P^{p_t} Q^{q_t}) P - P (P^{p_1} Q^{q_1} \ldots P^{p_t} Q^{q_t}) \\ &\qquad = \sum_{i=1}^{i=t} P^{p_1} Q^{q_1} \ldots P^{p_{i-1}} Q^{q_{i-1}} P^{p_i} (Q^{q_i} P - P Q^{q_i}) P^{p_{i+1}} \ldots P^{p_t} Q^{q_t}, \end{aligned}$$

and is of order $m\Sigma p + n\Sigma q - n + k$. Thus the terms of highest order in $F(P, Q)$ contribute the terms of highest order in the alternant

$$F(P, Q) P - P F(P, Q),$$

which are therefore, after reduction, $\Delta^s P - P \Delta^s$. This vanishes only if $\Delta P - P \Delta$, *i.e.* $a(Q^{m_1} P - P Q^{m_1})$ vanishes. But this is a non-vanishing operator of order $m_1 n - n + k$. Hence the alternant of $F(P, Q)$ and P does not vanish unless that of Q, P vanish. Hence $F(P, Q)$ can vanish identically, only if P, Q are commutative.

The problem of "solving" an identity $F(P, Q) = 0$, restricted as to its highest terms in the way described, *i.e.* of finding operators P, Q which satisfy it, thus forms part of the general problem of determining commutative operators. It is to be presumed that all such restricted identities are soluble, and no others, but these hypotheses are, as yet, unproved.

3. *Derivation of New Operators by Transference of Factors.*

Since $(P-h)\,y=0$ and $(Q-k)\,y=0$ have a common solution or solutions, we may suppose them to be a complete set of solutions of some equation $Ty=0$, and hence we must have

$$P-h\equiv R.T \quad \text{and} \quad Q-k\equiv S.T.$$

Since $P-h$ and $Q-k$ are commutative,

$$RTST=STRT, \quad i.e.\ RTS=STR,^* \quad i.e.\ TRTS=TSTR.$$

Thus the operators $P'-h\equiv TR$ and $Q'-k\equiv TS$ are commutative, and P', Q' themselves are commutative. But again, if h' and k' are properly related, $P'-h'\equiv R'T'$ and $Q'-k'\equiv S'T'$. By transference of the common factor T' we derive a new commutative pair P'', Q'' given by $P''-h'\equiv T'R'$ and $Q''-k'\equiv T'S'$, and so forth endlessly.

If we expand the characteristic identity $f(P,Q)=0$, in the form

$$\Sigma a(P-h)^p\,(Q-k)^q=0,$$

and observe that

$$T(P-h)^p\,(Q-k)^q=TRTRT\ldots RTST\ldots ST=TRTR\ldots TRTS\ldots TST$$
$$=(P'-h)^p\,(Q'-k)^q\,T,$$

we see that
$$Tf(P,\,Q)=f(P',\,Q')\,T,$$

and hence
$$f(P',\,Q')=0.$$

Thus the characteristic identity persists with unchanged form for the whole set of operators derived from a chosen P, Q by transference of common factors. So restricted, the operators presumably form a group.

If, after each transference of factors, we avail ourselves of our full freedom and choose each new h an arbitrary constant, the group will admit of continuous variation and have the character of a continuous group. Alternatively, we may fix k, say, throughout the generation of the group. The relation between h, k is algebraic and gives choice of n values of h against the fixed value of k. The group is finite and algebraic in character. It is possible to apply this notion to the determination of commutative pairs, but the analysis is long and difficult.

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* For, if $AB=0$, clearly $A=0$ or $B=0$, since the highest term in AB is the algebraic product of the highest terms in A and B. Hence, if $AC=BC$, necessarily $A=B$, unless $C=0$.

4. *Some Special Commutative Operators.*

We may illustrate the foregoing principles with certain operators constructed by aid of the special operator $\delta \equiv xD$.

If we write

$$P \equiv x^{-m}\delta(\delta-n)(\delta-2n)\ldots(\delta-mn+n),$$

$$Q \equiv x^{-n}\delta(\delta-m)(\delta-2m)\ldots(\delta-mn+m),$$

it is seen that

$$QP = x^{-(m+n)}\delta(\delta-n)\ldots(\delta-mn+n)(\delta-m)\ldots(\delta-mn+m)(\delta-mn) = PQ,$$

and that $$P^n = x^{-mn}\prod_{r=0}^{n-1}\prod_{s=0}^{m-1}(\delta-mr-ns) = Q^m.$$

This represents a generalisation of the commutative pair P, Q already mentioned in equations (1), for those can be rewritten as

$$xPx^{-1} = x^{-2}\delta(\delta-3), \qquad xQx^{-1} = x^{-3}\delta(\delta-2)(\delta-4).$$

We may generalise this latter pair in a somewhat different fashion.

Consider the identity $P^3 = Q^2$. This can be satisfied by writing

$$(3) \qquad P \equiv D^2 = x^{-2}\delta(\delta-1), \qquad Q \equiv D^3 = x^{-3}\delta(\delta-1)(\delta-2).$$

The factor $\delta-1$, common to P, Q, indicates the presence of a repeated common root. By transference of this factor we derive

$$(4) \qquad \begin{cases} P' \equiv (\delta-1)x^{-2}\delta = x^{-2}\delta(\delta-3), \\ Q' \equiv (\delta-1)x^{-3}\delta(\delta-2) = x^{-3}\delta(\delta-2)(\delta-4). \end{cases}$$

By transference of factors it is possible, then, to derive irreducible operators such as (4) from reducible operators such as (3). It is not clear what are the general implications of this principle. At any rate it is apparently not the case that every commutative pair can be derived from a reducible pair by transference of factors, and it is also essential for the success of the manœuvre that the factor transferred should represent the common root to its highest degree. If in (3) we had transferred the other common factor δ, we should have obtained

$$P'' = \delta x^{-2}(\delta-1) = x^{-2}(\delta-1)(\delta-2) = (D-x^{-1})^2 = xD^2x^{-1},$$

$$Q'' = \delta x^{-3}(\delta-1)(\delta-2) = x^{-3}(\delta-1)(\delta-2)(\delta-3) = (D-x^{-1})^3 = xD^3x^{-1}.$$

Here P'', Q'' do not differ in character from the P, Q from which they were derived and no advance has been made.

If we commence with the trio of reducible commutative operators

$$P \equiv D^3 = x^{-3}\delta(\delta-1)(\delta-2), \quad Q \equiv D^4 = x^{-4}\delta(\delta-1)(\delta-2)(\delta-3),$$

$$R \equiv D^5 = x^{-5}\delta(\delta-1)(\delta-2)(\delta-3)(\delta-4),$$

and transfer the factor $\delta-2$ representing the common triple factor D^3, we obtain

$$P' = x^{-3}\delta(\delta-1)(\delta-5), \quad Q' = x^{-4}\delta(\delta-1)(\delta-3)(\delta-6),$$

$$R' = x^{-5}\delta(\delta-1)(\delta-3)(\delta-4)(\delta-7),$$

which are irreducible and obey the identities $P'R' = Q'^2$ and $Q'R' = P'^3$, since P, Q, R obey similar identities. Generally, if we commence with the n operators D^n, D^{n+1}, ..., D^{2n-1} expressed in terms of δ, and transfer the factor $\delta-n+1$, which represents the multiple factor D^n, we obtain a set of operators mutually commutative and irreducible.

Our enquiry as to commutative operators ought therefore not to be limited merely to commutative pairs, but must take into account the existence of *sets* of mutually commutative operators. If P_m, Q_n, R_r is a mutually commutative trio, then the pair P, Q must appear among the commutative pairs of orders (m, n). But in view of the added conditions that they be also commutative with R, they will presumably not be the most general commutative pair of these orders. The search for the most general commutative *sets* is thus more restricted, and so presumably less difficult, than the search for the most general commutative *pairs*.

It will appear in the next section that in an irreducible commutative set of n operators none may have order less than n.

5. *Reducibility and Standard Form.*

The identity $PQ-QP = 0$ is not invalidated by any change of independent variable or by the substitutions

$$P' = \theta^{-1}P\theta, \quad Q' = \theta^{-1}Q\theta,$$

where θ is any function of x.

By aid of these two substitutions we may throw P, Q into the forms

$$D^m + \alpha_2 D^{m-2} + \ldots, \quad D^n + b_1 D^{n-1} + \beta_2 D^{n-2} + \ldots,$$

in which the leading coefficients are unity and the second coefficients are lacking or constant. For if

$$P \equiv \alpha_0 D^m + \alpha_1 D^{m-1} + \ldots, \quad Q \equiv \beta_0 D^n + \beta_1 D^{n-1} + \ldots,$$

then the coefficient of D^{m+n-1} in $QP-PQ$ is, as we have seen,

$$n\beta_0 \frac{d\alpha_0}{dx} - m\alpha_0 \frac{d\beta_0}{dx},$$

and vanishes. Hence we can write

$$\alpha_0 = a\xi^m, \qquad \beta_0 = b\xi^n.$$

With $\int dx/\xi$ as new independent variable, the leading coefficients of P, Q are now constants, and it is sufficient to write

$$P \equiv D^m + \alpha_1 D^{m-1} + \ldots, \qquad Q \equiv D^n + \beta_1 D^{n-1} + \ldots.$$

The coefficient of D^{m+n-2} in $QP-PQ$ is now

$$n \frac{d\alpha_1}{dx} - m \frac{d\beta_1}{dx}.$$

Since this must vanish, we can write

$$\alpha_1 = m\xi, \qquad \beta_1 = b_1 + n\xi.$$

If we put

$$P' = \exp\left(\int \xi\, dx\right) P \exp\left(-\int \xi\, dx\right), \qquad Q' = \exp\left(\int \xi\, dx\right) Q \exp\left(-\int \xi\, dx\right),$$

we have reduced P and Q to the promised form.

If we consider P, Q, the two most general operators in this standard form, the alternant $QP-PQ$ is now an operator of order $m+n-3$, *i.e.* $m+n-2$ equations secure the vanishing of every coefficient. But P, Q have now $m-1$, $n-1$ arbitrary coefficients which can presumably be determined without ambiguity from the equations.

If n is a multiple of m, say $n = mr$, then $Q-P^r$ is commutative with P and is of order less than n. We need not therefore study Q apart from this other operator of lower order. More generally, if P, Q are commutative, so are $f(P, Q)$, $g(P, Q)$. Our interest will therefore be confined to *irreducible* commutatives, *i.e.* to those not expressible as $f(R, S)$, $g(R, S)$ in terms of commutatives R, S of lower order. In particular we can ignore the case in which one of m, n is a multiple of the other, and therefore certainly the case in which $m = n$.

The case of $m = 1$ is thus excluded, and certainly too the case of $m = 0$, for in standard form P_0 is a mere constant. Hence in an irreducible commutative pair the least possible order is 2. So in a commutative trio P, Q, R the least possible order is 3; for suppose P to be of order 2. Then Q, R must be each of odd order, say

$$Q \equiv D^{2n+1}+\ldots, \qquad R \equiv D^{2n+2p+1}+\ldots,$$

and evidently R is reducible by means of QP^p.

So generally in a set of n commutative operators none can have order less than n, and if one have order n, the others must be congruent to $1, 2, \ldots, n-1 \pmod{n}$.

6. *One Operator Quadratic. Semi-Commutatives.*

We now discuss the particular case in which $m = 2$, and hence sufficiently

$$P \equiv D^2+2\theta_1.$$

If Q is irreducible, it is of odd order, and we have

$$Q_{2n+1} \equiv D^{2n+1}+b_1 D^{2n}+\ldots .$$

Since $Q_{2n+1}-b_1 P^n$ is also commutative with P, we may sufficiently take

$$Q_{2n+1} \equiv D^{2n+1}+\beta_2 D^{2n-1}+\ldots .$$

It is convenient here to introduce the notion of "semi-commutative." If of P, Q the operator of lower order (m) is P, then we say that Q is semi-commutative with P, if $QP-PQ$ is an operator of order not exceeding $m-2$.

The equations of condition imposed by the vanishing of the coefficients of $QP-PQ$ are now fewer by $m-1$; this is the number of independent coefficients in P. We may therefore prescribe P and investigate the most general irreducible operator Q, semi-commutative with it. In the restricted case of $m = 2$, Q is semi-commutative, if $QP-PQ$ reduces to the term independent of D, *i.e.* is no longer an operator proper, but a mere multiplying function of x.

The investigation falls into two parts: we first obtain a particular Q of prescribed odd order and then deduce the general Q of prescribed order.

Now, accents denoting differentiation,

$$D(D^2+2\theta_1)-(D^2+2\theta_1)D = 2\theta_1'.$$

Hence we may take D as a Q_1. Suppose, then, that we have found a Q_{2n-1}, *i.e.* suppose that

(5) $$Q_{2n-1}P-PQ_{2n-1} = 2\theta_n'.$$

Write $$R \equiv Q_{2n-1}P+PQ_{2n-1}.$$

Then
$$\begin{aligned} RP-PR &= (Q_{2n-1}P+PQ_{2n-1})P-P(Q_{2n-1}P+PQ_{2n-1}) \\ &= (Q_{2n-1}P-PQ_{2n-1})P+P(Q_{2n-1}P-PQ_{2n-1}) \\ &= 2\theta_n'P+2P\theta_n' \\ &= 4\theta_n'D^2+4\theta_n''D+2\theta_n'''+8\theta_1\theta_n'. \end{aligned}$$

But $(2\theta_n D+\theta_n')P-P(2\theta_n D+\theta_n') = -4\theta_n'D^2-4\theta_n''D-\theta_n'''+4\theta_1'\theta_n.$

Hence

$$(6)\qquad (R+2\theta_n D+\theta_n')P-P(R+2\theta_n D+\theta_n') = \theta_n'''+4\theta_1'\theta_n+8\theta_1\theta_n'.$$

Thus $R+2\theta_n D+\theta_n'$ is semi-commutative with P. We may write this operator in the form

$$(7)\qquad Q_{2n-1}P+PQ_{2n-1}+\theta_n D+D\theta_n,$$

which shows it to be *self-adjoint*.* By use of (5) we may write the operator as
$$2\left\{Q_{2n-1}P+\theta_n D-\tfrac{1}{2}\theta_n'\right\}.$$

Write now

$$(8)\qquad Q_{2n+1}\equiv Q_{2n-1}P+\theta_n D-\tfrac{1}{2}\theta_n' \quad (n=1,2,3,\ldots),$$

and
$$Q_1\equiv D.$$

We can in this way determine inductively a Q of prescribed odd order semi-commutative with a given P. The explicit expression for Q_{2n+1} is

$$(9)\quad DP^n+\theta_1 DP^{n-1}+\theta_2 DP^{n-2}+\ldots+\theta_n D-\tfrac{1}{2}(\theta_1'P^{n-1}+\theta_2'P^{n-2}+\ldots+\theta_n').$$

To obtain the general operator R_{2n+1} of order $2n+1$, semi-commutative with P, we must remember that the discussion of § 5 as to standard form and reducibility applies to semi-commutative pairs as well as to pairs fully commutative, since we there made use of the coefficients of D^{m+n-1}, D^{m+n-2} only in the alternant $QP-PQ$.

Hence R_{2n+1} is, save for $f(P)$, of the form $a_0D^{2n+1}+a_2D^{2n-1}+\ldots$. The term of highest order in Q_{2n+1} is D^{2n+1}, as is evident from (8). Thus $R_{2n+1}-a_0Q_{2n+1}$, semi-commutative with P, is of order $2n-1$, *i.e.* begins $a_1D^{2n-1}+a_4D^{2n-3}+\ldots$. Proceeding in this way we see that at length

$$R_{2n+1}-a_0Q_{2n+1}-a_1Q_{2n-1}-\ldots-a_{n-1}Q_3$$

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* We return to this point in § 9.

is linear and semi-commutative with P. Since identically

$$(aD+\beta)P-P(aD+\beta) = -2a'D^2-(a''+2\beta')D-\beta''+2a\theta_1',$$

it is clear that the only linear operator semi-commutative with P is $a_n D+a_{n+1}$. We may reject a_{n+1} and so every irreducible semi-commutative R_{2n+1} must be included in

(10) $$a_0 Q_{2n+1}+a_1 Q_{2n-1}+\ldots+a_n Q_1.$$

7. *Operators completely Commutative.*

The passage from operators semi-commutative with P to operators fully commutative therewith is effected, if we make proper choice of the function θ_1, still at our disposal, which enters into the definition of P.

If R_{2n+1} now denote the semi-commutative (10), we have that

(11) $$R_{2n+1}P-PR_{2n+1} = 2\{a_0\theta_{n+1}'+a_1\theta_n'+\ldots+a_n\theta_1'\}.$$

We can determine θ_n inductively by means of (6). The reduction-formula actually is

(12) $$\theta_{n+1}' = \tfrac{1}{4}\theta_n'''+\theta_1'\theta_n+2\theta_1\theta_n'.$$

The right-hand side of (11) equated to zero is a differential equation for θ_1 of order $2n+1$ and degree $n+1$. It can be shown to have the n integrating factors $\theta_1, \theta_2, \ldots, \theta_n$, and it is also integrable as it stands. It is therefore reducible to an equation of order n, and since x is not explicitly present, a change to variables θ_1', θ_1 depresses the order to $n-1$. The equation at this stage appears to be impervious to further straightforward treatment and a new line of attack must be developed.

By the results of § 2, h can always be found such that the differential equations $Qy = 0 = (P-h)y$ have a common solution. Thus, identifying Q with R_{2n+1}, we can always find an η such that

$$R_{2n+1}\eta = 0, \quad P\eta = h\eta.$$

From (9), we see that

$$Q_{2n+1}\eta = (h^n+\theta_1 h^{n-1}+\theta_2 h^{n-2}+\ldots+\theta_n)D\eta-\tfrac{1}{2}(\theta_1' h^{n-1}+\ldots+\theta_n')\eta.$$

Hence $$R_{2n+1}\eta \equiv \Theta D\eta-\tfrac{1}{2}\Theta'\eta = 0,$$

where

(13) $$\Theta \equiv a_0 h^n+a_1 h^{n-1}+\ldots+a_n+\theta_1(a_0 h^{n-1}+\ldots+a_{n-1}) +\theta_2(a_0 h^{n-2}+\ldots+a_{n-2})+\ldots+\theta_n a_0.$$

This differential equation for η gives

$$\eta^2 = \Theta \times constant.$$

Take the constant to be $1/a_0$, and we may factorise Θ and write

$$\eta^2 = (h-a_1)(h-a_2)\ldots(h-a_n), \tag{14}$$

where $a_1, a_2, \ldots, a_n$ are some functions of x. Hence, differentiating,

$$\frac{2\eta'}{\eta} = -\Sigma \frac{a'}{h-a}$$

and
$$\frac{2\eta''}{\eta} - 2\left(\frac{\eta'}{\eta}\right)^2 = -\Sigma \frac{a''}{h-a} - \Sigma \frac{a'^2}{(h-a)^2}.$$

But $\eta''/\eta = h - 2\theta_1$, since $P\eta = h\eta$, and we have the equation

$$0 = 4(h-2\theta_1) + \Sigma \frac{2a_r''}{h-a_r} + \Sigma \frac{a_r'^2}{(h-a_r)^2} - 2\Sigma \frac{a_r' a_s'}{(h-a_r)(h-a_s)}. \tag{15}$$

Now, by § 2, h satisfies an algebraic equation

$$(h-e_1)(h-e_2)\ldots(h-e_{2n+1}) = 0,$$

of degree $2n+1$. This is evidently the equation in h obtained by clearing (15) of fractions. Thus we shall have identically

$$\begin{aligned} &4(h-e_1)\ldots(h-e_{2n+1}) \\ &\equiv \sum_{r=1}^{n} a_r'^2 (h-a_1)^2 \ldots (h-a_{r-1})^2 (h-a_{r+1})^2 \ldots (h-a_n)^2 \\ &\quad + 4(h-2\theta_1) \prod_{r=1}^{n} (h-a_r)^2 + \phi(h) \prod_{r=1}^{n} (h-a_r), \end{aligned} \tag{16}$$

where ϕ is a function of h of degree not exceeding $n-1$ and involving a_r'', $a_r' a_s'$.

Put in succession $h = a_1, \ldots, h = a_n$, and we obtain the n equations

$$\frac{da_r}{2\surd\{(a_r-e_1)\ldots(a_r-e_{2n+1})\}} = \frac{dx}{(a_r-a_1)\ldots(a_r-a_{r-1})(a_r-a_{r+1})\ldots(a_r-a_n)} \tag{17}$$

$$(r = 1, 2, \ldots, n).$$

Define the hyperelliptic function $a = F(u)$ by inversion of the integral

$$u = \int^a \frac{da}{2\surd\{(a-e_1)\ldots(a-e_{2n+1})\}}. \tag{18}$$

If $\alpha_1, \ldots, \alpha_n$ are the values of $F(u)$ corresponding to arguments $u_1, \ldots, u_n$, we have from (17) the following n equations connecting these n arguments

$$(19) \qquad \begin{cases} \Sigma \{F(u)\}^r \, du = 0 & (r = 0, 1, \ldots, n-2), \\ \Sigma \{F(u)\}^{n-1} \, du = dx. \end{cases}$$

These equations suffice to interconnect $x, u_1, \ldots, u_n$, leaving one degree of freedom.

The problem has now been transferred from the domain of differential operators: further discussion rightly belongs to the theory of hyperelliptic functions.

It should be pointed out that we can express $\theta_1, \ldots, \theta_n$ in terms of these functions by identifying the expressions (13), (14) for Θ. We have

$$\theta_1 + a_1/a_0 = -\Sigma \alpha_r,$$

$$\theta_2 + (a_1/a_0)\,\theta_1 + a_2/a_0 = \Sigma \alpha_r \alpha_s, \text{ etc.}$$

If we compare coefficients of h^{2n} in (16), we have

$$-4\Sigma e = -8\theta_1 - 8\Sigma\alpha.$$

Hence $$a_1/a_0 = -\tfrac{1}{2}\Sigma e.$$

It is legitimate and usual to take $\Sigma e = 0$, so that $a_1 = 0$.

The other constants $a_2, a_3, \ldots$ can be explained by the fact that $\theta_2, \theta_3, \ldots$ which enter the analysis are defined only by their differential coefficients. They represent the arbitrary constants thus introduced.

The argument so far developed is not, however, competent to show that the operators which we have defined in terms of elliptic functions are *necessarily* commutative. We have, in point of fact, proved only that they are a class which necessarily contains the class of commutative operators: we have not shown that the two classes are coextensive. It is fortunately possible, with certain precautions, to reverse the argument of the preceding pages, thus proving the hyperelliptic operators definitely commutative. It is probably sufficient to sketch the reversed argument, as follows.

It will be found that, when the n expressions α_r are defined as in equations (17), then each root $h = \alpha_r$ of (16), now interpreted as an equation in h, is a double root. The equation has thus $2n$ known roots. The coefficients of h^{2n+1}, h^{2n} disappear, the latter in virtue of the relation

$$\theta_1 + a_1/a_0 + \Sigma\alpha_r = 0.$$

The equation in h is thus, at most, of degree $2n-1$, and so is re-established as an identity. Hence, if h is any e_r, then (15) is true, and η, defined by (14), satisfies the equation $P\eta = h\eta$. Differentiation of (14) gives (13), if the $n-1$ functions $\theta_2, \ldots, \theta_n$ are properly defined in terms of the functions a_r. If, then, Q is the operator (10), it follows that $Q\eta = 0$. Hence any such η is annihilated by the operator

$$Q^2-(P-e_1)\ldots(P-e_{2n+1}). \tag{20}$$

If ξ be a solution of $(P-h)\,y = 0$, distinct from η, we have

$$\xi'\eta-\xi\eta' = c.$$

Since still $P\xi = h\xi$, it follows, as for η, that

$$Q\xi = \Theta\xi'-\tfrac{1}{2}\Theta'\xi = a_0(\eta^2\xi'-\eta\eta'\xi) = ca_0\eta.$$

Hence $$Q^2\xi = ca_0\,Q\eta = 0.$$

Thus every such ξ is also annihilated by (20). The argument of § 2 shows that the ξ's and the η's are all linearly independent. We have thus $4n+2$ linearly independent expressions annihilated by (20). But since the highest terms of Q^2, P^{2n+1} are alike D^{4n+2}, it is clear that the operator (20) is at most of degree $4n+1$ and so vanishes identically. Since this is an identity of the form considered in § 2, it follows that P, Q are commutative.

8. *The Pair of Orders* (3, 2).

For the case $2n+1 = 3$, *i.e.* $n = 1$, the n hyperelliptic functions reduce to a single elliptic function. In fact, equation (18) gives

$$F(u) \equiv \wp(u+\text{constant}),$$

and equations (19) reduce to the single equation $du = dx$. We have also now $\theta_1 = -a$, and hence sufficiently

$$\left\{\begin{aligned} P &\equiv D^2-2\wp(x), \\ Q &\equiv D^3-3\wp(x)D-\tfrac{3}{2}\wp'(x). \end{aligned}\right. \tag{21}$$

In this case of $n=1$, the differential equation $\theta'_{n+1}=0$ (namely, the added condition securing that a semi-commutative is completely commutative) is at once integrable. For, from (12), it is

$$\tfrac{1}{4}\theta_1''' +3\theta_1'\theta_1 = 0,$$

which is at once integrable into

$$\theta_1'^2+4\theta_1^3=g_2\theta_1-g_3,$$

the differential equation of $-\wp(x)$. Equations (21) could thus have been obtained directly from this standpoint.

The functional identity is now

$$Q^2=(P-e_1)(P-e_2)(P-e_3),$$

or

$$4Q^2=4P^3-g_2P-g_3.$$

Hence also, by the general theory of § 2, the equations

$$(P-h)y=0=(Q-k)y$$

have a common solution, if

$$4k^2=4h^3-g_2h-g_3,$$

i.e. if h, k are respectively $\wp(a)$, $\frac{1}{2}\wp'(a)$, where a is any constant. Thus

$$[Q-\tfrac{1}{2}\wp'(a)]y=0=[P-\wp(a)]y$$

have a common solution, which is consequently a solution of

$$[D\{P-\wp(a)\}-\{Q-\tfrac{1}{2}\wp'(a)\}]y=0,$$

i.e. of

$$\frac{Dy}{y}=\frac{1}{2}\frac{\wp'(x)-\wp'(a)}{\wp(x)-\wp(a)}=\zeta(x+a)-\zeta(x)-\zeta(a).$$

Hence sufficiently

$$y=\exp\{-x\zeta(a)\}\,\sigma(x+a)/\sigma(x).$$

Since

$$\wp(-a)=\wp(a)\quad\text{and}\quad\wp'(-a)=-\wp'(a),$$

the other solution of $[P-\wp(a)]y=0$ is the solution common to this equation and to $[Q+\frac{1}{2}\wp'(a)]y=0$.

The complete solution of

$$[D^2-2\wp(x)-\wp(a)]y=0 \tag{22}$$

is thus

$$y=c_1\frac{\sigma(x+a)}{\sigma(x)}e^{-x\zeta(a)}+c_2\frac{\sigma(x-a)}{\sigma(x)}e^{x\zeta(a)}.$$

This, however, is well known, since (22) is exactly Lamé's equation.*

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* Halphen, *Fonctions Elliptiques*, 2nd Part, pp. 494 *et seq.*; Forsyth, *Theory of Functions*, 2nd Ed., p. 314.

In like fashion the complete solution of

(23) $$[D^3-3\wp(x)D-\tfrac{3}{2}\wp'(x)-\tfrac{1}{2}\wp'(a)]\,y = 0$$

is $$y\sigma(x) = c_1\sigma(x+a)\,e^{-x\zeta(a)}+c_2\sigma(x+b)\,e^{-x\zeta(b)}+c_3\sigma(x+c)\,e^{-x\zeta(c)},$$

where $\wp(a)$, $\wp(b)$, $\wp(c)$ are the roots of the cubic

$$\wp'^2(x) = \wp'^2(a),$$

whence $$a+b+c = 0 = \wp(a)+\wp(b)+\wp(c).$$

From a knowledge of its solutions we can factorise $P-\wp(a)$ either as

(24) $$\{D+\zeta(x+a)-\zeta(x)-\zeta(a)\}\,\{D-\zeta(x+a)+\zeta(x)+\zeta(a)\},$$

or as

(25) $$\{D+\zeta(x-a)-\zeta(x)+\zeta(a)\}\,\{D-\zeta(x-a)+\zeta(x)-\zeta(a)\}.$$

If we interchange the factors in (24), we obtain

$$\{D+\zeta(x)-\zeta(x+a)+\zeta(a)\}\,\{D-\zeta(x)+\zeta(x+a)-\zeta(a)\}.$$

But this is exactly (25) with $x+a$ written for x. Hence, if in the spirit of § 3 we "transfer" the factor common to $P-\wp(a)$ and $Q-\frac{1}{2}\wp'(a)$, the first of them becomes $D^2-2\wp(x+a)-\wp(a)$, or, let us say, $P_{x+a}-\wp(a)$. The principles of § 5 show that there can be only one operator (save for an added constant) commutative with this. Hence $Q-\frac{1}{2}\wp'(a)$ becomes by the transference

$$D^3-3\wp(x+a)D-\tfrac{3}{2}\wp'(x+a)-\text{constant},$$

or, let us say, $Q_{x+a}-C$.

Then by a fundamental property of a "transference" $P_{x+a}-\wp(a)$, $Q_{x+a}-C$ are connected by the same identity as are $P_x-\wp(a)$, $Q_x-\frac{1}{2}\wp'(a)$. But since the coefficients in the identity do not involve x, the identity connecting the *second* pair is the same as that connecting $P_{x+a}-\wp(a)$, $Q_{x+a}-\frac{1}{2}\wp'(a)$. In view of the form of the identity this necessitates

$$Q^2_{x+a} = \{Q_{x+a}-C+\tfrac{1}{2}\wp'(a)\}^2,$$

and hence $$C = \tfrac{1}{2}\wp'(a).$$

Hence the "transference" on $P-\wp(a)$ and $Q-\frac{1}{2}\wp'(a)$ alike is adequately represented by the substitution $x = x'+a$. We have therefore a vindication of the presumption of § 3, that repeated transference of common factors of $P-h$, $Q-k$ generates a continuous group; for it is the group of the substitutions

$$x = x'+a.$$

If we write $a = \omega_r$, a half period of the elliptic function $(r = 1, 2, 3)$, the operators reduce to Q, $P-e_r$. The roots of $Qy = 0$ are

$$\sqrt{\{\wp(x)-e_1\}}, \quad \sqrt{\{\wp(x)-e_2\}}, \quad \sqrt{\{\wp(x)-e_3\}}.$$

Those of $(P-e_r)y = 0$ previously given are now coalescent. An adequate pair is

$$\sqrt{\{\wp(x)-e_r\}}, \quad \{\zeta(x+\omega_r)+xe_r\}\sqrt{\{\wp(x)-e_r\}}.$$

Transference of end-factors is now equivalent to one of the substitutions

$$x = x'+\omega_r \quad (r = 1, 2, 3),$$

and hence, from the P, Q defined by equations (21), we derive three associated commutative pairs

$$P_r \equiv D^2-2\wp(x+\omega_r), \quad Q_r \equiv D^3-3\wp(x+\omega_r)D-\tfrac{3}{2}\wp'(x+\omega_r) \quad (r = 1, 2, 3).$$

Reiterated transference of factors is equivalent to a general substitution

$$x = x'+p_1\omega_1+p_2\omega_2+p_3\omega_3,$$

where p_1, p_2, p_3 are integers.

Hence by reason of the periodicity of $\wp(x)$ no new operators are introduced, and so (P, Q), (P_1, Q_1), (P_2, Q_2), (P_3, Q_3) constitute a closed group of four terms for any transference of factors. This substantiates the statement of the second half of § 3. It can be shown that for the pair of orders $(2n+1, 2)$ the corresponding group has 4^n terms. We may exhibit the characteristic relations between the members of this group, if we factorise them into linear operators.

If r, s, t are 1, 2, 3 in any order, write

$$rs \equiv D-\frac{\frac{1}{2}\wp'(x+\omega_s)}{\wp(x+\omega_s)-e_t}, \quad 0s \equiv D-\frac{\frac{1}{2}\wp'(x+\omega_s)}{\wp(x+\omega_s)-e_s}, \quad r0 \equiv D-\frac{\frac{1}{2}\wp'(x)}{\wp(x)-e_r}.$$

Then, writing P_0, Q_0 for P, Q, we have

$$(26)\qquad P_0 = 01\,.\,10+e_1 = 02\,.\,20+e_2 = 03\,.\,30+e_3,$$

$$Q_0 = 01\,.\,12\,.\,20 = 01\,.\,13\,.\,30 = 02\,.\,23\,.\,30 = 02\,.\,21\,.\,10$$

$$= 03\,.\,31\,.\,10 = 03\,.\,32\,.\,20,$$

$$P_1 = 10\,.\,01+e_1 = 13\,.\,31+e_2 = 12\,.\,21+e_3,$$

$$Q_1 = 10\,.\,02\,.\,21 = 10\,.\,03\,.\,31 = 12\,.\,23\,.\,31 = 12\,.\,20\,.\,01$$

$$= 13\,.\,30\,.\,01 = 13\,.\,32\,.\,21,$$

with similar expressions for P_2, Q_2, P_3, Q_3.

If we transform to $\wp(x)$ as independent variable, we can evidently express P, Q in terms of functions purely algebraic. For brevity, denote by α, β, γ the sums of products of $x-e_1$, $x-e_2$, $x-e_3$ taken respectively 3, 2, 1 at a time, we then find

$$(27)\qquad \begin{cases} P = 4(\alpha D^2+\tfrac{1}{2}\beta D-\tfrac{1}{6}\gamma), \\ Q = 4\sqrt{\alpha}(\alpha D^3+\tfrac{3}{2}\beta D^2+\tfrac{3}{4}\gamma D-\tfrac{3}{8}). \end{cases}$$

These may be factorised in various forms, of which the following are typical:

$$P-e_1 = 4(x-e_2)^{\frac{1}{2}}(x-e_3)^{\frac{1}{2}}D(x-e_2)^{\frac{1}{2}}(x-e_3)^{\frac{1}{2}}(x-e_1)^{\frac{3}{2}}D(x-e_1)^{-\frac{1}{2}},$$

$$Q = 4(x-e_3)^{\frac{1}{2}}(x-e_1)^{\frac{1}{2}}D(x-e_2)^{\frac{3}{2}}(x-e_3)^{\frac{1}{2}}D(x-e_3)^{\frac{1}{2}}(x-e_1)^{\frac{3}{2}}D(x-e_1)^{-\frac{1}{2}}.$$

It does not appear, however, that the discussion of the commutative operators is facilitated by considering them in these forms. The natural expression seems to be that already obtained for them in terms of elliptic or hyperelliptic functions.

It is sometimes useful as an arithmetical check to consider the degenerate forms which arise when $e_1 = e_2 = e_3 = 0$. We then have

$$P = 4(x^3D^2+\tfrac{3}{2}x^2D-\tfrac{1}{2}) = 4x(\delta+1)(\delta-\tfrac{1}{2}),$$

$$Q = 4x^{\frac{3}{2}}(x^3D^3+\tfrac{9}{2}x^2D^2+\tfrac{9}{4}xD-\tfrac{3}{8}) = 4x^{\frac{3}{2}}(\delta+\tfrac{3}{2})(\delta+\tfrac{1}{2})(\delta-\tfrac{1}{2}).$$

If we make the further substitution $x = x'^{-2}$, we obtain the P, Q of equations (1). We might equally, of course, have obtained them by putting $e_1 = e_2 = e_3 = 0$, *i.e.* $\wp(x) = x^{-2}$, in equations (21).

9. *Adjoint Operators.*

The notion of adjoint operators is of some importance in the present theory. If we denote by P', Q' the operators adjoint to any given operators P, Q, then it is known from the general theory of adjoint operators* that $P'Q'-Q'P'$ is the operator adjoint to $QP-PQ$. Hence, if P, Q are commutative, so also are P', Q'. For, if an operator vanish identically, so also must the adjoint operator.

Again, if the identity connecting P, Q is

$$f(P, Q) \equiv \Sigma a_{rs} P^r Q^s = 0,$$

then the operator adjoint to $f(P, Q)$ vanishes identically, and is

$$\Sigma a_{rs} Q'^s P'^r = \Sigma a_{rs} P'^r Q'^s,$$

since P', Q' are commutative. Thus P', Q' obey the same identity, as do P, Q, namely $f(P', Q') = 0$. The group of transformations for which the fundamental identity is invariant includes therefore both the "transference" and the "adjoint" transformations. For pairs of orders $(2n+1, 2)$ it is the case that the "adjoint" transformation can be expressed as a result of successive transferences. For the case (3, 2) this is clear from the formulæ (26), if we remember that the adjoint of, say, 01 . 12 . 20 is 02 . 21 . 10, obtained by simple reversal of the figures. A corresponding fact is probably true generally.

If P is self-adjoint, then Q and its adjoint Q' are each commutative with P, *i.e.* $Q \pm Q'$ are each commutative with P. Since the highest terms in Q, Q' are the same, save for a possible change of sign, one of $\frac{1}{2}(Q+Q')$, $\frac{1}{2}(Q-Q')$, say the latter, is of lower order than Q. Hence we may consider

$$Q \equiv \tfrac{1}{2}(Q+Q') + \tfrac{1}{2}(Q-Q'),$$

reducible to $\frac{1}{2}(Q+Q')$ which is self-adjoint. Thus in seeking operators commutative with a self-adjoint, we may limit ourselves to operators themselves self-adjoint. In particular, if P is in standard form $D^2+2\theta$, it is self-adjoint, and hence Q, commutative with P and in standard form, is at least reducible to a self-adjoint. But, when Q is in standard form, the identity, as we have seen, takes the form

$$Q^2 - f(P) = 0.$$

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* For the more important properties of adjoint operators, see Forsyth, *Theory of Differential Equations*, Vol. 4, p. 253 *et seq.*; Darboux, *Théorie des Surfaces*, Pt. 2, Ser. 4, Ch. 5.

The adjoint identity is therefore

$$Q'^2 - f(P) = 0.$$

Thus $Q^2 - Q'^2 = 0$, which may be written

$$(Q+Q')(Q-Q')+(Q-Q')(Q+Q') = 0.$$

Now, if R, S be two operators whose highest terms are respectively αD^r, βD^s, then the highest term of $RS+SR$ is $2\alpha\beta D^{r+s}$. Hence $RS+SR$ can vanish, only if one of R, S vanish. Thus one of $Q \pm Q'$ must vanish, and so Q in standard form is actually self-adjoint.

Certain of the foregoing arguments apply equally to semi-commutative operators, for if $QP-PQ$ is of order r, so is its adjoint $P'Q'-Q'P'$, and hence, if P, Q are semi-commutative, so, too, are P', Q'. This provides corroboration of the formula (7) of § 6 which presents Q_{2n+1}, semi-commutative with $D^2+2\theta_1$, in a form self-adjoint, if P, Q_{2n-1} are.

Darboux has shown* that the functions $\eta_1, \ldots, \eta_{2n+1}$, annihilated by a self-adjoint of order $2n+1$ with leading term D^{2n+1}, are connected by a quadratic relation with constant coefficients

$$F(\eta_1, \ldots, \eta_{2n+1}) = 0,$$

the same relation being also satisfied when $\eta_1, \ldots, \eta_{2n+1}$ are replaced by their r-th derivatives ($r = 1, 2, \ldots, n-1$); and further, if they be replaced by their n-th derivatives, the relation becomes $F =$ constant. If the self-adjoint is Q_{2n+1}, commutative with a quadratic P, and in standard form, the quadratic relations exceed in number those stated by Darboux for the general self-adjoint, and moreover appear in "normal" form. For we have seen that a complete set of solutions of $Q_{2n+1}y = 0$ is constituted by η_r ($r = 1, 2, \ldots, 2n+1$), the solution common to

$$Q_{2n+1}\, y = 0 = (P-e_r)\, y,$$

and that for every such η_r,

$$\eta_r^2 = e_r^n + \phi_1 e_r^{n-1} + \ldots + \phi_n,$$

where $\phi_1, \ldots, \phi_n$ are functions of x independent of r. If therefore Δ_r denote the minor of e_r^{2n} in

$$\Delta \equiv | e_r^{s-1} | \quad (r, s = 1, 2, \ldots, 2n+1),$$

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* *Théorie des Surfaces*, Pt. 2, p. 115 (with a slight change of notation).

we have $n+1$ relations

$$F_{s-1}(\eta_1, \ldots, \eta_{2n+1}) \equiv \sum_{r=1}^{2n+1} e_r^{s-1} \Delta_r \eta_r^2 = 0 \quad (s = 1, 2, \ldots, n),$$

$$F_n(\eta_1, \ldots, \eta_{2n+1}) \equiv \sum_{r=1}^{2n+1} e_r^n \Delta_r \eta_r^2 = \Delta.$$

If we differentiate twice any of these relations (save the last), we have

$$\Sigma e_r^{s-1} \Delta_r (\eta_r'^2 + \eta_r \eta_r'') = 0.$$

Since $$\eta_r'' = (e_r - 2\theta_1) \eta_r,$$

this gives $$\Sigma e_r^{s-1} \Delta_r \eta_r'^2 + \Sigma e_r^s \Delta_r \eta_r^2 = 0.$$

Hence we have $$F_{s-1}(\eta_1', \ldots, \eta_{2n+1}') = 0 \quad (s = 1, 2, \ldots, n-1),$$

$$F_{n-1}(\eta_1', \ldots, \eta_{2n+1}') = -\Delta,$$

i.e. the $n+1$ quadratic relations have reduced to n. We may proceed in this way obtaining $n-r+1$ quadratic relations between the r-th derivatives, and ending finally with the single relation

$$F_0(\eta_1^{(n)}, \ldots, \eta_{2n+1}^{(n)}) = (-)^n \Delta,$$

between the n-th derivatives. Evidently F_0, which persists throughout the series, may be taken as specially representing the quadratic form of Darboux.

ON THE THEORY OF FUNCTIONS OF TWO COMPLEX VARIABLES

By W. H. YOUNG.

[Received March 9th, 1922.—Read March 9th, 1922.]

1. Osgood[*] defines an analytic function of two complex variables

$$z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2,$$

as a function satisfying the following conditions:

(A) For every value of z_1 it is an analytic function of z_2, and for every value of z_2 it is an analytic function of z_1.

(B) It is bounded.

He then proves that an analytic function, so defined, can actually be expanded in an absolutely and uniformly convergent double power series in the domain of definition.

Osgood asks himself whether the condition (B) is not implicitly contained in (A). As far as I am aware, no light has been thrown on this question, except that given by Osgood himself in *Math. Ann.*, Vol. 53.[†]

In the following communication I propose to show that it is sufficient to replace condition (B) by a wider one (B′) given below.

The method employed, that of the theory of functions of several real variables, depends for its success on the notion of integration in the most modern sense. The condition (B′) is then the widest that can be obtained by this method. Also (B′), like (B), when coupled with (A), is necessary and sufficient.

Before obtaining (B′), I first prove Osgood's theorem by the new method, and this for two reasons: first because it is used in the later argument, and in this way the paper is made to a certain extent complete in itself; secondly, because the machinery involved being less intricate than for the more general result, light is thrown on the efficacy of the method, as well as, to use a terminology in vogue, on the depth at which the main theorem lies.

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[*] *Math. Ann.*, Vol. 52.

[†] This paper of Osgood's is interesting, but has not apparently led to further progress.

I may state as follows the principal result of the paper :—

THEOREM.—*If*

(A) *$U(x_1, y_1, x_2, y_2)+iV(x_1, y_1, x_2, y_2)$ is an analytic function of x_1+iy_1 and also of x_2+iy_2, and so U and V satisfy the Riemann-Cauchy equations*

$$\partial U/\partial x_1 = \partial V/\partial y_1, \quad \partial U/\partial x_2 = \partial V/\partial y_2,$$

$$\partial V/\partial x_1 = -\partial U/\partial y_1, \quad \partial V/\partial x_2 = -\partial U/\partial y_2,$$

in the interior of a domain D defined by

$$|x_1+iy_1| < R_1, \quad |x_2+iy_2| < R_2;$$

(B′) *the double integral*

$$\iint |U+iV|\, dz_1 dz_2$$

taken round the boundary of a domain D′, inside D, defined by

$$|x_1+iy_1| < r_1 < R_1, \quad |x_2+iy_2| < r_2 < R_2,$$

exists ;

then $U+iV$ is an analytic function of $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$, in the domain D′, in the sense that it can be expanded in the usual double power series converging doubly, uniformly and absolutely in the whole domain.

2. The simplicity of the result obtained in its final form depends on the use of a theorem of G. H. Hardy, by virtue of which the integral of the modulus of a function of a single complex variable, taken round the circumference of a circle in the region in which it is analytic, is a monotone increasing function of the radius of the circle.

But in the main reasoning, all turns on the use of a type of artifice whose efficacy I have now more than once had occasion to test, and on two properties of absolutely convergent integrals, the term integral being employed in the sense adopted for the first time by Lebesgue.*

The first of these two properties is contained in the theorem, originally formulated by Lebesgue for the case of a single real variable, that if

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* Lebesgue himself always uses the term *summable* to express that an unbounded function possesses an integral in this sense. It is to be hoped that the word *integration* will come to be used, without any special reference to its degree of generality, in the manner I have myself found convenient.

$$\iint |f(x, y)|\, dx\, dy$$

exists, then

$$\iint |f(x+h,\ y+k)-f(x,\ y)|\, dx\, dy$$

tends uniformly to zero as h and k approach zero in any manner.* The second property concerns the differential coefficients of an absolutely convergent double integral, and is contained in the theorem,† that except for a set of values of x of content zero, the same for all values of y,

$$\frac{\partial}{\partial x}\int^{x}\int^{y} f(\xi, \eta)\, d\xi\, d\eta,$$

the integral being absolutely convergent, exists, and is equal to

$$\int^{y} f(x, \eta)\, d\eta.$$

3. The artifice referred to above is the use of quadruple integrals of the type

$$\phi = \frac{1}{h_1 k_1 h_2 k_2}\int_{x_1}^{x_1+h_1}\int_{y_1}^{y_1+k_1}\int_{x_2}^{x_2+h_2}\int_{y_2}^{y_2+k_2} U\, dx_1\, dy_1\, dx_2\, dy_2,$$

$$\psi = \frac{1}{h_1 k_1 h_2 k_2}\int_{x_1}^{x_1+h_1}\int_{y_1}^{y_1+k_1}\int_{x_2}^{x_2+h_2}\int_{y_2}^{y_2+k_2} V\, dx_1\, dy_1\, dx_2\, dy_2.$$

We first assume U and V bounded.‡ In this case it is immediately obvious, by the theory of bounded sequences and their term-by-term integration, that ϕ and ψ satisfy the Riemann-Cauchy equations. Indeed it also follows that the differential coefficients of ϕ and ψ occurring in these equations are continuous functions of the ensemble of all the variables. We are thus enabled to follow Poincaré and his successors and employ Stokes's Theorem for a curved three-dimensional manifold situated in space of four dimensions and bounded by a curved surface, and so prove Cauchy's theorem for $\phi+i\psi$ in Poincaré's form :

$$\iint (\phi+i\psi)\, dz_1\, dz_2 = 0, \tag{1}$$

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* Lebesgue, *Leçons sur les séries trigonométriques*. Though the theorem would at first sight appear to lie somewhat deep, it is in point of fact an intuitive consequence of the definition by monotone sequences of an integral of a function of any number of variables, on which I have so often insisted.

† For references, cf. Hobson, *Theory of Functions of a Real Variable*, 2nd ed., p. 561.

‡ It then follows that they belong to Baire's first class of functions, *i.e.* each of them is an *ul*- and a *lu*-function, and therefore possesses generalised integrals.

extended over $|z_1| = r_1 < R_1, \quad |z_2| = r_2 < R_2.$

But $\phi \to U$, $\psi \to V$ boundedly as $(h_1, k_1, h_2, k_2) \to (0, 0, 0, 0)$. Hence, if in the equation (1) we make the h's and k's approach zero in any order, we have, by the theory of integration of bounded sequences:

$$\iint (U+iV)\,dz_1 dz_2 = 0. \tag{2}$$

The above argument holds for integration over the surface or surfaces forming the complete boundary of any three-dimensional manifold throughout which U and V are bounded. We may therefore apply Cauchy's theorem (2) to the contour of the double ring:

$$|z_1| \leqslant r_1 < R_1, \quad |z_2| \leqslant r_2 < R_2,$$

$$|z_1 - p_1| \geqslant c_1, \quad |z_2 - p_2| \geqslant c_2,$$

with $|p_1| < r_1, \quad |p_2| < r_2, \quad c_1 < |r_1 - p_1|, \quad c_2 < |r_2 - p_2|.$

For our present purpose we now substitute for $U+iV$ the expression

$$\bar{U} + i\bar{V} = (U+iV)/(z_1-p_1)(z_2-p_2),$$

which is legitimate, since evidently $\bar{U}$ and $\bar{V}$ satisfy the Riemann-Cauchy equations and are bounded in the double ring. Thus

$$\iint (U+iV)\,d\zeta_1 d\zeta_2/(\zeta_1-p_1)(\zeta_2-p_2) = \iint (U+iV)\,dz_1 dz_2/(z_1-p_1)(z_2-p_2), \tag{3}$$

round the circular contours

$$|\zeta_1 - p_1| = c_1, \quad |\zeta_2 - p_2| = c_2,$$

on the left of the equation, and round

$$|z_1| = r_1, \quad |z_2| = r_2,$$

on the right.

On the left-hand side we write

$$\zeta_1 = \xi_1 + i\eta_1, \quad \zeta_1 - p_1 = c_1 e^{i\theta_1}, \quad p_1 = a_1 + ib_1,$$

$$\zeta_2 = \xi_2 + i\eta_2, \quad \zeta_2 - p_2 = c_2 e^{i\theta_2}, \quad p_2 = a_2 + ib_2,$$

and we get $$-\int_0^{2\pi}\int_0^{2\pi} (U+iV)\,d\theta_1 d\theta_2.$$

Now integrating in the z_2-plane, we have

$$\int_0^{2\pi} U d\theta_2 = \int_0^{2\pi} \left\{ U_0 + \int_0^{c_2} \frac{\partial U}{\partial r_2} dr_2 \right\} d\theta_2 = \int_0^{2\pi} U_0 d\theta_2 + \int_0^{c_2} dr_2 \,.\, \frac{1}{r_2}\int_0^{2\pi} \frac{\partial V}{\partial \theta_2} d\theta_2$$
$$= 2\pi U_0 = 2\pi U\{\xi_1, \eta_1, a_2, b_2\},$$

since, if we write $$z_2-p_2 = r_2 e^{i\theta_2},$$

we have $$\partial U/\partial r_2 = \partial V/r_2\partial\theta_2,$$

and since $$\int_0^{2\pi} \frac{\partial V}{\partial\theta_2}\cdot d\theta_2 = \Big[V(r, \theta_2)\Big]_{\theta_2=0}^{\theta_2=2\pi} = 0.$$

Repeating the process by integration with respect to θ_1, we get for the real part of the left-hand side of (3)

$$4\pi^2 U(a_1, b_1, a_2, b_2),$$

and similarly for the imaginary part.

On the other hand, by expanding the right-hand side of (3) in ascending powers of z_1, z_2, we get the required analytical expression for $U+iV$, which proves the theorem (Osgood's, slightly modified in language):

THEOREM 1.—*If*

(A) $U(x_1, y_1, x_2, y_2)+iV(x_1, y_1, x_2, y_2)$ *is an analytic function of* x_1+iy, *and also of* x_2+iy_2, *so that* U *and* V *satisfy tke Riemann-Cauchy equations*

$$\partial U/\partial x_1 = \partial V/\partial y_1, \quad \partial U/\partial x_2 = \partial V/\partial y_2,$$
$$\partial V/\partial x_1 = -\partial U/\partial y_1, \quad \partial V/\partial x_2 = -\partial U/\partial y_2,$$

in the interior of a domain D *defined by*

$$|x_1+iy_1| < R_1, \quad |x_2+iy_2| < R_2;$$

(B) U *and* V *are bounded throughout the domain*;

then $U+iV$ *is an analytic function* $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$ *in the domain* D.

4. We here remark that whenever the conditions (A) hold, the Riemann-Cauchy equations may be written in the form

(A′) $$\partial U/\partial r_1 = \partial V/r_1\partial\theta_1, \quad \partial V/\partial r_1 = -\partial U/r_1\partial\theta_1,$$
$$\partial U/\partial r_2 = \partial V/r_2\partial\theta_2, \quad \partial V/\partial r_2 = -\partial U/r_2\partial\theta_2.$$

This follows from the theory of functions of a single complex variable, and has moreover been already implicitly understood in the above.

We shall assume then that U and V are functions of r_1, r_2, θ_1, θ_2, satisfying equations (A′), or changing the variables r_1, r_2 to $\mu_1 = \log(R_1/r_1)$, $\mu_2 = \log(R_2/r_2)$:

$$\partial U/\partial\mu_1 = -\partial V/\partial\theta_1, \quad \partial V/\partial\mu_1 = \partial U/\partial\theta_1,$$

$$\partial U/\partial\mu_2 = -\partial V/\partial\theta_2, \quad \partial V/\partial\mu_2 = \partial U/\partial\theta_2.$$

We shall also assume that U and V are both identically zero when either $r_1 = 0$ or $r_2 = 0$.*

We shall further suppose for the present that all the integrals

$$\int^{\mu_1}\int^{\mu_2} |U|\, dt_1 dt_2, \quad \int^{\mu_1}\int^{\theta_2} |U|\, dt_1 d\tau_2, \quad \int^{\theta_1}\int^{\mu_2} |U|\, d\tau_1 dt_2, \quad \int^{\theta_1}\int^{\theta_2} |U| d\tau_1 d\tau_2,$$

together with the same integrals with V instead of U, are bounded functions of $(\mu_1, \mu_2, \theta_1, \theta_2)$, this assumption taking the place of that of the boundedness of U and V. Both conditions are supposed satisfied within the region

$$|z_1| < R_1, \quad |z_2| < R_2.$$

We proceed to prove that under these circumstances, $U+iV$ is bounded, and hence, by Theorem 1, an analytic function of

$$z_1 = R_1 e^{-\mu_1 + i\theta_1}, \quad z_2 = R_2 e^{-\mu_2 + i\theta_2}.$$

For our purpose, we first prove equation (1) for

$$\phi = \int_{\mu_1}^{\mu_1+h_1}\int_{\mu_2}^{\mu_2+h_2}\int_{\theta_1}^{\theta_1+k_1}\int_{\theta_2}^{\theta_2+k_2} U\, dt_1 dt_2 d\tau_1 d\tau_2,$$

and for the corresponding quadruple integral ψ of V, by showing that ϕ and ψ are bounded and satisfy the Riemann-Cauchy equations.

That ϕ and ψ are bounded is immediately evident. For by the supposed existence and boundedness of

$$\int^{\theta_1}\int^{\theta_2} |U|\, d\tau_1 d\tau_2, \quad \int^{\theta_1}\int^{\theta_2} |V|\, d\tau_1 d\tau_2,$$

ϕ and ψ are absolutely convergent integrals, and accordingly continuous with respect to the ensemble of all the variables, and in particular, bounded.

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* That this involves no loss of generality is evident. We have at most to subtract from $U+iV$ the sum of two appropriate functions, one analytic in t_1 and constant in t_2, the other analytic in t_2 and constant in t_1, that is to say an analytic function of (t_1, t_2).

To prove that they satisfy the Riemann-Cauchy equations, we proceed as follows:

The double integral

$$\int_{\theta_1}^{\theta_1+k_1}\int_{\theta_2}^{\theta_2+k_2}\left[U\right]_{\mu_2}^{\mu_2+h_2} d\tau_1\, d\tau_2$$

is a continuous function of μ_2. For it is equal to

$$\int_{\theta_1}^{\theta_1+k_1}\int_{\theta_2}^{\theta_2+k_2}\int_{\mu_2}^{\mu_2+h_2}\frac{\partial U}{\partial t_2}\, d\tau_1\, d\tau_2 dt_2 = -\int_{\theta_1}^{\theta_1+k_1}\int_{\mu_2}^{\mu_2+h_2}\int_{\theta_2}^{\theta_2+k_2}\frac{\partial V}{\partial \tau_2}\, d\tau_1\, dt_2 d\tau_2$$

(changing the order of integration with respect to θ_2, μ_2, as we may, by the theory of functions of a single complex variable), that is, equal to

$$-\int_{\theta_1}^{\theta_1+k_1}\int_{\mu_2}^{\mu_2+h_2}\left[V\right]_{\theta_2}^{\theta_2+k_2} d\tau_1\, dt_2,$$

which, by the absolute convergence of $\int^{\theta_1}\int^{\mu_2} V\, d\tau_1\, dt_2$, is continuous with respect to μ_2.

The double integral

$$\int_{\theta_1}^{\theta_1+k_1}\int_{\theta_2}^{\theta_2+k_2}\left[U\right]_{\mu_2}^{\mu_2+h_2} d\tau_1\, d\tau_2$$

being thus bounded and continuous with respect to μ_2, so is

$$\int_{\mu_1}^{\mu_1+h_1}\int_{\theta_1}^{\theta_1+k_1}\int_{\theta_2}^{\theta_2+k_2}\left[U\right]_{\mu_2}^{\mu_2+h_2} dt_1\, d\tau_1\, d\tau_2$$

continuous with respect to μ_2, and hence equal to the differential coefficient of ϕ with respect to μ_2. That is to say we have

$$\partial\phi/\partial\mu_2 = \int_{\mu_1}^{\mu_1+h_1}\int_{\theta_1}^{\theta_1+k_1}\int_{\theta_2}^{\theta_2+k_2}\int_{\mu_2}^{\mu_2+h_2}\frac{\partial U}{\partial \mu_2}\, dt_1\, d\tau_1\, d\tau_2 dt_2.$$

Similarly, using the absolute convergence of

$$\int^{\mu_1}\int^{\mu_2} V\, dt_1\, d\tau_2, \quad \int^{\mu_1}\int^{\theta_1} U\, dt_1\, d\tau_2,$$

we see that

$$\partial\psi/\partial\theta_2 = \int_{\mu_1}^{\mu_1+h_1}\int_{\theta_1}^{\theta_1+k_1}\int_{\mu_2}^{\mu_2+h_2}\int_{\theta_2}^{\theta_2+k_2}\frac{\partial V}{\partial \tau_2}\, dt_1\, d\tau_1\, dt_2 d\tau_2.$$

Hence, noting that $\partial U/\partial t_2 = -\partial V/\partial \tau_2$,

and that order of integration with respect to μ_2, θ_2 is indifferent, we get

$$\partial\phi/\partial\mu_2 = -\partial\psi/\partial\theta_2.$$

Similarly, interchanging in the preceding argument the symbols ϕ and ψ, U and V, we should find

$$\partial\psi/\partial\mu_2 = \int_{\mu_1}^{\mu_1+h_1}\int_{\theta_1}^{\theta_1+k_1}\int_{\theta_2}^{\theta_2+k_2}\int_{\mu_2}^{\mu_2+h_2}\frac{\partial V}{\partial t_2}\,dt_1\,d\tau_1\,d\tau_2\,dt_2,$$

$$\partial\phi/\partial\theta_2 = \int_{\mu_1}^{\mu_1+h_1}\int_{\theta_1}^{\theta_1+k_1}\int_{\mu_2}^{\mu_2+h_2}\int_{\theta_2}^{\theta_2+k_2}\frac{\partial U}{\partial\tau_2}\,dt_1\,d\tau_1\,dt_2\,d\tau_2,$$

hence $$\partial\psi/\partial\mu_2 = +\partial\phi/\partial\theta_2.$$

Similarly, interchanging the indices 1 and 2, we prove the same equations for ϕ and ψ with respect to μ_1, θ_1.

Consequently $\phi+i\psi$ is an analytic function of

$$z_1 = R_1 e^{-\mu_1+i\theta_1}, \qquad z_2 = R_2 e^{-\mu_2+i\theta_2},$$

throughout the region $|z_1| < R_1, \quad |z_2| < R_2,$

and we have therefore

$$-4\pi^2\{\phi_0+i\psi_0\} = \iint \frac{\phi+i\psi}{(z_1-p_1)(z_2-p_2)}\,dz_1\,dz_2,$$

$\phi_0+i\psi_0$ standing for the value of $\phi+i\psi$ at the point (p_1, p_2) and the integration on the right-hand side being taken over the circumference of a double circle $$|z_1| \leqslant r_1 < R_1, \quad |z_2| \leqslant r_2 < R_2,$$

containing the point (p_1, p_2) as inner point.

We next remark that in the above equality the right-hand side is now

$$-\int_0^{2\pi}\int_0^{2\pi}\frac{\phi+i\psi}{\left(1-\frac{p_1}{z_1}\right)\left(1-\frac{p_2}{z_2}\right)}\,d\theta_1\,d\theta_2.$$

Also that, as $h_1 \to 0$ and then $h_2 \to 0$ suitably,

$$\phi/h_1h_2 \to \int_{\theta_1}^{\theta_1+k_1}\int_{\theta_2}^{\theta_2+k_2} U\{\mu_1, \mu_2, \tau_1, \tau_2\}\,d\tau_1\,d\tau_2,$$

the approach being bounded in virtue of absolute convergence of

$$\int^{\theta_1}\int^{\theta_2} U\,d\tau_1\,d\tau_2.$$

Similarly for ψ and V.

This gives the result that, as $h_1\to 0$, $h_2\to 0$,

$$(\phi+i\psi)/h_1h_2k_1k_2-[U+iV]\to\int_0^1\int_0^1\{(U+iV)(\mu_1,\mu_2,\theta_1+\kappa\lambda_1,\theta_2+\kappa\lambda_2)$$
$$-(U+iV)(\mu_1,\mu_2,\theta_1,\theta_2)\}\,d\lambda_1\,d\lambda_2.$$

Thus we have, dividing by $(z_1-p_1)(z_2-p_2)$ and integrating with respect to z_1, z_2:

$$\left|\frac{-4\pi^2(\phi_0+i\psi_0)}{h_1h_2k_1k_2}-\iint\frac{U+iV}{(z_1-p_1)(z_2-p_2)}\,dz_1\,dz_2\right|$$
$$\to\left|\iint\frac{dz_1\,dz_2}{(z_1-p_1)(z_2-p_2)}\int_0^1\int_0^1\{(U+iV)(\mu_1,\mu_2,\theta_1+\kappa_1\lambda_1,\theta_2+\kappa_2\lambda_2)\right.$$
$$\left.-(U+iV)(\mu_1,\mu_2,\theta_1,\theta_2)\}\,d\lambda_1\,d\lambda_2\right|$$
$$\leqslant B\left|\int_0^{2\pi}\int_0^{2\pi}d\theta_1\,d\theta_2\int_0^1\int_0^1\{(U+iV)(\mu_1,\mu_2,\theta_1+\kappa_1\lambda_1,\theta_2+\kappa_2\lambda_2)\right.$$
$$\left.-(U+iV)(\mu_1,\mu_2,\theta_1,\theta_2)\}\,d\lambda_1\,d\lambda_2\right|$$
$$\leqslant B\int_0^1\int_0^1 d\lambda_1\,d\lambda_2\int_0^{2\pi}\int_0^{2\pi}d\theta_1\,d\theta_2\,|(U+iV)(\mu_1,\mu_2,\theta_1+\kappa_1\lambda_1,\theta_2+\kappa_2\lambda_2)$$
$$-(U+iV)(\mu_1,\mu_2,\theta_1,\theta_2)|,$$

the reasoning being justified by virtue of the fact that this last integral exists.

But, by the theorem of Lebesgue's in its two-dimensional form given in § 2, the inside double integral tends boundedly to zero as limit when $\kappa_1\to 0$, $\kappa_2\to 0$.

Therefore

$$\operatorname*{Lt}_{h_1\to 0,\,h_2\to 0,\,k_1\to 0,\,k_2\to 0}\left|\frac{4\pi^2(\phi_0+i\psi_0)}{h_1h_2k_1k_2}+\iint\frac{U+iV}{(z_1-p_1)(z_2-p_2)}\,dz_1\,dz_2\right|=0. \quad (4)$$

To prove that Cauchy's equation in the form (1) holds for $U+iV$, we have then only to prove that

$$\operatorname*{Lt}_{h_1\to 0,\,h_2\to 0,\,k_1\to 0,\,k_2\to 0}\frac{\phi_0+i\psi_0}{h_1h_2k_1k_2}=U_0+iV_0,$$

where
$$\phi=\int_{\mu_1}^{\mu_1+h_1}\int_{\mu_2}^{\mu_2+h_2}\int_{\theta_1}^{\theta_1+k_1}\int_{\theta_2}^{\theta_2+k_2}U\,dt_1\,dt_2\,d\tau_1\,d\tau_2,$$

and similarly ψ is the same quadruple integral of V, and where

$$R_1e^{-\mu_1+i\theta_1}=p_1,\qquad R_2e^{-\mu_2+i\theta_2}=p_2$$

is a point inside the double circle

$$|z_1|=r_1<R_1,\qquad |z_2|=r_2<R_2.$$

The limit to consider is therefore the mixed quadruple differential coefficient at the point (p_1, p_2) of the function $\phi' + i\psi'$, where

$$\phi' = \int^{\mu_1}\int^{\mu_2}\int^{\theta_1}\int^{\theta_2} U\,dt_1\,dt_2\,d\tau_1\,d\tau_2,$$

and ψ' is the same quadruple integral of V.

It is first evident that if we differentiate ϕ' with respect to μ_1 and then with respect to μ_2, we get the inside double integral. For this double integral

$$S = \iint U\,d\tau_1\,d\tau_2$$

is continuous with respect to μ_1 and with respect to μ_2, as it is equal to

$$-\int^{\theta_1}\int^{\mu_2} V\,d\tau_1\,dt_2 + \int^{\theta_1}\int^{\mu_2}[V]_{\theta_2=a_2}\,d\tau_1\,dt_2 + \int^{\theta_1}\int^{\theta_2}[U]_{\mu_2=a_2}\,d\tau_1\,d\tau_2,$$

and also to

$$-\int^{\theta_2}\int^{\mu_1} V\,d\tau_2\,dt_1 + \int^{\theta_2}\int^{\mu_1}[V]_{\theta_1=a_1}\,d\tau_2\,dt_1 + \int^{\theta_2}\int^{\theta_1}[U]_{\mu_1=a_1}\,d\tau_2\,d\tau_1.$$

This follows by double integration of

$$\partial U/\partial\mu_2 = -\partial V/\partial\theta_2, \qquad \partial U/\partial\mu_1 = -\partial V/\partial\theta_1,$$

between the limits a_2, θ_2 and a_2, μ_2; a_1, θ_1 and a_1, μ_1, respectively, noting that change of order of integration with respect to variables of same index is allowed. By the absolute convergence of the double integrals of V in the above expressions, the required continuity of S follows.

It also follows therefore that S is the differential coefficient with respect to μ_2 of the triple integral

$$\int^{\mu_2}\int^{\theta_1}\int^{\theta_2} U\,dt_2\,d\tau_1\,d\tau_2,$$

and that this triple integral is a continuous function of μ_1, and is hence equal to the differential coefficient of ϕ' with respect to μ_1; so that we have as stated

$$\frac{\partial^2\phi'}{\partial\mu_1\,\partial\mu_2} = S = \int^{\theta_1}\int^{\theta_2} U\,d\tau_1\,d\tau_2.$$

It remains to show that $\partial^2 S/\partial\theta_1\partial\theta_2$ is equal to U.

Now we have as above

$$S = -\int^{\theta_2}\int^{\mu_1} V\,d\tau_2\,dt_1 + \int^{\theta_2}\int^{\mu_1}[V]_{\theta_1=a_1}\,d\tau_2\,dt_1 + \int^{\theta_2}\int^{\theta_1}[U]_{\mu_1=a_1}\,d\tau_2\,d\tau_1,$$

hence by the second theorem referred to in § 2, holding good for absolutely convergent double integrals,

$$\frac{\partial S}{\partial \theta_2} = -\int^{\mu_1} V dt_1 + \int^{\mu_1} [V]_{\theta_1=a_1} dt_1 + \int^{\theta_1} [U]_{\mu_1=a_1} d\tau_1,$$

except for a set of values of θ_2 of zero content the same for all values of μ_1; in fact, the first term is correct except for a set of values of θ_2 the same for all values of μ_1, the second except for a set of the same type, and the third, being itself independent of μ_1, gives a set of exceptional values of θ_2 of zero content equally independent of μ_1.*

But the right-hand side is equal to $\int^{\theta_1} U d\tau_1$, which, on the other hand, we know by the same theorem to be equal to $\partial S/\partial\theta_2$ except for a set of values of θ_2 of zero content the same for all values of θ_1.

The two sets of exceptional values of θ_2 must be identical, that is to say we have

$$\frac{\partial S}{\partial \theta_2} = \int U d\tau_1$$

except for a set of values of θ_2 of zero content the same for all values of θ_1 and for all values of μ_1.

Differentiating with respect to θ_2 we get, since U is continuous with respect to θ_1,

$$\partial^2 S/\partial\theta_1\partial\theta_2 = U,$$

except for the set of values of θ_2 above mentioned, the same for all values of θ_1 and for all values of μ_1.

By interchanging the indices 1 and 2 in the preceding argument, we also see that

$$\partial^2 S/\partial\theta_2\partial\theta_1 = U$$

except for a set of values of θ_1 of zero content the same for all values of θ_2 and for all values of μ_2.

The same reasoning holds if we interchange U and V, ϕ' and ψ'. We therefore see, returning to ϕ and ψ, that

$$\operatorname*{Lt}_{h_1\to 0,\ h_2\to 0,\ k_1\to 0,\ k_2\to 0} (\phi+i\psi)/h_1 h_2 k_1 k_2$$

is equal to $U+iV$ except for a set of values of θ_2 of zero content the same for all values of θ_1 and for all values of μ_1, while the same limit, interchanging the order with respect to k_1, k_2, is equal to $U+iV$ except for a set of values of θ_1 the same for all values of θ_2 and for all values of μ_2.

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* This third partial set might depend on the particular choice of the constant a_1. This clearly does not affect the argument.

Now the order of proceeding to the limit with respect to k_1, k_2 must be indifferent, as (4) is independent of the order in which k_1, k_2 tend to zero. It follows that the two sets of exceptional points must coincide, which they can only do if they disappear.

In other words, we have, by (4), as desired, for the point (p_1, p_2):

$$\operatorname*{Lt}_{h_1\to 0,\ h_2\to 0,\ k_1\to 0,\ k_2\to 0} (\phi_0+i\psi_0)/h_1 h_2 k_1 k_2 = U_0+iV_0.$$

As a result of proceeding to the fourfold limit, we have therefore

$$-4\pi^2(U_0+iV_0) = \iint \frac{U+iV}{(z_1-p_1)(z_2-p_2)}\,dz_1\,dz_2.$$

But by hypothesis the double integral

$$\iint |U+iV|\,dz_1 dz_2$$

taken over the double circle

$$|z_1| = r_1 < R_1, \qquad |z_2| = r_2 < R_2$$

is bounded and less than K_1. Therefore the corresponding integral

$$\iint \frac{|(U+iV)|}{|(z_1-p_1)(z_2-p_2)|}\,dz_1 dz_2 < \frac{K_1}{(r_1-|p_1|)(r_2-|p_2|)},$$

or, by the above equality,

$$|U_0+iV_0| < \frac{K_2}{(r_1-|p_1|)(r_2-|p_2|)},$$

where K_2 is independent of the position of the point (p_1, p_2) inside the circle $|z_1| = r_1$, $|z_2| = r_2$.

If, then, we vary the point (p_1, p_2), we see that $|U+iV|$ is bounded throughout a region

$$|z_1| \leqslant \rho_1 < r_1 < R_1, \qquad |z_2| \leqslant \rho_2 < r_2 < R_2,$$

and is consequently, by our Theorem I, an analytic function of the pair of complex variables (z_1, z_2), throughout this region.

We may remark that in the above the double circle (r_1, r_2) may be taken as near as we please to the bounding circles (R_1, R_2), and the double circle (ρ_1, ρ_2) as near as we please to (r_1, r_2). *With our conditions, $U+iV$ is therefore analytic with respect to (z_1, z_2) in any open region which lies, together with its boundary, completely inside the fundamental region* $|z_1| < R_1$, $|z_2| < R_2$.

5. We proceed to show that our conditions may be replaced by the following:

(B′) *The double integral*

$$\int_0^{2\pi}\int_0^{2\pi} |U+iV|\, d\theta_1 d\theta_2,$$

taken round a double circle

$$|z_1| = r_1 < R_1, \quad |z_2| = r_2 < R_2,$$

exists.

It is easily seen that under the condition (B′) our conditions of the last section are fulfilled in the region $|z_1| < r_1$, $|z_2| < r_2$.

In fact (1) we know that if $\rho_2 < r_2$,

$$\int_0^{2\pi} |(U+iV)(r_1, r_2, \theta_1, \theta_2)|\, d\theta_2 \geqslant \int_0^{2\pi} |(U+iV)(r_1, \rho_2, \theta_1, \theta_2)|\, d\theta_2,$$

by the theory of functions of a single complex variable, and therefore if $\iint |U+iV|\, d\theta_1 d\theta_2$ exists, *a fortiori* so does

$$\int_0^{2\pi}\int_0^{2\pi} |(U+iV)(r_1, \rho_2, \theta_1, \theta_2)|\, d\theta_2 d\theta_1.$$

We may therefore change the order of integration in the latter integral, and as for $\rho_1 < r_1$,

$$\int_0^{2\pi} |(U+iV)(r_1, \rho_2, \theta_1, \theta_2)|\, d\theta_1 \geqslant \int_0^{2\pi} |(U+iV)(\rho_1, \rho_2, \theta_1, \theta_2)|\, d\theta_1,$$

we similarly see that the double integral

$$\int_0^{2\pi}\int_0^{2\pi} |(U+iV)(\rho_1, \rho_2, \theta_1, \theta_2)|\, d\theta_1 d\theta_2,$$

taken along any double circle

$$|z_1| = \rho_1 \leqslant r_1, \quad |z_2| = \rho_2 \leqslant r_2$$

exists, and *a fortiori* the absolute convergence of

$$\int^{\theta_1}\int^{\theta_2} |U+iV|\, d\theta_1 d\theta_2$$

taken along any arc of such a circle, is established. This is the necessary and sufficient condition for the absolute convergence of the corresponding integrals of U and V separately.

Further (2) we easily verify that, speaking generally, if $P(r, \theta)$ and

$iQ(r, \theta)$ are the real and imaginary parts of an analytic function of a single complex variable $z = re^{i\theta}$, and if we suppose the latter function to be zero at the origin, then we have, for $\rho < r$,

$$\int_0^\rho |P(t, \theta)|\, dt/t \leqslant K \int_0^{2\pi} |P(r, \theta)|\, d\theta,$$

and the same for Q, the integral on the right being taken over the circle of radius r, and K being a constant depending on ρ.

For expanding P and Q in the form

$$P = \sum_1^\infty (a_n r^n \cos n\theta + b_n r^n \sin n\theta),$$

$$Q = \sum_1^\infty (a_n r^n \sin n\theta - b_n r^n \cos n\theta),$$

we see that we have

$$\int_0^\rho |P(t, \theta)|\, dt/t \leqslant \int_0^\rho \sum_1^\infty \{|a_n| + |b_n|\}\, t^{n-1}\, dt \leqslant \sum_1^\infty \rho^n \{|a_n| + |b_n|\}.$$

Hence as $|a_n| = \left| \frac{1}{\pi r^n} \int_0^{2\pi} [P(r, \theta)] \cos n\theta\, d\theta \right| \leqslant \frac{1}{\pi r^n} \int_0^{2\pi} |P(r, \theta)|\, d\theta,$

$$|b_n| \leqslant \frac{1}{\pi r^n} \int_0^{2\pi} |P(r, \theta)|\, d\theta,$$

we get by substituting in the preceding equality

$$\int_0^\rho |P(t, \theta)|\, dt/t \leqslant \frac{2}{\pi} \int_0^{2\pi} |P(r, \theta)| \sum_1^\infty (\rho/r)^n\, d\theta = \frac{2r}{\pi(r-\rho)} \int_0^{2\pi} |P(r, \theta)|\, d\theta \qquad (\rho < r),$$

which verifies our inequality.

Applying this, it is evident that if we suppose that $U + iV$ is identically zero when $r_1 = 0$ and when $r_2 = 0$, as is legitimate by a preceding remark, the existence of the integral

$$\int_0^{2\pi} d\theta_2 \int_0^{2\pi} |U(r_1, r_2, \theta_1, \theta_2)|\, d\theta_1,$$

implies that of $$\int_0^{2\pi} d\theta_2 \int_0^{\rho_1} |U(t_1, r_2, \theta_1, \theta_2)|\, \frac{dt_1}{t_1},$$

and of $$\int d\theta_1 \int |U(r_1, t_2, \theta_1, \theta_2)|\, dt_2 \quad (\rho_1 < r_1,\ \rho_2 < r_2),$$

and hence also of $$\int_0^{\rho_1} dt_1 \int_0^{\rho_2} |U(t_1, t_2, \theta_1, \theta_2)|\, dt_2,$$

a fortiori the three double integrals

$$\int^{\rho_1}\int^{\rho_2} |\,U\,| \frac{dt_1}{t_1}\frac{dt_2}{t_2}, \qquad \int^{\rho_1}\int^{\theta_2} |\,U\,| \frac{dt_1}{t_1}\,d\tau_2, \qquad \int^{\theta_1}\int^{\rho_2} |\,U\,|\,d\tau_1\frac{dt_2}{t_2}$$

are bounded in any region

$$|z_1| \leqslant \rho_1 < r_1, \qquad |z_2| \leqslant \rho_2 < r_2.$$

That is to say, with a slight change of notation, the conditions of § 4 hold in this region.

Now inspection shows that for the final statement of § 4, we only use these conditions for the closed region

$$|z_1| \leqslant r_1, \qquad |z_2| \leqslant r_2.$$

If then we replace these conditions by the condition (B′) our statement becomes:

Under the conditions (B′), $U+iV$ *is analytic with respect to* z_1, z_2 *in any open region which lies, together with its boundary, completely inside the double circle* (r_1, r_2), *along which the integration referred to takes place.*

It is immediately evident that if this double circle may be chosen to lie as near as we please to the bounding circles (R_1, R_2) of the region in which U and V satisfy the Riemann-Cauchy equations, $U+iV$ is analytic with respect to (z_1, z_2) in the whole open region

$$|z_1| < R_1, \qquad |z_2| < R_2.$$

In particular, if a function of two complex variables satisfies the Riemann-Cauchy equations in the whole double complex plane and possesses an absolutely convergent integral round any double circle however large, it is an integral function of the pair of variables, in the sense that it is analytic with respect to the pair of variables in the whole double complex plane.

One further remark should be made, namely, that the corresponding result to that we have obtained appears to hold for any number of variables, and to be provable in the same manner.

ON THE CONDITIONS FOR CAUCHY'S THEOREM

By S. POLLARD.

[Read February 10th, 1921.—Revised January 19th, 1923.]

Introduction.

1. Let C be any closed plane curve with no multiple points. By a standard theorem, C encloses a connected domain Δ which is its interior. Let further z be a complex variable, and $f(z)$ any function of it which exists and is single valued at every point of C and Δ. We do *not* assume that $f(z)$ is analytic: if x and y are respectively the real and imaginary parts of z, then $f(z)$ is to be considered as having the quite general form $P(x, y)+iQ(x, y)$, where $P(x, y)$ and $Q(x, y)$ are any single-valued real functions of x and y whatever.

Cauchy's well-known theorem states that, under certain conditions, the integral of $f(z)$ round the contour C is zero; the conditions being approximately:

(i) $f'(z)$ *exists and is finite at all points of the interior* Δ.

(ii) $f(z)$ *is continuous (in* z) *on* C *so far as values at points within and on it are concerned.**

(iii) C *is a curve of bounded variation.*

Our object is to make these conditions precise, and to extend them wherever it is possible. To this end we proceed to state the various points of interest, which we shall review in order.

Before doing so, however, let us make one or two observations. The theorem we are considering is, like Green's theorem, one of those which connect properties of a boundary with those of the region to which it belongs. And the appropriate conditions may be classified under the following heads:—

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* It may be completely irregular as regards the values at points outside C.

(*a*) Conditions affecting the region itself.

(*b*) Conditions ensuring a proper connexion between the region and its boundary.

(*c*) A special condition to be satisfied by the boundary.

(*a*) it is quite obvious we must have, for the somewhat absurdly patent reason that otherwise the theorem could not concern the region at all. (*b*) is necessary to ensure the possibility of the inference from the properties of the region to those required for the boundary. But the reason for (*c*) is not quite so apparent. We require it because in theorems of this kind it is not possible to obtain the most general results by an argument directly concerned with the boundary; this, in the general case, being just too complicated for the success of the available methods. The argument has first to be applied to the special case of a polygon; and then extended to the general case by means of a passage to the limit, the boundary in the latter being regarded as the limit of a sequence of approximative polygons. Passages to the limit not usually being valid without the introduction of appropriate conditions, there will in all probability be a condition required here. This is (*c*).

Take now the conditions of type (*a*), *i.e.* those concerned with the interior Δ. They are somewhat remarkable in that the sufficient ones are considerably wider than those which are necessary.[*] For the sufficient conditions known at present enable Cauchy's theorem to be proved, not only for the contour C, but for every contour of a certain type contained inside it. And, this being so, it can be shown[†] that $f(z)$ is necessarily expansible as a power series in the immediate neighbourhood of every point of the interior Δ, a fact which is considerably more than is required by any of the conditions initially assumed.

Thus conditions of type (*a*) can be considered in one or other of two ways:—

(*α*) As to whether they are *complete*, *i.e.* both necessary and sufficient, and, incidentally, inclusive of all other conditions.

(*β*) As to how far they are the most general conditions directly giving rise to the theorem, no regard being paid to indirect results which may be obtained by means of the latter itself.

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[*] Usually the reverse is the case.

[†] By Morera's theorem, No. **1**, p. 87.

Of complete conditions there is really only one, namely, that $f(z)$ is regular at every point of Δ. If the theorem is to hold, this must be true; and if this is true, then the theorem is demonstrated with ease.* With this point we shall not concern ourselves further.

Of conditions of the other kind there are several: Goursat's, to the effect that $f'(z)$ exists at every point of Δ; Montel's, to the effect that $P(x, y)$ and $Q(x, y)$ have first order partial derivatives which are bounded and satisfy Cauchy's equations

$$P_x(x, y) = Q_y(x, y), \quad Q_x(x, y) = -P_y(x, y); \tag{1}$$

and some which we are about to develop.

In discussing these conditions the main points of interest are the method by means of which the theorem is obtained from them, and the extent to which they are "pseudo-distinct."†

The methods in existence are two: one due to Goursat and employing the principle of repeated subdivision (the main subsidiary result of this method is Goursat's famous Lemma); and one initiated by Riemann and based on the two-dimensional form of Green's theorem. These two methods are apparently quite separate‡ and the interest lies in seeing which method gives rise to the most general set of conditions.

The distinctness of the various conditions is obviously a matter for consideration. It needs no little care, and also, we may say, is a matter for more discussion than we give, the means at present available for dealing with it not being completely satisfactory. The discussion given here is based on the idea of differentiability; and it is shown that Goursat's conditions, if they hold at a single point only, require $P(x, y)$ and $Q(x, y)$ to be differentiable there, while this is not so for the type of conditions considered by the writer.

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* This seems to be the reason why Weierstrass's theory of functions is in no way inferior to that which follows from Cauchy's theorem, although, since it starts with the full assumption that $f(z)$ is expansible in the neighbourhood of every point as a power series, it seems to be a good deal less general. The fact is simply that Weierstrass's assumption is, in virtue of the result described in § 7, implicitly contained in that which gives rise to Cauchy's theorem.

† Conditions are said to be "pseudo-distinct" when they cannot be reconciled apart from Cauchy's theorem itself.

‡ It does not seem possible for Green's theorem in the plane to be proved in anything like a general form by the method of repeated subdivision.

We may say that the second type of conditions, when regarded as affecting only a single point, is undoubtedly distinct from the first. It is only when the conditions are taken as affecting a connected domain that the second type becomes obtainable from the first. Though even then this is only so (as far as can be seen at present) in virtue of Cauchy's theorem itself.*

We pass now to conditions (b). At the outset we may say that we postulate continuity in order to ensure that, if Π is a variable polygon tending to the curve C, then

$$\int_{\Pi} f(z)\,dz \to \int_{C} f(z)\,dz.$$

This is most easily established when $f(z)$ is continuous on C.

It may (in certain cases it certainly will) hold when $f(z)$ is not continuous on C; but the method of proof is not likely to be altered in any essential. The result, when it is obtained in any discontinuous case, will be obtained by showing that the discontinuities present, though they may complicate, nevertheless do not invalidate the argument used in the continuous case.

We therefore, for the present, confine ourselves to the continuous case —regarding it as typical.

The prime difficulty is, as one might expect, to obtain the passage to the limit indicated above—or the slightly more general one

$$\int_{C'} f(z)\,dz \to \int_{C} f(z)\,dz,$$

where C' is some contour of simple type (not necessarily polygonal) contained within C for which Cauchy's theorem has already been proved.

It looks easy enough; but, in reality, it is one of the most difficult things there are in our theory. It is one of those results which are discovered by geometrical intuition; and, as anyone acquainted with Analysis

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* The initial step towards establishing the dependence of the two sets of conditions (without Cauchy's theorem) would be to show that Cauchy's equations (1), when holding at every point of a connected domain, imply that $P(x, y)$ and $Q(x, y)$ are differentiable in that domain.

Another step in a similar direction would be to show that $f'(z)$ cannot exist at every point of a connected domain without being continuous in that domain.

To the best of the writer's knowledge, no one has yet come near showing either of the above.

Situs will readily admit, results of this kind have a way of defying analysis. Instead of being an obvious detail to be left for the reader to work out for himself, it is an intricacy the clue to which is by no means easy to find. As far as we can see, the only way of obtaining a correct proof of the passage to the limit is by means of some rather delicate theorems of de la Vallée Poussin, the development of which requires great care and unusual nicety of thought.

After all, there must be a difficulty somewhere. Cauchy's theorem is an amazing theorem, with many consequences,* and we cannot obtain it from nothing.

There are, apparently, only two writers who make any serious attempt at a proof: Goursat,† and Watson.‡ Goursat's proof, as it stands, when the stringent requirements of Analysis Situs are taken into account, is obviously insufficient; and Watson's proof introduces a limitation which we shall show is not required.

We give what we believe to be a correct proof in § 8.

It may be observed in passing that Goursat's method of proof (of the main theorem) avoids, to a certain extent, the somewhat awkward passage to the limit under discussion. The proof can be applied directly to the original contour provided that this is not cut by transversals parallel to one or other of the axes in more than a finite number of points. This constitutes a great point in favour of Goursat's proof.

It requires, however, several extra conditions. One is that just mentioned: it is shown (§ 10) to be somewhat narrow in that contours of bounded variation§ need not satisfy it. The other is to the effect that $f'(z)$ exists on C as well as in its interior.

A brief discussion of Goursat's proof (of the main theorem) is given in § 12.

Watson's proof (of the validity of the passage to the limit), we should say, does not purport to be completely general. His aim is to demonstrate a possibility—to show that it is really possible to give an account of Cauchy's theorem which leaves nothing to geometrical intuition—rather than to conduct an exhaustive investigation.

For simplicity, his account, as far as Analysis Situs is concerned, is based on the work of Ames. Now Ames's proof of the fundamental theorem of closed curves makes use of a certain assumption, to the effect that the curve considered can be divided into a finite number of arcs each of which has an equation of one or other of the forms

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* Practically the whole of the vast structure of the theory of Functions of a Complex Variable, not to mention a good deal of the modern theory of numbers, rests upon Cauchy's theorem.

† No. **1**.

‡ No. **6**.

§ See a few paragraphs further on.

$$y = \phi(x), \quad x = \psi(y);$$

an assumption which we shall describe as taking the curve to have the "Ames property." This assumption is admitted by Watson throughout.

The question naturally arises as to whether the assumption made by Ames is really necessary after all, and whether it is not introducing an undue restriction on the conditions for Cauchy's theorem.

There are two points to be considered :—

(i) is Ames's assumption really required for the truth of the fundamental theorem of closed curves ?

(ii) if it is not, is it a consequence of any of our conditions, so that, although we can prove the Analysis Situs theorems without it, yet nothing is gained thereby when we come to consider Cauchy's theorem ?

On the first point we may answer in the negative at once. Correct proofs of the theorem on closed curves in its most general form have been given by Jordan,[*] Brouwer,[†] and others. The second point is not so simple.

It is fairly certain that a restriction which is essential is that the curve should be of bounded variation. This must be so, as we show later, if the integral round the contour is to exist at all generally. And it might be imagined, this restriction being quite a severe one, that it automatically included the assumption required by Ames. This we show, in § 11, not to be the case. A curve is there given which is of bounded variation, but does not possess the Ames property.

The interest now centres, naturally, in finding a contour C' within C, such that

$$(a) \quad \int_{C'} f(z)\,dz = 0,$$

$$(b) \quad \int_{C'} f(z)\,dz \rightarrow \int_{C} f(z)\,dz,$$

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[*] *Cours d'Analyse*, t. 1, 3rd ed. (1909), pp. 91–98.

[†] "Beweis des Jordanschen Kurvensatzes", *Math. Annalen*, Vol. 69 (1910), pp. 169–175. Brouwer's is a proof of wonderful elegance and simplicity.

on the sole assumptions that C is of bounded variation and $f(z)$ continuous on it, values at points outside C being excluded.

The complete details of this will be found in § 8. It may be not out of place, however, to give here a few indications of the lines along which we proceed, especially as the matter is a little involved.

To simplify matters, let us say at once that the continuity of $f(z)$ ensures that (b) holds provided only that:

(α) $C' \to C$.

(β) C' is of uniformly bounded variation, *i.e.* as C' tends to C there is a fixed number G which the sum of its x and y variations (and, incidentally, its length) does not exceed.

C' is most simply obtained as a polygon Π composed of parts of the perimeters of squares enclosing C. If the squares are small enough, and every one contains a point of C, then the resulting polygon will lie as near to C as we please, and (α) is obtained.

For (β) we make use of the fact that, if O is the middle point of an arc PQ of length s, then a square with its centre at O and its side greater than s will completely enclose PQ. Since the side of the square may be as near s as we please, and therefore its perimeter as near $4s$, the whole contour C can evidently be enclosed in squares the total length of whose perimeters is less than k times the length of the contour, k being any constant greater than 4. (β) follows at once.

It remains to discuss the conditions of type (c), *i.e.* the special conditions affecting the boundary. The first factor to be taken into account in determining them is evidently that they must be such as to render the integral of $f(z)$ existent. This factor, as it happens, proves sufficient of itself to settle the matter.

We have already assumed that $f(z)$ is continuous on the contour. Thus the question resolves itself into the determination of what conditions must be satisfied by the contour in order that the integral upon it of every continuous $f(z)$ shall exist.

For this we make use of a result obtained by the writer in the course of an investigation of the Stieltjes integral. It runs as follows:

The necessary and sufficient condition that the Stieltjes integral $\int \theta(t)\, d\psi(t)$ *shall exist for all continuous integrands* $\theta(t)$ *is that* $\psi(t)$ *be of bounded variation.**

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* "The Stieltjes Integral and its Generalisations", *Quarterly Journal of Pure and Applied Mathematics*, Vol. 49 (1920), p. 83.

Now, if $\theta(t) = F\{\psi(t), \chi(t)\}$, the Stieltjes integral given above and the curvilinear integral $\int F(x, y)\,dx$ taken along

$$x = \psi(t), \quad y = \chi(t),$$

are identical.

It follows at once, by steps which the reader can supply for himself, that, if every continuous function is to have an integral on the contour, then the latter must be of bounded variation. This is the condition required.*

Analysis of the usual conditions affecting the integrand.

2. The proof of Cauchy's theorem usually recognised as standard is that due to Goursat. It rests on the hypothesis of the existence of $f'(z)$; one that we shall refer to as hypothesis (i).

Leaving aside all questions as to the extent of the region over which $f'(z)$ exists, and omitting all details as to the nature of the contour, etc., let us examine this hypothesis (i) with a view to ascertaining its exact significance.

To begin with, by the definition of a differential coefficient, we have

$$f(z') - f(z) = f'(z)(z'-z) + \epsilon\,|z'-z|, \tag{2}$$

ϵ being a function of z' and z with the limit zero as z' tends to z.

Write $f(z)$, $f'(z)$ in full as complex functions of x and y:

$$f(z) = P(x, y) + iQ(x, y), \quad f'(z) = P_1(x, y) + iQ_1(x, y).$$

Then, from (2),

$$P(x', y') + iQ(x', y') - P(x, y) - iQ(x, y)$$
$$= \{P_1(x, y) + iQ_1(x, y)\}\,\{x'-x+i(y'-y)\} + \epsilon\sqrt{\{(x'-x)^2 + (y'-y)^2\}}.$$

Equating real and imaginary parts, we get

$$P(x', y') - P(x, y) = P_1(x, y)(x'-x) - Q_1(x, y)(y'-y) + \epsilon_1\rho, \tag{3}$$

$$Q(x', y') - Q(x, y) = Q_1(x, y)(x'-x) + P_1(x, y)(y'-y) + \epsilon_2\rho, \tag{4}$$

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* It must be understood quite clearly that there is no *absolute* condition which the contour must satisfy: the conditions we seek are always those appropriate to the particular restriction made upon $f(z)$. If we care to impose additional restrictions upon $f(z)$ (*e.g.* that it be of bounded variation) then we can relax the conditions to be satisfied by the contour.

where ϵ_1, ϵ_2 are the real and imaginary parts of ϵ, and

$$\rho = \sqrt{\{(x'-x)^2+(y'-y)^2\}} = |z'-z|.$$

Since ϵ_1 and ϵ_2 tend to zero with ϵ, and therefore with ρ, (3) and (4) show that $P(x, y)$ and $Q(x, y)$ are differentiable functions of the two variables x, y.

(i) *thus involves, besides the well known equalities*

$$P_x(x, y) = \quad Q_y(x, y) \quad [= P_1(x, y)],$$

$$Q_x(x, y) = -P_y(x, y) \quad [= Q_1(x, y)],$$

the fact that $P(x, y)$ *and* $Q(x, y)$ *are differentiable functions of* x, y.

Some considerations on differentiability.

3. The function $\phi(x, y)$ is differentiable at (x, y) if

$$\phi(x+h, y+k)-\phi(x, y) = A.h+B.k+\epsilon\rho,$$

where A and B are independent of h and k, ρ stands for $\sqrt{(h^2+k^2)}$, and ϵ is a function of h and k which tends to zero with ρ.

It is necessary for this, but not sufficient, that the derivatives $\phi_x(x, y)$, $\phi_y(x, y)$ exist. For, putting $k = 0$, we get

$$\phi(x+h, y)-\phi(x, y) = A.h+\epsilon h,$$

i.e.

$$\frac{\phi(x+h, y)-\phi(x, y)}{h} = A+\epsilon$$

$$\rightarrow A,$$

as $h \rightarrow 0$. Thus

$$\phi_x(x, y) = A.$$

Similarly

$$\phi_y(x, y) = B.$$

That the mere existence of $\phi_x(x, y)$ and $\phi_y(x, y)$ is not sufficient is shown by the following example, which we quote from Pierpont,[*]

$$\phi(x, y) = \frac{xy}{\sqrt{(x^2+y^2)}} \quad (x \neq 0, \text{ or } y \neq 0)$$

$$= 0 \qquad (x = y = 0).$$

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[*] *Theory of Functions of Real Variables*, Vol. 1, p. 270.

Evidently $$\phi_x(0, 0) = \phi_y(0, 0) = 0.$$

But $\phi(x, y)$ is not differentiable at the origin; for if it were we should have

$$\frac{hk}{\sqrt{(h^2+k^2)}} = 0.h+0.k+\epsilon\rho,$$

i.e. putting $$h = \rho\cos\theta, \quad k = \rho\sin\theta,$$

$$\frac{\cos\theta\sin\theta}{\rho} = \epsilon\rho,$$

or $$\cos\theta\sin\theta \to 0,$$

as $\rho \to 0$, for all values of θ; which is not so.

It is sufficient for the differentiability of $\phi(x, y)$, but not necessary, that $\phi_x(x, y)$ and $\phi_y(x, y)$, in addition to existing, be continuous. This, as far as sufficiency is concerned, is a well known deduction from the Mean Value theorem, namely, that

$$\phi(x+h, y+k)-\phi(x, y) = h\phi_x(x+\theta_1 h, y+\theta_2 k)+k\phi_y(x+\theta_1 h, y+\theta_2 k),$$

where θ_1 and θ_2 lie between 0 and 1.

For the second part, that it is not necessary for $\phi_x(x, y)$ and $\phi_y(x, y)$ to be continuous, we employ an actual example. Take

$$\phi(x, y) = \psi(x)+\psi(y),$$

where $$\psi(x) = x^2\sin\frac{1}{x} \quad (x \neq 0)$$

$$= 0 \qquad x = 0.$$

Then $$\phi(h, k)-\phi(0, 0) = h^2\sin\frac{1}{h}+k^2\sin\frac{1}{k}$$

$$= 0.h+0.k+\epsilon\rho,$$

where $$\epsilon = \frac{1}{\rho}\left\{h^2\sin\frac{1}{h}+k^2\sin\frac{1}{k}\right\}$$

$$= \frac{h}{\rho}h\sin\frac{1}{h}+\frac{k}{\rho}k\sin\frac{1}{k}$$

$$\to 0,$$

as $\rho \to 0$. Thus $\phi(x, y)$ is differentiable at the origin.

But
$$\phi_x(x, y) = 2x \sin\frac{1}{x} - \cos\frac{1}{x} \quad (x \neq 0)$$
$$= 0 \quad (x = 0),$$

and thus is not continuous at the origin. Neither is $\phi_y(x, y)$, which is given by the same expression, but with y instead of x.

Thus the differentiability of $\phi(x, y)$ forms a sort of mean between the existence merely, and the existence and continuity of $\phi_x(x, y)$ and $\phi_y(x, y)$. It happens to be just the idea for the investigation of the conditions under which a function $f(z)$ of the complex variable z has a unique derivative $f'(z)$. Without the idea of differentiability, we can only state, as is usually stated (or, if not explicitly stated, implied) in text-books, that, for $f'(z)$ to exist, it is necessary that

$$P_x(x, y), \quad P_y(x, y), \quad Q_x(x, y), \quad Q_y(x, y)$$

exist and satisfy Cauchy's equations; and it is sufficient that they exist, and are continuous, and satisfy Cauchy's equations. Conditions which are both necessary and sufficient cannot be given.

The necessary and sufficient condition for the existence of $f'(z)$.

4. *It is necessary and sufficient for the existence of $f'(z)$ that $P(x, y)$ and $Q(x, y)$ be differentiable and such that their first order derivatives satisfy Cauchy's equations* (1).

We have already shown (§ 2) that it is necessary. The reversal of the analysis there given will be found to show at once that it is sufficient.

Cauchy's equations by themselves are not sufficient.

5. We shall show, by an example, that the first order derivatives of $P(x, y)$ and $Q(x, y)$ may exist at a point and satisfy Cauchy's equations, without entailing the existence of $f'(z)$. Take

$$P(x, y) = \frac{x^2+y^2}{x-y} \quad (x \neq y)$$
$$= 0 \quad (x = y),$$
$$Q(x, y) = \frac{x^2+y^2}{x+y} \quad (x \neq -y)$$
$$= 0 \quad (x = -y).$$

Then

$$\left.\begin{array}{ll} P_x(0,\,0)=+1, & P_y(0,\,0)=-1 \\ Q_x(0,\,0)=+1, & Q_y(0,\,0)=+1 \end{array}\right\}, \tag{5}$$

and Cauchy's equations are satisfied for $z = 0$. But $f'(z)$ does not exist for this value of z; since, if it did, then $P(x, y)$ would be differentiable at the origin; which, by the method of § 3, is easily shown not to be the case.

A new condition.

6. By means of the theory of integration it can be shown that hypothesis (i), to the effect that $f'(z)$ exists, is replaceable by

(i)′ *The real and imaginary parts of $f(z)$ have partial derivatives which satisfy Cauchy's equations at all points within the contour, and are integrable over every rectangle within the contour—integrability being understood either in the sense of Riemann, or in the more general sense of Lebesgue.*

This is the condition which is obtained at once when Cauchy's theorem is treated by the methods of Real Variable theory, *i.e.* as a result involving x and y, and not as one involving z.

It is stated in a restricted form by Montel,* who considers only the case in which the derivatives of $P(x, y)$ and $Q(x, y)$ are bounded.

Distinctness of the new condition.

7. We shall show that the new condition is distinct from the old in that a function may possess integrable partial derivatives and yet have points at which it is not differentiable. And we shall do so by the production of an actual example.

Take the function, discussed in § 3, given by

$$\phi(x,\,y)=\frac{xy}{\sqrt{(x^2+y^2)}} \qquad (x\neq 0,\ \textit{or}\ y\neq 0)$$

$$=0 \qquad (x=y=0).$$

It is not differentiable at the origin.

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* No. 3. Montel assumes that $P_x(x, y)$, $P_y(x, y)$, etc., are bounded, and then goes on to find conditions which are both necessary and sufficient for the truth of Cauchy's theorem. His work is interesting in that he does not use the converse of Cauchy's theorem.

Now
$$\phi_x(x, y) = \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}} \quad (x \neq 0, \ or \ y \neq 0)$$
$$= 0 \quad (x = y = 0),$$
and
$$\phi_y(x, y) = \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}} \quad (x \neq 0, \ or \ y \neq 0)$$
$$= 0 \quad (x = y = 0).$$
Evidently
$$|\, \phi_x(x, y) \,| \leqslant 1, \qquad |\, \phi_y(x, y) \,| \leqslant 1$$
throughout. And the only discontinuity of $\phi_x(x, y)$ and also of $\phi_y(x, y)$, is at the origin. Thus $\phi_x(x, y)$ and $\phi_y(x, y)$ are each bounded functions with only a finite number of discontinuities, and are therefore integrable.[*] $\phi(x, y)$ is an "exceptional" function of the kind we require.

If an exceptional function with more than a single point of non-differentiability is required, one can be obtained quite easily, by means of Hankel's well known method of condensing the singularities, as follows.

Let (a_n, b_n), $n = 1, 2, \ldots$, be the points of any enumerable two-dimensional set. Write
$$\rho_n = \sqrt{(a_n^2 + b_n^2)},$$
and take positive constants $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$,

such that the series
$$\sum_{n=1}^{\infty} \lambda_n, \quad \sum_{n=1}^{\infty} \lambda_n \rho_n$$
converge. And form the series
$$\sum_{n=1}^{\infty} \lambda_n \phi(x - a_n, \ y - b_n),$$
where $\phi(x, y)$ is the function just considered.

Put
$$R = \sqrt{[(x-a_n)^2 + (y-b_n)^2]},$$
$$r = \sqrt{(x^2+y^2)}.$$
Then
$$|\, \phi(x-a_n, \ y-b_n) \,| < \frac{R^2}{R} = R \leqslant r + \rho_n;$$
$\sum_{n=1}^{\infty} \lambda_n \phi(x-a_n, \ y-b_n)$ converges absolutely for all values of x and y, and represents a function, which we will denote by $\Phi(x, y)$, all over the plane.

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[*] In the restricted sense of Riemann, as well as in the general sense of Lebesgue.

This function $\Phi(x, y)$ is an exceptional function with the points (a_n, b_n) as points of non-differentiability.

For, since

$$|\phi_x(x-a_n,\ y-b_n)| \leqslant 1, \qquad |\phi_y(x-a_n,\ y-b_n)| \leqslant 1,$$

the series

$$\sum_{n=2}^{\infty} \lambda_n \phi_x(x-a_n,\ y-b_n), \qquad \sum_{n=2}^{\infty} \lambda_n \phi_y(x-a_n,\ y-b_n)$$

converge absolutely and uniformly over the whole plane.

Every term of each of these series is continuous at (a_1, b_1), and so they represent respectively x and y derivatives at (a_1, b_1) of

$$\Psi(x, y) = \Phi(x, y) - \lambda_1 \phi(x-a_1,\ y-b_1).$$

Thus $\Psi(x, y)$ has continuous derivatives at (a_1, b_1) and must be differentiable there. It follows at once that $\Phi(x, y)$ cannot be differentiable there; as otherwise $\phi(x-a_1,\ y-b_1)$ would be, which is not the case. Hence (a_1, b_1) is a point of non-differentiability. Similarly, so are all the points (a_n, b_n).

Proof of Cauchy's Theorem under the new conditions.

8. (A). *For a rectangle.*

Let R be any rectangle for which the conditions are satisfied, (a, b) its lower left-hand corner, and (α, β) its upper right-hand corner. Since $P_x(x, y)$ is integrable over the interior of R, which we will call D, we have, by Fubini's theorem:*

(*a*) $\int_a^\alpha P_x(x, y)\, dx$ exists for almost all values of y in (b, β);†

(*b*) $$\iint_D P_x(x, y)\, dx\, dy = \int_b^\beta dy \int_a^\alpha P_x(x, y)\, dx,$$

the inner integral being considered only where it exists.

Now Lebesgue has shown that, if a function possesses an integrable derivative, then it is an integral of this derivative,‡

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* See de la Vallée Poussin, *Intégrales de Lebesgue*, p. 53.

† *I.e.* the values of y in (b, β) for which this integral is non-existent, if there are any at all, form a set of measure zero.

‡ Lebesgue, *Leçons sur l'intégration*, p. 123; de la Vallée Poussin, *Cours d'Analyse*, t. 1, 3rd ed. (1914), p. 272.

Thus
$$\int_a^\alpha P_x(x, y)\,dx = P(\alpha, y) - P(a, y), \tag{6}$$

wherever this integral exists. Moreover, since in calculating the value of

$$\int_b^\beta dy \int_a^\alpha P_x(x, y)\,dx,$$

we may attribute to the inner integral any value we please at the points where it does not exist in the usual sense, we may take (6) to hold throughout without affecting (*b*). Thus

$$\begin{aligned}\iint_D P_x(x, y)\,dx\,dy &= \int_b^\beta [P(\alpha, y) - P(a, y)]\,dy \\ &= \int_R P(x, y)\,dy.\end{aligned}$$

Similarly
$$\iint_D Q_x(x, y)\,dx\,dy = \int_R Q(x, y)\,dy,$$

$$\iint_D P_y(x, y)\,dx\,dy = -\int_R P(x, y)\,dx,$$

$$\iint_D Q_y(x, y)\,dx\,dy = -\int_R Q(x, y)\,dx,$$

R being taken counter clockwise throughout.

Whence

$$\begin{aligned}\int_R f(z)\,dz &= \int_R [P(x, y) + iQ(x, y)]\,(dx + i\,dy) \\ &= \int_R P(x, y)\,dx - Q(x, y)\,dy + i\int_R Q(x, y)\,dx + P(x, y)\,dy \\ &= \iint_D [-P_y(x, y) - Q_x(x, y)]\,dx\,dy + i\iint_D [-Q_y(x, y) + P_x(x, y)]\,dx\,dy \\ &= 0,\end{aligned}$$

in virtue of Cauchy's equations (1).

(B) *For any polygon* Π *whose sides are parallel to one or other of the axes.*

Such a polygon can be decomposed into the sum of a finite number of rectangles R, the boundaries of which, when all described in the same sense, give the boundary of Π as the net result. This being so, we get

$$\int_{\Pi} f(z)\,dz = \Sigma \int_{R} f(z)\,dz = 0,$$

and the theorem is proved for the polygon Π.

(C) *For any simple contour C of bounded variation.*

C being of bounded variation, it is, by a well-known result, rectifiable. Let L be its length. Take any initial point p_1 and obtain the points $p_2, p_3, \ldots, p_n$ whose distances from p_1 measured along the curve are respectively $L/n, 2L/n, \ldots, (n-1)L/n$, n being a positive integer to be determined later on. Denote the arcs $p_1p_2, p_2p_3, \ldots, p_np_1$ by $\gamma_1, \gamma_2, \ldots, \gamma_n$ respectively.

Let δ be the minimum distance between non-consecutive γ's. Divide each γ, in any way, into arcs (at least two) of length less than $\delta/\sqrt{2}$. Corresponding to each of these small arcs construct a square with its sides parallel to one or other of the axes, the centre of the square being at the centre of the arc (regard being had to distances measured along it), and its side being just greater than the length of the arc and less than $\delta/\sqrt{2}$.* The square evidently encloses the whole arc.

Since the diagonals of all these squares are less than $\delta/2$, no square attached to one γ can touch a square attached to a non-consecutive γ.

The squares attached to any given γ overlap two by two and form together a connected region ρ entirely enclosing γ and with its outer boundary consisting of a polygon P, whose sides are parallel to one or other of the axes. These regions ρ form, in de la Vallée Poussin's terminology, a "chain" of which they are the "links." Since non-consecutive links do not overlap, the chain is what is described as "regular." Since ρ_n overlaps ρ_1, the chain is "closed."

Now for regular closed chains de la Vallée Poussin has established the following valuable result:†

If there are five or more links in the chain, then the chain forms a ring bounded on the outside by a polygon π_0, *and on the inside by a second polygon* π_i. *Both* π_0 *and* π_i *contain parts of every link.*

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* And also, in view of what is to come, less than the length of γ_r. This is evidently possible since γ_r is divided into at least two parts.

† *Cours d'Analyse*, t. 1, 3rd ed. (1914), p. 377.

By taking $n \geqslant 5$, we ensure the presence of at least five links, and may therefore apply de la Vallée Poussin's result.

The inner polygon π_i we take as our polygon Π. That it lies within C is established by de la Vallée Poussin in the course of the investigation from which we have quoted. Thus

$$\int_{\Pi} f(z)\,dz = 0.$$

It remains to show that, by choice of n, we can ensure that

$$\left| \int_C f(z)\,dz - \int_{\Pi} f(z)\,dz \right| < \epsilon, \tag{7}$$

where ϵ is any positive number given in advance. For, this being so, we have

$$\left| \int_C f(z)\,dz \right| < \epsilon,$$

and, ϵ being arbitrary and the integral fixed, we must have

$$\int_C f(z)\,dz = 0.$$

Lemma I.—*However many squares are used in the process of obtaining the polygon* Π, *the total length of their perimeters, and therefore the length of* Π, *need not exceed* $5L$.

For every square is associated with a definite arc of the contour C; and it is constructed so that its side is just greater than the length of the arc. Its perimeter is thus just greater than four times the length of the arc, and can certainly be taken less than five times the length of the arc. Since exactly one square is associated with each arc, the total length of the perimeters need not exceed $5L$.

Lemma II.—*Two consecutive regions* ρ *need not overlap except at points belonging to their end squares.*

Let ρ_r be the region enclosing the arc γ_r, and ρ_{r+1} that enclosing the consecutive arc γ_{r+1}. Suppose the end square σ of ρ_r which abuts on γ_{r+1}, and the corresponding end square τ of ρ_{r+1}, already constructed, in accordance with the directions laid down. They are to be made so small that σ contains no points of γ_{r+1} outside τ, and τ contains no points of γ_r outside σ.

The points of γ_r not belonging to the arc associated with σ will be at a positive distance from τ and also from γ_{r+1}. Thus, if the remaining squares of ρ_r are made sufficiently small, they will all be at a positive distance from τ and from γ_{r+1}. And if the remaining squares of ρ_{r+1} are made sufficiently small they will not meet any of these last mentioned squares. Our lemma is demonstrated.

It follows very simply from this last lemma that the removal from any region ρ of the parts common to it and one or other of the adjoining regions cannot divide its boundary into more than two parts. Thus the part of Π belonging to any particular region ρ forms a connected piece. We denote the piece belonging to ρ_r by π_r, and its end points (Π being described in the same sense as c) by q_r, q_{r+1}.

Denote, now, by c_r the closed (but not necessarily simple) curve formed by γ_r, the line $p_{r+1}q_{r+1}$, π_r described in the sense opposite to γ_r, and the line $q_r p_r$. Evidently we have

$$\int_C f(z)\,dz - \int_\Pi f(z)\,dz = \Sigma \int_{c_r} f(z)\,dz,$$

the lines $p_r q_r$ being described twice, once in each direction, and the integrals along them destroying one another.

Since some of the lines $p_r q_r$ may lie partly outside C, and so the behaviour of $f(z)$ on them is not altogether assured, we replace $f(z)$ by an integrand $\phi(z)$ obtained as follows.

On γ_r and π_r, $\phi(z) = f(z)$. On $p_r q_r$, $\phi(z)$ is the linear function which coincides with $f(z)$ at p_r and at q_r.

Evidently $$\Sigma \int_{c_r} f(z)\,dz = \Sigma \int_{c_r} \phi(z)\,dz.$$

Now let f_r be the value of $f(z)$ at p_r, and η_r the maximum value of

$$| f(z) - f_r |$$

as z varies on c_r. η_r will naturally tend to zero as π_r is made to approach γ_r, on account of the continuity of $f(z)$, and therefore of $\phi(z)$, on the contour as far as points within it are concerned.

We have $$\phi(z) = f_r + [\phi(z) - f_r],$$

whence $$\int_{c_r} \phi(z)\,dz = \int_{c_r} f_r\,dz + \int_{c_r} [\phi(z) - f_r]\,dz.$$

Now c_r is closed. Thus

$$\int_{c_r} f_r\,dz = f_r \int_{c_r} 1\,.\,dz$$

$$= 0.$$

Hence $$\int_{c_r} \phi(z)\,dz = \int_{c_r} [\phi(z) - f_r]\,dz,$$

$$\left| \int_{c_r} \phi(z)\,dz \right| < l_r \eta_r,$$

where l_r is the length of c_r.

Denote by η the greatest of the η_r's, and we have

$$\left| \Sigma \int_{c_r} \phi(z)\,dz \right| \leqslant \eta \Sigma l_r.$$

Now let us find the value of Σl_r. Any particular l_r consists of γ_r, c_r and the straight lines $p_r q_r$ and $p_{r+1} q_{r+1}$. Thus

$$\Sigma l_r = \Sigma \gamma_r + \Sigma c_r + 2\Sigma p_r q_r$$

$$< L \;+ 5L + 2\Sigma p_r q_r.$$

Now q_r lies on a square whose centre is on γ_r and whose side is less than L/n, the length of γ_r. No point of γ_r is distant from p_r by more than L/n. Thus

$$p_r q_r < 2L/n,$$

$$\Sigma p_r q_r < 2L,$$

and so $$\Sigma l_r < 10L.$$

This gives

$$\left| \int_c f(z)\,dz - \int_{\text{II}} f(z)\,dz \right| = \left| \Sigma \int_{c_r} \phi(z)\,dz \right| \leqslant 10L\,.\,\eta,$$

and so, since η tends to zero as n tends to infinity and therefore π_r approaches γ_r, we can secure (7) by choice of n. The proof is completed.

An extension of the new conditions.

9. If we examine the proof just given, we shall see that it depends essentially on the fact that there are two functions, which we will call

$\lambda(x, y)$ and $\mu(x, y)$, such that

$$\iint_D \lambda(x, y)\,dx\,dy = \int_b^\beta [P(\alpha, y)-P(a, y)]\,dy \left(= \int_R P(x, y)\,dy\right) \tag{8}$$

$$= \int_a^\alpha [Q(x, \beta)-Q(x, b)]\,dx \left(= -\int_R Q(x, y)\,dx\right), \tag{9}$$

$$\iint_D \mu(x, y)\,dx\,dy = \int_b^\beta [Q(\alpha, y)-Q(a, y)]\,dy \left(= \int_R Q(x, y)\,dy\right) \tag{10}$$

$$= -\int_a^\alpha [P(x, \beta)-P(x, b)]\,dx \left(= \int_R P(x, y)\,dx\right). \tag{11}$$

Under the conditions considered in § 8 we were able to obtain the functions $\lambda(x, y)$ and $\mu(x, y)$ by taking

$$\lambda(x, y) = P_x(x, y), \qquad \mu(x, y) = Q_x(x, y).$$

But this fact does not in any way imply that the functions $\lambda(x, y)$ and $\mu(x, y)$ cannot be obtained in other ways.

Let us investigate the conditions they require.

Assume that (8) is satisfied, both sides being existent and equal.

Since
$$\iint_D \lambda(x, y)\,dx\,dy$$
exists, we know, by Fubini's theorem, that
$$\int_a^\alpha \lambda(x, y)\,dx$$
exists for all values of y between b and β save (at most) those of a set of measure zero, and also that

$$\iint_D \lambda(x, y)\,dx\,dy = \int_b^\beta dy \int_a^\alpha \lambda(x, y)\,dx,$$

the inner integral being considered only where it exists.*

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* *I.e.* we take as the integrand of the outer integral the function which is equal to $\int_a^\alpha \lambda(x, y)\,dx$ when this exists and to zero when it does not. This is an artifice often used in the Lebesgue theory, and is made permissible by the fact that we can alter the value of an integrand in any way we like (even to making it infinite) at each of the points of a set of measure zero without affecting in the slightest the value of its integral.

Taking into account (10), we have

$$\int_b^\beta dy \int_a^\alpha \lambda(x, y)\,dx = \int_b^\beta [P(\alpha, y) - P(a, y)]\,dy.$$

Since the same argument holds with β replaced by any number β' between it and b, we have

$$\int_b^{\beta'} dy \int_a^\alpha \lambda(x, y)\,dx = \int_b^{\beta'} [P(\alpha, y) - P(a, y)]\,dy.$$

Now the integrand of the right-hand side is continuous. Thus

$$\frac{d}{d\beta'} \int_b^{\beta'} [P(\alpha, y) - P(a, y)]\,dy = P(\alpha, \beta') - P(a, \beta').$$

Also, by Lebesgue's fundamental theorem on the differentiability of an indefinite Lebesgue integral, we have

$$\frac{d}{d\beta'} \int_b^{\beta'} dy \int_a^\alpha \lambda(x, y)\,dx = \int_a^\alpha \lambda(x, \beta')\,dx$$

or all values of y between b and β save those of a set of measure zero.

Hence $$\int_a^\alpha \lambda(x, y)\,dx = P(\alpha, y) - P(a, y)$$

for all values of y between b and β save those of a set of measure zero.*

The complete investigation of this last equation seems too difficult a matter to be embarked on here.† We content ourselves with finding particular circumstances in which it is readily shown to hold.

It is certainly satisfied if, for the non-exceptional values of y, $P(x, y)$ is the indefinite integral of $\lambda(x, y)$; *i.e.* in virtue of a certain condition appropriate to this, if $P(x, y)$ is, for the stated y's, an *absolutely con-*

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* It will be noted that it is *linear* measure that is meant here.

† The difficulty lies in the fact that we may get different exceptional sets of values of y corresponding to different values of α. If we are to get anything out of the equation, it is evident we shall have to proceed by replacing α by a variable number α' between it and a, just as we replaced β by a variable number β' between it and b; and the exceptional sets for all the values of α' concerned might add up to a set of non-zero measure.

tinuous function of x, and $\lambda(x, y)$ is equal to $P_x(x, y)$ except at a set of measure zero.*

Conversely, if $P(x, y)$ is, for all values of y in (b, β) save those of a set of measure zero, an absolutely continuous function of x, and $P_x(x, y)$, considered where it exists, is integrable in D; then we have

$$\iint_D P_x(x, y)\,dx\,dy = \int_R P(x, y)\,dy,$$

as we have only to reverse the previous analysis to show.

Under exactly similar conditions we can obtain the corresponding results for $Q_x(x, y)$, $P_y(x, y)$, and $Q_y(x, y)$. And we are then able to enunciate the following extension of the condition (i)′ of § 6.

(i)″. (*a*) *Save for a set of values of y of measure zero, $P(x, y)$ and $Q(x, y)$ are absolutely continuous functions of x within the contour, with the corresponding condition with x and y interchanged.*

(*b*) *$P_x(x, y)$, $P_y(x, y)$, $Q_x(x, y)$, $Q_y(x, y)$, considered where they exist, are integrable over every rectangle lying within the contour.*

(*c*) $$P_x(x, y) = Q_y(x, y), \qquad Q_x(x, y) = -P_y(x, y)$$

save for a set of values of (superficial) measure zero.

A contour of bounded variation which is cut by a transversal in an infinity of points.

10. Consider the contour consisting of

$$y = x^2 \sin\frac{1}{x} \quad \left(0 \leqslant x \leqslant \frac{1}{2\pi}\right),$$

which we will call c_1, and the three lower sides of the rectangle whose corners are $(0, 0)$, $(0, -2)$, $(1/2\pi, -2)$, $(1/2\pi, 0)$, which we will call c_2.

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* De la Vallée Poussin, *Cours d'Analyse*, t. 1, 3rd ed., p. 279. It should be remarked that one of the properties of an absolutely continuous function is that of possessing a unique derivative at all the points of its interval of existence, save (possibly) those of a set of measure zero. This justifies our use of the derivative $P_x(x, y)$, which otherwise might not exist.

c_1 and c_2 obviously do not intersect and so form a proper contour. c_1 is a curve which is well known to be of bounded variation; that the variation of c_2 is bounded goes without saying. The contour considered is thus of bounded variation.

It is cut by the x-axis in the points

$$x = 0, \qquad x = \frac{1}{r\pi} \quad (r = 2, 3, \ldots),$$

which are infinite in number.

A contour of bounded variation which does not possess the property required by Ames.

11. Take numbers t_n $(n = 0, 1, 2, \ldots)$ as follows

$$t_0 = 1 > t_1 > t_2 > t_3 > \ldots > t_n > \ldots \to 0;$$

and define two functions $g(t)$ and $h(t)$ by taking

$$g(t_n) = \frac{1}{3 \, . \, 2^n}$$

when n is even and other than zero,

$$= \frac{1}{2^n}$$

when n is odd; $$g(0) = g(t_0) = 0,$$

$$h(t_n) = \frac{1}{n} \, g(t_n)$$

when n is other than zero;

$$h(0) = h(t_0) = 0,$$

and making both $g(t)$ and $h(t)$ linear in the intervals (t_{n+1}, t_n).

Let (Γ) be the curve given by

$$x = g(t), \qquad y = h(t),$$

where t varies from 0 to 1.

In order that it may be a closed contour of bounded variation we must have

(*a*) $g(t)$ and $h(t)$ are continuous for $0 \leqslant t \leqslant 1$.

(*b*) $g(t)$ and $h(t)$ are of bounded variation for $0 \leqslant t \leqslant 1$.

(*c*) There cannot be two values t' and t'' of t satisfying $0 \leqslant t' < t'' < 1$, and such that

$$g(t') = g(t''), \quad h(t') = h(t'')$$

hold simultaneously.

(*d*) $$g(1) = g(0) \quad \text{and} \quad h(1) = h(0).$$

For (*a*) we may note that, apart from the origin, the two functions each consist of linear (and therefore continuous) pieces continuously joined together, and are certainly continuous. At the origin continuity is established at once by the inequalities

$$0 \leqslant h(t) \leqslant g(t) \leqslant \frac{1}{2^n} \quad (0 \leqslant t < t_n).$$

For (*b*) it is sufficient to show that the actual values of the total variations of the functions concerned in the interval (0, 1) are finite. That of $g(t)$ is

$$\frac{1}{2} + \left(\frac{1}{2} - \frac{1}{3.2^2}\right) + \left(\frac{1}{2^3} - \frac{1}{3.2^3}\right) + \left(\frac{1}{2^3} - \frac{1}{3.2^4}\right) + \left(\frac{1}{2^5} - \frac{1}{3.2^4}\right) + \dots$$

$$= 2\left(\frac{1}{2} + \frac{1}{2^3} + \dots\right) - 2\left(\frac{1}{3.2^2} + \frac{1}{3.2^4} + \dots\right)$$

$$= 2.\frac{1}{2}.\frac{4}{3} - 2.\frac{1}{3.2^2}.\frac{4}{3}$$

$$= \frac{4}{3} - \frac{2}{9}$$

$$= \frac{10}{9}.$$

That of $h(t)$ is

$$\frac{1}{2}+\left(\frac{1}{2}-\frac{1}{2.3.2^2}\right)+\left(\frac{1}{3.2^3}-\frac{1}{2.3.2^2}\right)+\left(\frac{1}{3.2^3}-\frac{1}{4.3.2^4}\right)+\dots$$

$$=2\left(\frac{1}{2}+\frac{1}{3.2^3}+\frac{1}{5.2^5}+\dots\right)-2\left(\frac{1}{2.3.2^2}+\frac{1}{4.3.2^4}+\dots\right)$$

$$<2\left(\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\dots\right)$$

$$=\frac{4}{3}.$$

For (*c*) we shall show that, if

$$g(t')=g(t''),$$

then $$h(t')\neq h(t'').$$

From the first of these it follows that t' and t'' must be separated by at least one of the points t_n. Let it be t_ν, so that

$$t'<t_\nu \quad \text{and} \quad t''>t_\nu.$$

From the definition of the functions $g(t)$ and $h(t)$ we obtain at once

$$h(t')<\frac{1}{\nu}\times g(t'), \qquad h(t'')\geqslant\frac{1}{\nu-1}\times g(t'');$$

and therefore $$h(t')\neq h(t'').$$

This shows that (Γ) is a proper contour in that it does not cross itself.

(*d*) is given at once by the definition of the functions $g(t)$ and $h(t)$.

(Γ) is thus a closed curve of bounded variation which does not cross itself, and is, therefore, a suitable contour for Cauchy's theorem.

If possible, let it possess the Ames property; and let $T_1, T_2, \dots, T_m$ be the parameters of the points separating the various pieces. Of these there will be a pair T' and T'' such that

$$T'\leqslant 0\leqslant T''.$$

Let γ be the arc given by $$0\leqslant t\leqslant T''.$$

Then γ has an equation of one or other of the forms

$$y=\phi(x), \qquad x=\psi(y),$$

where ϕ and ψ are one-valued.

We shall show that neither of these is possible.

Let μ be the first value of n for which

$$\frac{1}{3\,.\,2^n} < T''.$$

Since $$g(t_{\mu+1}) = \frac{1}{2^{\mu+1}} > g(t_\mu) \quad\text{and}\quad g(t_{\mu+2}) = \frac{1}{3\,.\,2^{\mu+2}} < g(t_\mu),$$

there is a value $\bar{t}$ of t in the interval $(t_{\mu+2},\ t_{\mu+1})$ such that

$$g(\bar{t}) = g(t_\mu).$$

If $\bar{y}$ and y_μ are the values of y corresponding to $\bar{t}$ and t_μ respectively, and x is the common value of $g(\bar{t})$ and $g(t_\mu)$; then $\bar{y}$ and y_μ both satisfy $y = \phi(x)$. But we have shown in (*c*) that we cannot have simultaneously

$$g(\bar{t}) = g(t_\mu) \quad\text{and}\quad h(\bar{t}) = h(t_\mu).$$

It follows that $\bar{y} \neq y_\mu$ and ϕ cannot be one-valued.

In exactly the same way we can show that ψ cannot be one-valued.

The curve (Γ) does not possess the property required by Ames.

On the proof by Goursat's lemma.

12. By far the most compact proof of Cauchy's theorem is that due to Goursat. It obtains the result, as it were, in one step—not even the deduction from a special contour to the general one being required. As Goursat's proof is obviously an advantageous one to use, it seems worth while to give a little consideration to the conditions under which it applies. Not that the validity of Cauchy's theorem will be in any way affected by our consideration, but simply that it is interesting to know the exact circumstances under which the shortest and most direct proof can be used.

What the conditions are will be apparent at once to anyone who is familiar with the proof, and we can state them without discussion. They are:

(*a*) *$f'(z)$ exists at all points within and on the contour, values of $f(z)$ at points outside the contour being excluded.*

(*b*) *The contour is of bounded variation, and is not cut by a transversal parallel to one or other of the axes in more than a finite number of points.*

The new point is evidently the second half of (*b*). We would like to know, as far as possible, its exact meaning. In particular, we would like to know whether it implies the Ames property or not.

As to the exact meaning of (*b*), that is to say its expression in terms of known properties of the functions involved in the equations of the contour, nothing much seems to be obtainable at present. But whether or not it implies the Ames property we can determine at once, by means of the contour of the preceding section.

It might be imagined that, if the contour is cut by transversals in only a finite number of points, then it ought to be capable of subdivision into a finite number of pieces cut by transversals in only one point, *i.e.* it possesses the Ames property. But the contour we give contradicts this. It is cut by no transversal parallel to the y-axis in more than three points, and, as we have seen, it does not possess the Ames property.

The number of points in which the contour is cut by the transversal $x = a$ is given by the number of roots of the equation

$$g(t) = a.$$

Let ν be the greatest value of n for which

$$\frac{1}{2^n} > a.$$

Then an inspection of the graph of $g(t)$ will show that the roots must lie in $(t_{\nu+1}, t_{\nu-2})$, *i.e.* there are not more than *three* roots.

The contour satisfies (*b*) for transversals parallel to the y-axis. Similarly, it does so for those parallel to the x-axis. But it does not possess the Ames property.

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A THEOREM CONCERNING FOURIER SERIES

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(*Communicated by* G. H. HARDY.)

[Received December 25th, 1921.—Read January 12th, 1922.]

1. Let $f(x)$ be a summable function, of which we form the Fourier series,

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx),$$

and let s_ν denote the first $\nu+1$ terms,

$$s_\nu = \tfrac{1}{2}a_0 + \sum_{m=1}^{\nu} (a_m \cos mx + b_m \sin mx).$$

Then at a point x, where

$$S = \tfrac{1}{2}\{f(x+0)+f(x-0)\}$$

exists, we have by the well known theorem of Fejér,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} (s_\nu - S) = 0. \tag{1}$$

Similarly Lebesgue has proved that *almost everywhere*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} \{s_\nu - f(x)\} = 0.$$

In the case of functions whose squares are summable, Hardy and Littlewood[*] have obtained the further results

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} \{s_\nu - f(x)\}^2 = 0, \tag{2}$$

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} | s_\nu - f(x) | = 0, \tag{3}$$

each of which is true *almost everywhere*.

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[*] *Comptes Rendus*, t. 156 (1913), p. 1307.

These last relations show that the average value of $s_\nu - f(x)$ tends to zero, not because of the cancelling of positive and negative terms, but because the number of terms in the sum $\sum_{\nu=0}^{n} \{s_\nu - f(x)\}$ which are not themselves small, is small compared with n. By considering the average value of higher powers of $|s_\nu - f(x)|$, it is possible to draw further inferences about the size of the terms in this sum.

We may remark that, if

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu - f(x)|^\alpha = 0,$$

for $\alpha = \alpha_0$, then the relation is also true for $0 < \alpha \leqslant \alpha_0$; for, by a well known inequality due to Hölder,

$$\sum_{\nu=0}^{n} |s_\nu - f(x)|^\alpha \leqslant \left\{ \sum_{\nu=0}^{n} |s_\nu - f(x)|^{\alpha_0} \right\}^{\alpha/\alpha_0} \left\{ \sum_{\nu=0}^{n} 1 \right\}^{(\alpha_0-\alpha)/\alpha_0},$$

and so
$$\frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu - f(x)|^\alpha \leqslant \left\{ \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu - f(x)|^{\alpha_0} \right\}^{\alpha/\alpha_0}.$$

Our principal object in this note is to prove the following theorem :—

THEOREM 1.—*If s_ν is the sum of the first $\nu+1$ terms of the Fourier series of a summable function $f(x)$, and for a particular value of x,*

$$\int_0^\delta |f(x+t)+f(x-t)-2f(x)|\,dt = o(\delta), \tag{4}$$

$$\int_0^\delta \{f(x+t)+f(x-t)-2f(x)\}^2 dt = O(\delta), \tag{5}$$

when $\delta \to 0$, then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu - f(x)|^k = 0,$$

for every positive value of k.

For a function $f(x)$ of summable square, the conditions of the theorem are satisfied *almost everywhere*. For

$$\int_0^b \{f(x+t)+f(x-t)-2f(x)\}^2 dt$$
$$\leqslant 2\int_0^\delta \{f(x+t)-f(x)\}^2 dt + 2\int_0^\delta \{f(x-t)-f(x)\}^2 dt,$$

and (taking the first term on the right only)

$$\frac{1}{\delta}\int_0^\delta \{f(x+t)-f(x)\}^2\,dt$$
$$= \frac{1}{\delta}\int_0^\delta \{f(x+t)\}^2\,dt - \frac{2}{\delta}f(x)\int_0^\delta f(x+t)\,dt + \{f(x)\}^2,$$

and $\lim \frac{1}{\delta}\int_0^\delta f(x+t)\,dt = f(x)$, $\lim \frac{1}{\delta}\int_0^\delta \{f(x+t)\}^2\,dt = \{f(x)\}^2$

almost everywhere, so that the right-hand side is $o(1)$. The other condition is satisfied almost everywhere, since $f(x)$ is summable.* Hence we shall prove that, *if the square of $f(x)$ is summable, then*

$$\lim_{n\to\infty} \frac{1}{n+1}\sum_{\nu=0}^{n} |s_\nu - f(x)|^k = 0,$$

for almost all values of x, and all positive values of k.

2. We now proceed to the proof of Theorem 1. It is evident, from the remark which precedes the enunciation, that it is sufficient to prove the theorem when k is an even positive integer $2p$. We may without loss of generality take the particular value of x considered to be $x = 0$. We may suppose further that $f(x) = 0$ for $\eta < |x| < \pi$, so that we may take the square of $f(x)$ to be summable throughout $(-\pi, \pi)$, and not merely in the neighbourhood of $x = 0$. In fact,

$$f(x) = f_1(x) + f_2(x), \tag{6}$$

where $f_1(x) = 0$ for $\eta < |x| < \pi$, $f_2(x) = 0$ for $|x| < \eta$.

If we denote the corresponding partial sums by s'_ν, s''_ν, we have

$$s_\nu = s'_\nu + s''_\nu,$$

and by an inequality due to Minkowski,†

$$\left\{\sum_{\nu=0}^{n} |s_\nu - f(0)|^{2p}\right\}^{1/2p} \leqslant \left\{\sum_{\nu=0}^{n} |s'_\nu - f(0)|^{2p}\right\}^{1/2p} + \left\{\sum_{\nu=0}^{n} |s''_\nu|^{2p}\right\}^{1/2p}. \tag{7}$$

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* See H. Lebesgue, "Sur la convergence des séries de Fourier", *Math. Annalen*, Bd. 61, pp. 276, 277.

† See F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnus*, Paris, 1913, p. 45.

Now for $x=0$, $$\lim_{\nu\to\infty} s''_\nu = 0.$$

Hence $$\lim_{n\to\infty}\frac{1}{n+1}\sum_{\nu=0}^{n}|s''_\nu|^{2p}=0,$$

so that $$\lim_{n\to\infty}\frac{1}{n+1}\sum_{\nu=0}^{n}|s_\nu-f(0)|^{2p}=0,$$

if only $$\lim_{n\to\infty}\frac{1}{n+1}\sum_{\nu=0}^{n}|s'_\nu-f(0)|^{2p}=0.$$

Writing $$2\phi(t)=f(t)+f(-t)-2f(0),$$

the conditions of the theorem take the form

$$\int_0^\delta |\phi(t)|\,dt = o(\delta), \tag{8}$$

$$\int_0^\delta |\phi(t)|^2\,dt = O(\delta). \tag{9}$$

Now

$$\begin{aligned} s_\nu-f(0) &= \frac{1}{\pi}\int_0^\pi \phi(t)\frac{\sin(\nu+\frac12)t}{\sin\frac12 t}\,dt \\ &= \frac{1}{\pi}\int_0^\pi \sin\nu t\cot\tfrac12 t\,\phi(t)\,dt+\frac{1}{\pi}\int_0^\pi \cos\nu t\,\phi(t)\,dt \\ &= \frac{1}{\pi}\int_0^\delta \sin\nu t\cot\tfrac12 t\,\phi(t)\,dt+\frac{1}{\pi}\int_\delta^\pi \sin\nu t\cot\tfrac12 t\,\phi(t)\,dt \\ &\qquad +\frac{1}{\pi}\int_0^\pi \cos\nu t\,\phi(t)\,dt \\ &= \alpha_\nu+\beta_\nu+\gamma_\nu. \end{aligned} \tag{10}$$

Using the Minkowski inequality again,

$$\begin{aligned} &\left\{\sum_{\nu=0}^{n}|s_\nu-f(0)|^{2p}\right\}^{1/2p} \\ &\leqslant \left\{\sum_{\nu=0}^{n}|\alpha_\nu|^{2p}\right\}^{1/2p}+\left\{\sum_{\nu=0}^{n}|\beta_\nu|^{2p}\right\}^{1/2p}+\left\{\sum_{\nu=0}^{n}|\gamma_\nu|^{2p}\right\}^{1/2p}, \end{aligned} \tag{11}$$

and our problem is reduced to the consideration of the three sums

$$\sum_{\nu=0}^{n}|\alpha_\nu|^{2p},\quad \sum_{\nu=0}^{n}|\beta_\nu|^{2p},\quad \sum_{\nu=0}^{n}|\gamma_\nu|^{2p}.$$

3. (i) $\sum\limits_{\nu=0}^{n} |\alpha_\nu|^{2p}$.—For $0 < t < \delta$,

$$|\sin \nu t . \tfrac{1}{2} \cot \tfrac{1}{2} t| \leqslant \nu \{1 + o(1)\},$$

where the symbol o refers to $\delta \to 0$, and implies a constant independent of ν. Hence

$$|\alpha_\nu| = \frac{2}{\pi} \left| \int_0^\delta \sin \nu t . \tfrac{1}{2} \cot \tfrac{1}{2} t . \phi(t)\, dt \right|$$

$$\leqslant \frac{2\nu}{\pi} \{1 + o(1)\} \int_0^\delta |\phi(t)|\, dt$$

$$= \nu . o(\delta),$$

and

$$\left\{ \sum_{\nu=0}^{n} |\alpha_\nu|^{2p} \right\}^{1/2p} \leqslant \left\{ \sum_{\nu=0}^{n} \nu^{2p} \right\}^{1/2p} o(\delta)$$

$$< (n^{2p+1})^{1/2p}\, o(\delta),$$

or simply

(12) $$\left\{ \sum_{\nu=0}^{n} |\alpha_\nu|^{2p} \right\}^{1/2p} = n^{1+1/2p}\, o(\delta).$$

(ii) $\sum\limits_{\nu=0}^{n} |\gamma_\nu|^{2p}$.—We have

$$\gamma_\nu = \frac{1}{\pi} \int_0^\pi \cos \nu t\, \phi(t)\, dt,$$

so that

$$\phi(x) \sim \gamma_0 + 2 \sum_1^\infty \gamma_\nu \cos \nu x,$$

and, since ϕ is of summable square,

(13) $$\gamma_0^2 + 2 \sum_{\nu=1}^{\infty} \gamma_\nu^2 = \frac{1}{\pi} \int_0^\pi \{\phi(t)\}^2\, dt.$$

Hence

$$\sum_{\nu=0}^{\infty} |\gamma_\nu|^{2p} \leqslant \left\{ \sum_{\nu=0}^{\infty} |\gamma_\nu|^2 \right\}^p = O(1),$$

and so

(14) $$\left\{ \sum_{\nu=0}^{\infty} |\gamma_\nu|^{2p} \right\}^{1/2p} = O(1).$$

(iii) $\sum\limits_{\nu=0}^{n} |\beta_\nu|^{2p}$.—We proceed now to prove that

$$\sum_{\nu=1}^{\infty} |\beta_\nu|^{2p} = O\left(\frac{1}{\delta}\right).$$

Let

(15) $$c_\nu(s) = \frac{2}{\pi} \int_0^s \sin \nu t\, \phi(t)\, dt.$$

Then $c_\nu(s)$ is the typical Fourier coefficient of $\phi_s(x)$, where

$$\phi_s(x) = \phi(x) \quad (0 < x < s),$$

$$\phi_s(x) = 0 \qquad (s < x < \pi),$$

and
$$\phi_s(-x) = -\phi_s(x).$$

From

(16)
$$\phi_s(x) \sim \sum_{\nu=1}^{\infty} c_\nu(s) \sin \nu x,$$

we deduce
$$\phi_s(x+y) \sim \sum_{\nu=1}^{\infty} c_\nu(s)(\sin \nu x \cos \nu y + \cos \nu x \sin \nu y),$$

and

(17)
$$\phi_{2,s}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi_s(y)\, \phi_s(x+y)\, dy = \sum_{\nu=1}^{\infty} \{c_\nu(s)\}^2 \cos \nu x,$$

this series being absolutely convergent.

Similarly
$$\phi_{3,s}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi_{2,s}(y)\, \phi_s(x+y)\, dy = \sum_{\nu=1}^{\infty} \{c_\nu(s)\}^3 \sin \nu x,$$

and so, generally,
$$\phi_{2p,s}(x) = \sum_{\nu=1}^{\infty} \{c_\nu(s)\}^{2p} \cos \nu x,$$

the functions $\phi_{n,s}(x)$ being defined in succession by the formulæ

(18)
$$\phi_{n+1,s}(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi_{n,s}(y)\, \phi_s(x+y)\, dy \quad (n = 1, 2, \ldots),$$

where we write for uniformity

$$\phi_{1,s}(x) = \phi_s(x).$$

In particular

(19)
$$\sum_{\nu=1}^{\infty} \{c_\nu(s)\}^{2p} = \phi_{2p,s}(0).$$

Let

(20)
$$\int_0^s |\phi(t)|\, dt = \rho(s), \qquad \int_0^s |\phi(t)|^2\, dt = \theta(s).$$

Then
$$|\phi_{2,s}(x)| \leqslant \frac{1}{\pi} \left\{ \int_{-\pi}^{\pi} \{\phi_s(y)\}^2\, dy \int_{-\pi}^{\pi} \{\phi_s(x+y)\}^2\, dy \right\}^{\frac{1}{2}}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} |\phi_s(y)|^2\, dy = \frac{2}{\pi}\, \theta(s).$$

Hence
$$|\phi_{3,s}(x)| \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} |\phi_{2,s}(y)|\,|\phi_s(x+y)|\,dy$$
$$\leqslant \frac{2}{\pi^2}\,\theta(s)\int_{-\pi}^{\pi} |\phi_s(x+y)|\,dy$$
$$= \left(\frac{2}{\pi}\right)^2 \theta(s)\,\rho(s),$$

and proceeding in this way, we get
$$|\phi_{2p,s}(x)| \leqslant \left(\frac{2}{\pi}\right)^{2p-1} \theta(s)\,\{\rho(s)\}^{2p-2}.$$

In particular this is true for $x=0$, which gives

(21)
$$\sum_{\nu=1}^{\infty} \{c_\nu(s)\}^{2p} \leqslant \left(\frac{2}{\pi}\right)^{2p-1} \theta(s)\,\{\rho(s)\}^{2p-2}.$$

Now
$$\beta_\nu = \frac{2}{\pi}\int_{\delta}^{\pi} \sin \nu t\,.\,\tfrac{1}{2}\cot\tfrac{1}{2}t\,.\,\phi(t)\,dt$$
$$= -\tfrac{1}{2}\cot\tfrac{1}{2}\delta\,.\,c_\nu(\delta) + \tfrac{1}{4}\int_{\delta}^{\pi} c_\nu(t)\,\mathrm{cosec}^2\,\tfrac{1}{2}t\,.\,dt,$$

on integrating by parts; and so
$$\left|\sum_{\nu=1}^{\infty} \xi_\nu \beta_\nu\right| \leqslant \tfrac{1}{2}\cot\tfrac{1}{2}\delta \sum_{\nu=1}^{\infty} |\xi_\nu c_\nu(\delta)| + \tfrac{1}{4}\int_{\delta}^{\pi} \mathrm{cosec}^2\,\tfrac{1}{2}t \sum_{\nu=1}^{\infty} |\xi_\nu c_\nu(t)|\,dt,$$

for any values of the ξ's for which the right-hand side exists.

Suppose
$$\sum_{\nu=1}^{\infty} |\xi_\nu|^{2p/(2p-1)} = 1.$$

Then
$$\operatorname*{Max}_{\xi} \left|\sum_{\nu=1}^{\infty} a_\nu \xi_\nu\right| = \left\{\sum_{\nu=1}^{\infty} |a_\nu|^{2p}\right\}^{1/2p},$$

if the right-hand side is convergent; and this is so if
$$a_\nu = c_\nu(t).$$

Hence

(22)
$$\left\{\sum_{\nu=1}^{\infty} |\beta_\nu|^{2p}\right\}^{1/2p}$$
$$\leqslant \tfrac{1}{2}\cot\tfrac{1}{2}\delta \left\{\sum_{\nu=1}^{\infty} |c_\nu(\delta)|^{2p}\right\}^{1/2p} + \tfrac{1}{4}\int_{\delta}^{\pi} \mathrm{cosec}^2\,\tfrac{1}{2}t \left\{\sum_{1}^{\infty} |c_\nu(t)|^{2p}\right\}^{1/2p} dt,$$

a relation which can also be obtained by an extension of the Minkowski inequality. Now

$$\rho(s) < K_1 s, \qquad \theta(s) < K_2 s. \tag{23}$$

Hence
$$\sum_{\nu=1}^{\infty} \{c_\nu(s)\}^{2p} \leqslant \left(\frac{2}{\pi}\right)^{2p-1} K_2 s (K_1 s)^{2p-2},$$

and so
$$\left\{ \sum_{\nu=1}^{\infty} \{c_\nu(s)\}^{2p} \right\}^{1/2p} < K_3 s^{(2p-1)/2p},$$

where each K is an absolute constant. Hence

$$\begin{aligned} \left\{ \sum_{\nu=1}^{\infty} |\beta_\nu|^{2p} \right\}^{1/2p} &\leqslant \frac{K'}{\delta}\, \delta^{(2p-1)/2p} + K' \int_\delta^\pi t^{-1/2p-1}\, dt \\ &< K' \delta^{-1/2p} + K' . 2p . \delta^{-1/2p} \\ &< Kp\, \delta^{-1/2p}. \end{aligned} \tag{24}$$

We have thus obtained an equality for each of the three sums in question; and combining the results (11), (12), (14), (24), we have

$$\left\{ \sum_{\nu=0}^{n} |s_\nu - f(0)|^{2p} \right\}^{1/2p} < n^{1+1/2p}\, o(\delta) + Kp\, \delta^{-1/2p} + O(1),$$

or

$$\left\{ \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu - f(0)|^{2p} \right\}^{1/2p} < n\delta\, \epsilon(\delta) + \frac{Kp}{(n\delta)^{1/2p}} + \frac{K}{n^{1/2p}}, \tag{25}$$

where
$$\epsilon(\delta) = o(1) \quad \text{when} \quad \delta \to 0.$$

The conclusion is now immediate: take

$$\delta = b/n,$$

where b is large enough to make

$$Kp\, b^{-1/2p} < \tfrac{1}{2}\epsilon,$$

and then n large enough to make

$$b\, \epsilon(\delta) + K n^{-1/2p} < \tfrac{1}{2}\epsilon,$$

where now
$$\epsilon(\delta) = \epsilon\left(\frac{b}{n}\right) \to 0 \quad \text{when} \quad n \to \infty.$$

Hence
$$\overline{\lim_{n\to\infty}} \left\{ \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu - f(0)|^{2p} \right\} < \epsilon^{2p},$$

and so, since ϵ is as small as we please,

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu - f(0)|^{2p} = 0.$$

Theorem 1 is therefore established.

4. By taking the number η of equation (6) sufficiently small, we can make the constant K_1 of (23), and so the K of (25), as small as we please. Then the preceding analysis easily leads to the following result:—*to every* ϵ *there corresponds a* $p_0 = p_0(\epsilon)$ *such that*

$$\frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu - f(0)|^{2p} < \epsilon^p . p!, \tag{26}$$

for $p > p_0$ *and all values of* n.

From this inequality we can deduce immediately the following theorem:—

THEOREM 2.—*If* $f(x)$ *is subject to the conditions of Theorem* 1, *then*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} e^{a|s_\nu - f(x)|} = 1, \tag{27}$$

for all values of a.*

In fact (taking again the particular value of x considered to be $x = 0$)

$$\frac{1}{n+1} \sum_{\nu=0}^{n} e^{a|s_\nu - f(0)|} = 1 + \sum_{p=1}^{\infty} \frac{a^p}{p!} \frac{1}{n+1} \sum_{\nu=0}^{n} |s_\nu - f(0)|^p.$$

If we take ϵ so small that $|\epsilon a| < 1$, then it follows from (26) that the series on the right converges uniformly with respect to n; so that the limit of the sum when $n \to \infty$ is the sum of the limit of the separate terms; *i.e.*

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{\nu=0}^{n} e^{a|s_\nu - f(0)|} = 1.$$

We conclude with the following remark. Let $\Omega(n, l)$ be the number of terms in $\sum_{\nu=0}^{n} |s_\nu - f(0)|$ for which

$$|s_\nu - f(0)| > l.$$

---

* From Theorem 2 we conclude immediately the following corollary: if the square of $f(x)$ is summable, then $s_n = o(\log n)$ almost everywhere. This is a particular case of a theorem proved by Hardy [*Proc. London Math. Soc.*, Ser. 2, Vol. 12 (1913), pp. 365–372].

Then under the conditions of Theorem 1, there is a number $l' = l'(a)$ such that

$$\Omega(n, l) < (n+1)\, e^{-al}, \tag{28}$$

for $l > l'$.

This is a corollary to (26); in fact it follows from that relation that

$$\Omega(n, l)\, l^p < (n+1)\, \epsilon^p p\,!,$$

for $p > p_0$. Hence

$$\Omega(n, l) < (n+1)\, \frac{\epsilon^p p\,!}{l^p},$$

and choosing p so as to make $\frac{\epsilon^p p\,!}{l^p}$ a minimum, we obtain a result of the form (28).

ON TRIGONOMETRIC SERIES INVOLVING ALGEBRAIC NUMBERS

By L. J. MORDELL.

[Received June 23rd, 1922.—Read November 9th, 1922.]

IN a very recent number of the *Göttingen Nachrichten*, Siegel in a paper entitled[*] "Über die Diskriminanten total reeller Körper" has found some rather interesting trigonometric series involving algebraic numbers. Apart from their own interest, the series seem to have important applications; for example, Siegel proves Minkowski's theorem that the discriminant of an algebraic field is greater than unity for the particular case of a totally real field, *i.e.* when the field and all of its conjugates are real.

Let, then, K be a totally real algebraic field of degree $n \geqslant 1$, $d \geqslant 1$ its discriminant, and $a_1, a_2, \ldots, a_n$ the base of an ideal A in K. Denote the n conjugates[†] of a number a in K by $a^{(1)}, a^{(2)}, \ldots, a^{(n)}$, and suppose that (A_k^l) is the matrix reciprocal to (a_l^k), $(l, k = 1, 2, \ldots, n)$. It is well known then that $A_1, A_2, \ldots, A_n$ are the base of a fractional ideal $(1/AD)$ in K, where D is the *grund ideal* whose norm is d, the discriminant of K. His formulæ involve three sets of n variables, say

$$s^{(k)},\ w^{(k)},\ x^{(k)} \quad (k = 1, 2, \ldots, n),$$

occurring in a function $F(s, w, k)$, and the symbol NF indicates the product of all the expressions obtained from F by giving the variables s, w, x and also any number of K occurring in F upper indices (1), (2), ..., (n) simultaneously. Then Siegel's[‡] result is

$$\sum_{A \mid \mu}^{|\mu| \prec x} N\left\{e^{2\pi i \mu w}(x - |\mu|)\right\} = \frac{1}{N(A)\, d^{\frac{1}{2}}} \sum_{(1/AD) \mid \lambda} N\left[\frac{\sin \pi(w-\lambda)x}{\pi(w-\lambda)}\right]^2,$$

---

[*] Read November, 1921.

[†] $a^{(1)}$ is the same as a.

[‡] In the *Mathematische Annalen*, Vol. 87 (1922), he has shown that an immediate deduction from this result is Minkowski's theorem on linear forms, *i.e.* if $\xi_1 = a_{11}x_1 + \ldots + a_{1n}x_n$, etc. are n linear forms with real coefficients and determinant $D > 0$, and if $t_1, t_2, \ldots, t_n$ are any n positive numbers with product D, then n integers $x_1, x_2, \ldots, x_n$ not all zero can be found to satisfy the n inequalities

$$|\xi_1| \leqslant t_1, \quad |\xi_2| \leqslant t_2, \quad \ldots, \quad |\xi_n| \leqslant t_n.$$

where $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ are n positive numbers,

$w^{(1)}, w^{(2)}, \ldots, w^{(n)}$ are n real numbers,

the symbol $\sum\limits_{A \mid \mu}$ indicates that μ runs through all the numbers of the ideal A [whose norm is $N(A)$], and $|\mu| \prec x$ that μ is such that the n differences

$$x^{(1)}-|\mu^{(1)}|, \quad \ldots, \quad x^{(n)}-|\mu^{(n)}|$$

are all positive. From the special case

$$w^{(1)} = w^{(2)} = \ldots w^{(n)} = 0, \quad A = 1, \quad x^{(1)} = x^{(2)} = \ldots x^{(n)} = 1,$$

$$d^{\frac{1}{2}} = 1 + \sum_{(1/D) \mid \lambda}' N\left(\frac{\sin \pi\lambda}{\pi\lambda}\right)^2,$$

the Σ' denoting that $\lambda = 0$ is excluded in the summation, showing immediately that $d > 1$ if $n > 1$.

He proves it by contour integration applied to an expansion communicated to him by F. Bernstein. I shall now prove it in a much simpler way which also shows how any number of such formulæ can be written down. Changing A into $1/AD$, the result becomes

$$\frac{1}{N(A)d^{\frac{1}{2}}} \sum_{(1/AD) \mid \mu}^{\mu \prec x} N\left\{e^{2\pi i \mu w}(x-|\mu|)\right\} = \sum_{A \mid \lambda} N\left[\frac{\sin \pi (w-\lambda)\, x}{\pi(w-\lambda)}\right]^2.$$

The values of λ on the right-hand side (say R) are given by

$$\lambda^{(1)} = \tau_1 a_1 + \tau_2 a_2 + \tau_3 a_3 + \ldots,$$

where $\tau_1, \tau_2, \ldots$ take all integer values from $-\infty$ to ∞ and $a_1, a_2, a_3, \ldots$ are the base of the ideal A. The series R converges absolutely and uniformly in the real variables w of which it is a continuous function, since if we take any positive integers $q_1, q_2, \ldots, q_n$, the number of solutions in λ of the n inequalities $q \leqslant |w-\lambda| < q+1$ is finite and independent of q and w. Hence R is less than a multiple of the convergent series $N \sum\limits_{q=0}^{\infty} q^{-2}$, where when $q = 0$, the corresponding term is replaced by x^2 since $\sin^2 \pi\kappa x < \pi^2\kappa^2 x^2$ if $0 < \kappa < 1$.

Moreover this also applies to the series obtained by differentiating R any number of times with respect to the w's, since by Leibniz's theorem for the differentiation of a product

$$\frac{d^\sigma}{dw^\sigma}\left[\frac{\sin \pi (w-\lambda)\, x}{\pi(w-\lambda)}\right]^2 = O\left[\frac{1}{(w-\lambda)^2}\right],$$

if λ is large. No difficulty arises if $0 < |w-\lambda| < 1$.

Further, if new variables ξ are introduced by means of

$$w^{(1)} = \xi_1 a_1^{(1)} + \xi_2 a_2^{(1)} + \xi_3 a_3^{(1)} + \dots,$$
$$w^{(2)} = \xi_1 a_1^{(2)} + \xi_2 a_2^{(2)} + \xi_3 a_3^{(2)} + \dots,$$
$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

R becomes a periodic function in each of the variables $\xi_1, \xi_2, \dots$ with a period unity in each of them, since a change of ξ_1 into ξ_1+1 is equivalent to a change of τ_1 in the summation to τ_1+1. Hence S can be expanded in the usual Fourier series for $\xi_1, \xi_2, \dots$,

$$S = \sum_{\rho_1, \rho_2, \dots}^{\pm\infty} A_{\rho_1, \rho_2, \dots} e^{2\pi i(\rho_1\xi_1 + \rho_2\xi_2 + \dots)},$$

where $A_{\rho_1, \rho_2, \dots} = \displaystyle\int_0^1 \int_0^1 \sum_{A \mid \lambda} N \left[\frac{\sin \pi(w-\lambda)x}{\pi(w-\lambda)} \right]^2 e^{-2\pi i(\rho_1\xi_1 + \rho_2\xi_2 + \dots)}\, d\xi_1 d\xi_2 \dots .$

The right-hand side is the same as

$$A_{\rho_1, \rho_2, \dots} = \iint_{-\infty}^{\infty} N \left(\frac{\sin \pi w x}{\pi w} \right)^2 e^{-2\pi i(\rho_1\xi_1 + \rho_2\xi_2 + \dots)}\, d\xi_1 d\xi_2 \dots,$$

since the series under the integral sign converge uniformly in ξ.

Replacing the integration in ξ by integration for w, we have

$$dw_1 dw_2 \dots = | a_h^k |\, d\xi_1 d\xi_2 \dots$$
$$= N(A) d^{\frac{1}{2}}\, d\xi_1 d\xi_2 \dots,$$

and $\rho_1\xi_1 + \rho_2\xi_2 + \dots = \sigma_1 w_1 + \sigma_2 w_2 + \dots$ say.

Hence $A_{\rho_1, \rho_2, \dots} = \dfrac{1}{N(A) d^{\frac{1}{2}}} N \displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 \pi w x}{\pi^2 w^2} e^{-2\pi i \sigma w}\, dw$

$$= \frac{1}{N(A) d^{\frac{1}{2}}} N \int_0^{\infty} \frac{2 \sin^2 \pi w x \cos 2\pi\sigma w}{\pi^2 w^2}\, dw.$$

The right-hand integral is

$$\frac{1}{2\pi} \int_0^{\infty} \frac{2\cos 2\sigma w - \cos 2w(x+\sigma) - \cos 2w(x-\sigma)}{w^2}\, dw,$$

and since $\displaystyle\int_0^{\infty} \frac{1-\cos \kappa w}{w^2}\, dw = \frac{\pi}{2} | \kappa |,$

if κ is real, it becomes, when σ and x are both positive,

$$-\tfrac{1}{2}\left[2\sigma - \tfrac{1}{2}(x+\sigma) - \tfrac{1}{2}(x-\sigma)\right] = x - \sigma,$$

if $x > \sigma$. If, however, $\sigma > x$, the corresponding value is

$$-\tfrac{1}{2}\left[2\sigma - \tfrac{1}{2}(x+\sigma) - \tfrac{1}{2}(\sigma - x)\right] = 0.$$

Hence, since the integral is an even function of σ,

$$A_{\rho_1, \rho_2, \ldots} = \frac{1}{N(A)d^{\frac{1}{2}}} N(x - |\sigma|) \text{ or } 0,$$

according as $|\sigma| \prec x$ or not, so that

$$\frac{1}{N(A)d^{\frac{1}{2}}} \sum_{|\sigma| \prec x} N(x - |\sigma|)\, e^{2\pi i (\rho_1 \xi_1 + \rho_2 \xi_2 + \ldots)} = \sum_{A \mid \lambda} N \left[\frac{\sin \pi (w - \lambda) x}{\pi (w - \lambda)}\right]^2.$$

But as $$\rho_1 \xi_1 + \rho_2 \xi_2 \ldots = \sigma_1 w_1 + \sigma_2 w_2 + \ldots,$$

we have $$\rho_1 = \sigma_1 a_1^{(1)} + \sigma_2 a_1^{(2)} + \sigma_3 a_1^{(3)} + \ldots,$$

$$\rho_2 = \sigma_1 a_2^{(1)} + \sigma_2 a_2^{(2)} + \sigma_3 a_2^{(3)} + \ldots,$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

so that by solving for σ_1, σ_2, etc.,

$$\sigma_1 = \rho_1 A_1 + \rho_2 A_2 + \rho_3 A_3 \ldots,$$

$$\sigma_2 = \rho_1 A_1^{(2)} + \rho_2 A_2^{(2)} + \rho_3 A_3^{(2)} \ldots.$$

But as A_1, A_2, … are the base of the ideal $1/AD$ and ρ_1, ρ_2, … are any integers, σ_1, σ_2, … are a conjugate set of numbers, σ_1 being any number of the ideal $(1/AD)$, and the required result follows,

$$\frac{1}{N(A)d^{\frac{1}{2}}} \sum_{(1/AD) \mid \sigma}^{\sigma \prec x} N(x - |\sigma|)\, e^{2\pi i (\sigma_1 w_1 + \sigma_2 w_2 \ldots)} = \sum_{A \mid \lambda} N \left[\frac{\sin \pi (w - \lambda) x}{\pi (w - \lambda)}\right]^2.$$

As this result is practically equivalent to the expansion

$$\sum_{|\sigma| < x} (x - |\sigma|) e^{2\pi i \sigma w} = \sum_{\lambda}^{\pm\infty} \left[\frac{\sin \pi (w - \lambda) x}{\pi (w - \lambda)}\right]^2 \quad (x > 0)$$

in one real variable w, or to the fact that if x and σ are both real

$$\int_{-\infty}^{\infty} \frac{\sin^2 \pi w x}{\pi^2 w^2} e^{-2\pi i \sigma w} dw = |x| - |\sigma| \text{ or } 0,$$

according as $|\sigma| < |x|$ or $|\sigma| > |x|$, it is clear that any number of similar expansions can be written down.

If the field is not totally real, $|w - \lambda|$ occurs in the corresponding formulæ, which are not so simple as those proved above.

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CORRIGENDA.

Page 295. In equation (9 . 4), for " $\Theta\, u_q$ " read " $\Theta^f u_q$."

Page 299. In the diagram, for " $\Delta^2 y^3$ " read " $\Delta^2 y_3$ "

Printed by C. F. Hodgson & Son, 2 Newton Street, Kingsway, W.C.2.

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RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1921–JUNE, 1922.

Thursday, November 17*th*, 1921.

Annual General Meeting.

Mr. H. W. RICHMOND, President, in the Chair.

Present eighteen members.

Dr. J. Prescott was elected a member of the Society.

Messrs. H. D. Anthony, W. F. Beard, H. Bohr, J. C. Burkill, A. Buxton, J. J. Castelain, W. L. Marr, W. N. Richardson, G. Smeal, G. C. Steward, C. E. Wright, and Miss G. D. Sadd were nominated for election.

The Treasurer presented the accounts for the year. Lt.-Col. A. Cunningham was appointed Auditor.

The Officers and Council for the ensuing year were elected. The list is as follows:—President, Mr. H. W. Richmond; Vice-Presidents, Mr. J. E. Campbell, Mr. A. L. Dixon, Prof. W. H. Young; Treasurer, Dr. A. E. Western; Secretaries, Prof. G. H. Hardy, Dr. G. N. Watson; other members of the Council, Dr. T. J. I'A. Bromwich, Prof. L. N. G. Filon, Prof. H. Hilton, Miss H. P. Hudson, Prof. A. E. Jolliffe, Mr. J. E. Littlewood, Mr. E. A. Milne, Dr. J. W. Nicholson, Mr. F. B. Pidduck.

Prof. H. Hilton read a paper "On Plane Curves of Degree $2n$ with Tangents having Bi-n-Point Contact."*

Dr. W. F. Sheppard read a paper "Reciprocal Correspondence of Differences and Sums."*

Dr. T. Stuart made a communication "The Parametric Solutions and Minimum Numerical Solutions of $x^4+y^4+z^4=u^4+v^4$," which was followed by a discussion.

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* Printed in this volume.

The following papers were communicated by title from the Chair:—

- *An Example of an Orthogonal Development whose Sum is everywhere different from the Developed Function: Dr. S. Banach.
- *Expansions and Functions reduced to Zero by the Operator sinh $D-cD$: Prof. A. C. Dixon.
- A Table of Values of 30 Eulerian Numbers, based on a New Method: Mr. C. Krishnamachary and Mr. M. Bhimasena Rao.
- On the Number of Solutions in Positive Integers of the Equation $yz+zx+xy=n$: Mr. L. J. Mordell.
- *Sur les séries entières à coefficients entiers: Mr. G. Pólya.
- Some Applications of Integral Equations to the Theory of Differential Equations: Prof. H. J. Priestley.
- Generating Regions of a Quadric in Space of n Dimensions: Mr. R. Vythynathaswamy.
- *Proofs of some Formulæ enunciated by Ramanujan: Mr. B. M. Wilson.

ABSTRACTS.

Inverse Correspondence of Differences and Sums

W. F. SHEPPARD.

The main purpose is to find the sums corresponding to divided or adjusted differences, *i.e.* to the differences which occur in Newton's interpolation formula for data corresponding to values of a variable at irregular intervals. The inquiry is made by two methods (a) by treating summation as the inverse of differentiation, (b) by considering the set of quantities which correspond inversely to the set of differences. The converse problem of finding a set of quantities corresponding inversely to a given set of linear functions of sums, such as the moments, is also considered.

On the Number of Solutions in Positive Integers of the Equation $yz+zx+xy=n$.

L. J. MORDELL.

The number of solutions, reckoning as $\frac{1}{2}$ those for which one of the unknowns is zero, is three times the number of classes of binary quadratics of determinant $-n$, the classes $(k, 0, k)$ and $(2k, k, 2k)$ being reckoned as $\frac{1}{2}$ and $\frac{1}{3}$ respectively. For example, if $n = 19$, there are twelve solutions, six of which arise from the permutations of 1, 3, 4; three from the permutations of 1, 1, 9; and three from the *six* permutations of 0, 1, 19. Also there are four classes of binary forms, namely (1, 0, 19), (2, 1, 10), and $(4, \pm 1, 5)$.

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* Printed in this volume.

The result for n a prime was given in a slightly different form by Bell in the *Tôhoku Mathematical Journal,* May 1921, Vol. 19, pp. 105–116. His method of proof is entirely different from mine, and his remarks indicate that he is not aware of such a simple solution when n is composite.

My paper will, I hope, be published in the *American Journal of Mathematics.*

Some Applications of Integral Equations to the Theory of Differential Equations

H. J. PRIESTLEY.

Section I establishes the fundamental existence theorem and expresses the solution of the general linear equation as the solution of a Volterra Integral Equation.

Section II applies the Integral Equation to the investigation of those solutions of a second order equation, of Fuchs' type, which remain regular at the singularities of the equation. The results are summarised in the *Records of Proceedings* of the meeting on January 13th, 1921.

Section III discusses the equation

$$\frac{d}{dx}\left[\phi(x)\frac{dy}{dx}\right]+\psi(x)y=f(n)y, \tag{1}$$

where $f(n)$ is a rational algebraic function of n with real coefficients. It is shown that if $\phi(x)$ contains a factor $(x-c)^k$, where $k \leqslant 2$ and if $\psi(x)$ is regular at $x = c$, then, if

$$\operatorname*{Lt}_{x\to c}\frac{(x-c)^2\,\psi(x)}{\phi(x)} \leqslant 0, \qquad \operatorname*{Lt}_{x\to c}\frac{(x-c)^2\,[\psi(x)-f(n)]}{\phi(x)} \leqslant 0,$$

the solutions $y_n(x)$ of (1) which satisfy

$$y_n(c) = 0 \quad \text{or} \quad \phi(c)\frac{d}{dc}\,y_n(c) = 0,$$

and
$$p y_n(a) + q \frac{d}{da} y_n(a) = 0,$$

where p and q are real constants, are real for all real values of x.

Further there is only one such solution $y_n(x)$ for each possible value of n.

Finally, any function which is continuous and has continuous first and second derivatives in the range $a \leqslant x \leqslant c$, and which satisfies the same conditions as $y_n(x)$ at $x = a$ and $x = c$, can be expanded in a series of $y_n(x)$.

A Method of Solving certain Linear Partial Differential Equations

T. W. CHAUNDY.

The method for certain classes of linear equation obtains a solution in series of a generalised hypergeometric type, expresses this series as a definite integral by use of the integrals for the Γ- and the β-functions, and replaces arbitrary powers of certain expressions by arbitrary functions of them. In this way solutions are obtained with arbitrary functions equal in number to the order of the equation. The method is applied to equations of only two terms and to Laplace's equation and certain extensions of it.

Note on Gauss's Quadratic Identity

PANDIT OUDH UPADHYAYA.

Gauss discovered the identity

$$4X = Y^2 - (-1)^{\frac{1}{2}(p-1)} p Z^2,$$

where p is an odd prime,

$$X = x^{p-1} + x^{p-2} + \ldots + x + 1,$$

and Y and Z are polynomials in x with integral coefficients.*

Legendre proved that $Y \equiv 2(x-1)^{\frac{1}{2}(p-1)} \pmod{p}$, and erroneously stated in his *Théorie des Nombres* (3rd ed., 1830, § 512) that, for all values of p, Y might be found by expanding $2(x-1)^{\frac{1}{2}(p-1)}$, and reducing each coefficient to its absolutely least residue, mod. p. He afterwards corrected his mistake. It is now known that Legendre's rule holds good up to $p = 31$ inclusive, and that it fails for $p = 61$.* The author has proved that this rule holds good for $p = 37$, and that it fails for $p = 41$. The coefficient of x^{15} in Y for $p = 41$ is 29, while, if Legendre's rule applied, it would be -12.

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* *Disquisitiones Arithmeticae*, § 357.

Thursday, December 15th, 1921.

Mr. J. E. CAMPBELL, Vice-President, in the Chair.

Present thirty-five members and five visitors.

Messrs. H. D. Anthony, W. F. Beard, H. Bohr, J. C. Burkill, A. Buxton, J. J. Castelain, W. L. Marr, W. N. Richardson, G. Smeal, G. C. Steward, C. E. Wright, and Miss G. D. Sadd, were elected members of the Society.

Messrs. K. P. Dé, J. M. Keynes, J. Littlejohn, C. A. Stewart, and Miss M. T. Budden were nominated for election.

Messrs. J. C. Burkill, I. O. Griffith, E. A. Milne, and Miss G. D. Sadd were admitted into the Society.

The Auditor's report was received and the accounts adopted. A vote of thanks to the Auditor (Lt.-Col. Cunningham) was adopted unanimously.

Mr. J. H. Jeans delivered a lecture "The New Dynamics of the Quantum Theory."

The following papers were communicated by title from the Chair :—

†The Relations between Apolarity and Clebsch's Mapping of the Cubic Surface in a Plane : Prof. W. P. Milne.

The General Theory of Notational Relativity : Mr. H. F. Shaffer.

(1) Note on Gauss's Quadratic Identity, (2) A General Formula in Cubic Forms : Pandit Oudh Upadhyaya.

Gibbs's Phenomenon in Fourier's Series and Integrals : Prof. J. R. Wilton.

On certain Types of Plane Unicursal Sextic Curves : Miss G. D. Sadd.

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* Mathews, *Theory of Numbers*, 1892, §§ 193–5, and references there given.

† Printed in this volume.

SPECIAL GENERAL MEETING.

Mr. J. E. CAMPBELL, Vice-President, in the Chair.

Present thirty-five members.

The following Resolutions were put to the meeting and carried unanimously:—

(1) That the following new Article be adopted:—

20A.—The Council shall have power to agree with any Mathematical Society situate outside the United Kingdom, that members of such Society who are resident outside the United Kingdom, and who are members of the London Mathematical Society, shall be liable to pay an Entrance Fee and an Annual Subscription or a Life Composition Fee of such amounts as may be agreed in lieu of the Entrance Fee and Annual Subscription specified in Articles Nos. 19 and 20, and the Life Composition Fee for the time being payable under the By-laws. Provided that such reduced amounts shall not be less than one-half of the respective amounts which would be payable apart from this Article. Provided also that such other Society shall agree to admit members of the London Mathematical Society not resident in the country in which such Society is situate on such reduced terms below their ordinary Entrance Fee and ordinary Subscription as may be considered satisfactory by the Council. Any such arrangement shall continue for such period as may be agreed by the Council on behalf of the Society and by such other Society.

(2) That the following new Article be adopted:—

20B.—The Council shall have power to reduce the Entrance Fee and Annual Subscription payable by members resident outside the United Kingdom, and elected in or after November, 1921, to such amounts as the Council shall from time to time think fit. Provided that there shall not be more than twelve members at any one time who shall be entitled to reduction under this Article.

ABSTRACTS.

On certain Types of Plane Unicursal Sextic Curves

Miss G. D. SADD.

The sextics dealt with in this paper are those of zero deficiency which have all their double points concentrated at either a double or triple point of the curve. There are four cases to consider, since the third branch at the triple point may meet each of the others in one, two, or three points at the singularity.

The equations of these sextics have been obtained from the following considerations. In every case there is a pencil of conics meeting two of the branches four times each at the singularity. By choosing this pencil suitably, and considering its intersections with the general sextic, simple relations between the coefficients are obtained. If two branches of the sextic have (at least) 8-point contact, there is also a pencil of cubics meeting each of these branches in eight points. A suitable choice of this pencil results in a further simplification. Other relations are obtained by "analysis", sufficient in all to determine the equation of the sextic completely.

The pencil of cubics has also been used to find the parametral coordinates of any point on the curve, whence it becomes evident that there is no real unicursal sextic having all its double points coincident at a single double point of the curve.

The Parametric Solutions of the Indeterminate Quartic $x^4+y^4+z^4 = 2w^4$ *in the Rational Field.*

T. Stuart.

A solution of this equation, homogeneous quadratics in two parameters, has been previously obtained by many arithmeticians.* These are all embraced by the formulation

$$(p^2+2pq)^4+(q^2+2pq)^4+(p^2-q^2)^4 \equiv 2\,(p^2+pq+q^2)^4,$$

or variants thereof, and are clearly connected by $x \pm y \pm z = 0$. This solution is thus only a special solution, and, geometrically, represents a planar locus, viz. $w^2 = x^2 \pm xy + y^2$, $z = \pm(x \pm y)$; consequently no new parametric solution can be deduced. The author, utilising the above well known solution, has obtained *one* rational solution of degree 18, parametric in two variables, viz.

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* A. Cunningham, *Math. Questions, Educ. Times*, 1908, and also *Messenger of Mathematics*, 1908–9; F. Ferrari, *L'intermédiaire des math.*, Vol. 16 (1909), p. 83; E. Miot, *L'intermédiaire des math.*, Vol. 18 (1911), pp. 27, 28; A. Martin, *L'intermédiaire des math.*, Vol 2 (1910), p. 351; A. Gérardin, *Assoc. franç.*, 1910; *Sphinx-Oedipe*, 1910, 1911, 1913.

$$x = (15,\ 60,\ 50,\ 4,\ -154,\ -172,\ -142,\ 4,\ 23,\ 24)(a,\ b)^9,$$

$$y = (24,\ 23,\ 4,\ -142,\ -172,\ -154,\ 4,\ 50,\ 60,\ 15)(a,\ b)^9,$$

$$z = (t^2+2t\tau)\{(9,\ 52,\ 38,\ 52,\ -14,\ 52,\ 38,\ 52,\ 9)(a,\ b)^8\},$$

$$w = (t^2+t\tau+\tau^2)\{(21,\ 20,\ 46,\ 20,\ 74,\ 20,\ 46,\ 20,\ 21)(a,\ b)^8\},$$

where $$a = t^2-\tau^2, \quad b = \tau^2+2t\tau.$$

From this solution the author at once deduces, using his chain formulæ, *eight new parametric solutions.* In none of these solutions are the variables connected by the relation $x \pm y \pm z = 0$.

A very slight modification of the formulæ give, at once, corresponding solutions of the equation

$$x^4+y^4+z^4 = 2A^2w^4,$$

provided A is of the form $\alpha^2+\alpha\beta+\beta^2$ or composed of prime factors of this form; and also in other special cases, given by quartic forms. The author hopes that he will soon present to the Society an exhaustive paper on the complete solution of

$$x^4+y^4+z^4 = 2V^2,$$

with x, y, z quadratics in two parameters. This will include, as a particular case, the problem discussed above, and will, it is hoped, contribute materially to the final elucidation of Euler's problem

$$x^4+y^4+z^4 = w^4,$$

which has hitherto baffled analysis.

The Eight Chain Formulæ can be easily obtained; but their exact formulation in a symmetrical form still presents difficulties which the author has not been able to overcome.

Special Note.

The two minimum numerical solutions obtainable by this formula are

$$116745^4+155873^4+575528^4 = 2\,(484813)^4,$$

$$(237843505)^4+(127201328)^4+(946086833)^4 = 2\,(796419057)^4.$$

Thursday, January 12th, 1922.

Mr. J. E. CAMPBELL, Ex-President, and Mr. A. L. DIXON,
Vice-President, in the Chair.

Present nineteen members and two visitors.

Messrs. K. P. Dé, J. M. Keynes, C. A. Stewart, and Miss M. T. Budden were elected members of the Society.

Messrs. W. R. Dean and B. C. Laws were nominated for election.

Prof. G. N. Watson (Hon. Sec.) reported as to the number of members. The number on November 1st, 1919, was 312. 41 new members were elected in the Session 1919–20, and 25 in the Session 1920–21. 14 have died and 15 have resigned in the course of those two Sessions. The number on November 1st, 1921, was 349.

Miss G. D. Sadd read a paper "Rational Plane Sextic Curves."*

Mr. J. E. Campbell read a paper "On a Class of Surfaces in Euclidean Space which Generate an Expression for the Space Time Interval in Einstein's Geometry of a Particular Form."*

Prof. G. H. Hardy communicated a paper by Dr. F. Lettenmeyer "Neuer Beweis des allgemeinen Kroneckerschen Approximationssatzes."*

Dr. T. Stuart made a communication "Parametric Solutions of certain Diophantine Equations."

The following papers were communicated by title from the Chair:—

*A Theorem concerning Fourier's Series: T. Carleman (communicated by G. H. Hardy).

*Apolarity and the Weddle Surface: W. P. Milne.

ABSTRACTS.

On certain Types of Plane Unicursal Sextic Curves

Miss G. D. SADD.

The sextics discussed are those which have all their ten double points concentrated at a single point of the curve, which may be either a double or a triple point. The equation of the former type, obtained by the method of analysis, shows that there are only two distinct curves, and these are both unreal, as is evident from the expressions for the parametral coordinates.

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* Printed in this volume.

Of the three types of rational plane sextic having a single triple point and no other point-singularities, one only is discussed in detail, *i.e.* the one whose third branch does not touch the other two at the singularity. There are again only two distinct curves of this type, both real.

Neuer Beweis des allgemeinen Kroneckerschen Approximationssatzes

FRITZ LETTENMEYER (*communicated by* G. H. HARDY).

The paper contains a new and particularly simple proof of the following theorem of Kronecker :—

If $\theta_1, \theta_2, \ldots, \theta_m$ *are linearly independent irrationals, i.e. if there is no relation*

$$a_1\theta_1 + a_2\theta_2 + \ldots + a_m\theta_m + a_{m+1} = 0$$

in which the a's are integers, not all zero, and if (x) *denotes the least positive residue of* x *to modulus unity, then the points*

$$P_n = [(n\theta_1), (n\theta_2), \ldots, (n\theta_m)] \quad (n = 1, 2, \ldots)$$

lie everywhere dense in the "cube"

$$0 \leqslant x_\nu \leqslant 1 \quad (\nu = 1, 2, \ldots, m).$$

The central idea of the proof is as follows. We consider the aggregate of vectors

$$(P_n, P_{n+r}) \qquad (n = 1, 2, \ldots;\ r = 1, 2, \ldots),$$

and those among them whose length is less than ϵ: these we call "ϵ-vectors." We then show that it is in general possible to find m ϵ-vectors issuing from P_1 *and not all lying in the same* $(m-1)$-fold. From these we construct a "parallelogram-net" of points P_n; and every point of the cube is within a distance ϵ of one of the lattice-points of the net. This proves the theorem in the general case. The exceptional cases are examined, and shown to correspond to linear relations between the θ's.

A theorem concerning Fourier's series

T. CARLEMAN (*communicated by* G. H. HARDY).

Suppose that $f(x)$ is real, summable in $(0, 2\pi)$, and continuous for $x = a$, and that s_n is the sum of the first $n+1$ terms of its Fourier's series for $x = a$. It was shown by Fejér that

$$\frac{1}{n} \sum_{\nu=0}^{n} [s_\nu - f(a)] \to 0,$$

when $n \to \infty$.

Later, it was shown by Hardy and Littlewood that

$$\frac{1}{n} \sum_{\nu=0}^{n} [s_\nu - f(a)]^2 \to 0, \qquad \frac{1}{n} \sum_{\nu=0}^{n} | s_\nu - f(a) | \to 0.$$

Here it is shown, by a quite different method, that

$$\frac{1}{n} \sum_{\nu=0}^{n} | s_\nu - f(a) |^p \to 0$$

for every $p > 1$, and indeed

$$\frac{1}{n} \sum_{\nu=0}^{n} e^{c | s_\nu - f(a) |} = 1$$

for every $c > 0$. These results state the more the larger are p and c. The hypothesis of continuity for $x = a$ is also generalised.

The interest of such results lies in the light they throw on the possible modes of oscillation of s_n about its average value $f(a)$.

Thursday, February 9th, 1922.

Mr. H. W. RICHMOND, President, in the Chair.

Present eighteen members and three visitors.

Messrs. W. R. Dean and B. C. Laws were elected members of the Society.

Messrs. R. F. Budden, A. E. R. Church, H. J. Davis, H. V. Lowry, W. P. D. MacMahon, J. L. Navarro, S. W. P. Steen, and T. Thompson were nominated for election. Prof. Dr. W. Wirtinger, of the University of Vienna, was nominated for election under By-law 20B.

Prof. H. Hilton read a paper "Conics on the Pseudosphere."

Mr. W. P. D. MacMahon read a paper "The Design of Repeating Polygons in Euclidean Space of Two Dimensions."

Mr. J. E. Littlewood (with Prof. G. H. Hardy) made a communication "Dirichlet's Series with Lines of Singularities."

ABSTRACTS.

Dirichlet's Series with Lines of Singularities

G. H. HARDY *and* J. E. LITTLEWOOD.

In a remarkable memoir published recently in the *Abhandlungen aus dem math. Seminar der Hamburgischen Universität* (Vol. 1, pp. 54–76), Herr E. Hecke has considered the analytic properties of the function $f_1(s) = f_1(s, \theta)$ defined, when $\sigma = \mathfrak{R}(s) > 1$, by the Dirichlet's series

$$f_1(s) = \sum_1^\infty \frac{a_n}{n^s}.$$

Here θ is a quadratic irrational, and

$$a_n = a_n(\theta) = \{n\theta\} = n\theta - [n\theta] - \tfrac{1}{2}.$$

The properties of the function are intimately connected with the problem of the distribution of the numbers $n\theta$ to modulus 1. Hecke shows that $f_1(s)$ is *meromorphic* all over the plane, and that its only possible singularities are simple poles at the points

$$s = -2m + 2ka\pi i \quad (m = 0, 1, 2, \dots;\ k = \dots, -1, 0, -1, \dots),$$

where a is a number depending on θ.

It is interesting to discuss the properties of $f_1(s)$ for general values of θ. We suppose then that θ is any irrational between 0 and 1, and that

$$\theta = \frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\dots,$$

$$\theta = \frac{1}{a_1+\theta_1}, \qquad \theta_1 = \frac{1}{a_2+\theta_2}, \qquad \dots.$$

We say that θ is *of class* λ if λ is the least number such that

$$\frac{(\theta\theta_1 \dots \theta_{n-1})^{\lambda+\epsilon}}{\theta_n} \to 0$$

for every positive ϵ, or (what is the same thing) such that

$$n^{1+\lambda+\epsilon}\,|\sin n\theta\pi| \to 0$$

for every positive ϵ. A quadratic irrational is of class 0; and any algebraic number is of finite class, in virtue of a classical theorem of Liouville.

We write

$$f_1(s)=\Sigma\frac{a_n}{n^s},\qquad f_2(s)=\Sigma\frac{a_n^2-\frac{1}{12}}{n^s},\qquad f_3(s)=\tfrac{1}{12}\Sigma\frac{4a_n^3-a_n}{n^s},\qquad \ldots,$$

the functions of a_n which occur in the numerator being substantially Bernoullian functions. Our conclusions are that

(*a*) *$f_p(s)$ is regular for $\sigma > \sigma_p = 1-\frac{p}{1+\lambda}$*;

(*b*) *if $\lambda > 0$, the line $\sigma=\sigma_p$ is a singular line for $f_p(s)$.**

In particular, $f_1(s)$ is regular for $\sigma > \lambda/(1+\lambda)$, and $\sigma=\lambda/(1+\lambda)$ is a singular line when $\lambda>0$. It is also the line of convergence of the series. It appears then that the case considered by Hecke is entirely exceptional.

It appears very probable that $\sigma=\sigma_p$ is singular even when $\lambda=0$, except when θ is quadratic; but this we are unable to prove.

The Theory of Closed Repeating Polygons in Euclidean Space of Two Dimensions

W. P. D. MACMAHON.

The object of these investigations is to determine the conditions that must be satisfied by a closed rectilinear figure in order that it may be assembled, in side-to-side contact with other closed rectilinear figures identical with it, so as to cover continuously two-dimensional Euclidean space.

In the present communication such polygons only are considered as have no angle equal to 180°.

(1) An analysis of the different kinds of side-to-side contact to which a repeating polygon may be subjected in an assemblage, shows that a closed polygon is a repeat when and only when

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* We have proved this completely only when $p=1$ or $p=2$.

(*a*) its angles are such that they may be distributed into *sets* of 2, 3, or 4 differently lettered angles, the sum of the angles in *each* set being equal to π or 2π (this is called the law of angle distribution);

(*b*) certain lateral conditions closely associated with the law of angle distribution are satisfied.

A general table of these sets for all orders of repeat is drawn up from which among others the following facts are immediately in evidence.

(i) Every triangle and every quadrilateral is a repeat.

(ii) No general case exists for repeats whose order is greater than the fourth.

(iii) Repeats of order $2m$ ($m > 2$) exhibit three distinct types; those of order $2m+1$ ($m > 1$) two distinct types only.

(iv) Every pentagon having two alternate sides equal and parallel is a repeat.

(v) Every hexagon having either two alternate or two opposite sides equal or parallel is a repeat.

(vi) A convex repeat polygon of order greater than the sixth does not exist.

(vii) A repeat polygon having κ re-entering angles does not exist for an order greater than $2\kappa+2$ or less than $2\kappa+6$.

(viii) A repeat polygon of order $2m$ has either $m-3$, $m-2$, or $m-1$ re-entering angles.

(ix) A repeat polygon of order $2m+1$ has either $m-2$ or $m-1$ re-entering angles.

(x) Five consecutive orders of polygon exhibit, as repeats, polygons with any specified order of concavity.

According to the lateral conditions satisfied, the various types of repeat admit of sub-division. An example of each type or sub-type, from the third to the sixth order, is constructed and assembled.

On Conics on the Pseudosphere

HAROLD HILTON.

There is a geometry of algebraic curves on the pseudosphere, for which the constant Gaussian curvature $K = -1/R^2$, somewhat similar to that of algebraic curves on the plane and sphere. The equation of such a curve in geodesic polar coordinates is obtained on replacing r by tanh r/R in the polar equation of a plane algebraic curve.

The properties of pseudospherical conics are analogous to those of sphero-conics. The principal types are the circle, ellipses with two (real) foci, ellipses with two foci and directrices, hyperbolas with two foci and directrices, hyperbolas with no foci but two or four directrices, conics with one focus, conics with one focus and directrix, and conics with neither focus nor directrix.

Thursday, March 9th, 1922.

Mr. H. W. RICHMOND, President, in the Chair.

Present twenty-one members.

Messrs. R. F. Budden, A. E. R. Church, H. J. Davis, H. V. Lowry, W. P. D. MacMahon, J. L. Navarro, S. W. P. Steen, and T. Thompson were elected members of the Society.

Prof. Dr. W. Wirtinger was elected under By-law 20*b*.

Messrs. W. Cowan, H. Cramér, H. A. Hazden, J. B. Homer, H. C. Plummer, and D. Wootton were nominated for election.

Prof. Dr. L. Fejér of Budapest, Prof. Dr. E. Lindelöf of Helsingfors, Prof. Dr. F. Riesz of Szeged, and Prof. Dr. W. Sierpinski of Warsaw were nominated under By-law 20*b*.

The President stated that the date of Prof. Hardy's lecture would be May 11th.

Prof. W. H. Young read a paper "On the Theory of Functions of Two Complex Variables."*

Col. R. L. Hippisley read a paper "The Nodes of the Three-Bar Sextic."*

Lt.-Col. A. Cunningham read a paper "On Least Primitive Roots."*

Mr. R. F. Whitehead made a communication "The Number of Solutions in Positive Integers of the Equation $yz+zx+xy=n$."

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* Printed in this volume.

The following papers were communicated by title from the Chair :—

The Determination of the Criterion to prevent "Hunting" in Hartnell's Governor : T. Stuart.

The Average Value of a Functional : N. Wiener.

ABSTRACTS.

The Nodes of the Three-Bar Sextic

Col. R. L. HIPPISLEY, C.B.

A proof is given of the theorem that the line joining the other extremities of the principal chords passing through a node passes through the other two nodes. From this the sum of the arguments of the elliptic functions by which the coordinates of the nodes can be expressed is determined. A relation is also given between the two arguments for a node. A relation between the areas of the parallelograms of the triple generation theorem when the tracing point is at a node is given, and certain theorems regarding the Simson lines of the nodes.

On Least Primitive Roots

Lt.-Col. ALLAN CUNNINGHAM, R.E. ; H. J. WOODALL ; and T. G. CREAK.

The paper contains a table of the least primitive roots, both positive and negative, of the 2800 odd primes < 25410.

The table shows two important results—

(1) The *maximum* least primitive roots for the 2800 primes are $+31$ (occurs *twice* only) and -39 (*twice*).

(2) The small numbers $\ngtr 12$ supply 2637 of the positive roots and 2688 of the negative roots out of the totals of 2800.

As the use of primitive roots in computation involves very frequent multiplication by the root itself, these results are of great practical importance. These results seem new, inasmuch as Jacobi used 29 primitive roots >100

in computing the *Canon Arithmeticus*, which extends only to primes < 1000, for which ± 21 is the highest really required: and Desmarest in a table giving *one* primitive root for all primes < 10000 gives 36 roots > 1000.

The process used by the joint authors for finding the least primitive roots is described: it is far shorter than that described in Mathews's *Theory of Numbers*, Vol. 1, Art. 20.

On the Number of Solutions in Positive Integers of the Equation

$$yz+zx+xy = n.$$

R. F. WHITEHEAD.

In an abstract of a forthcoming paper, which appeared in the *Records of Proceedings* for November 17th, 1921, Mr. Mordell has stated that the number of solutions is equal to three times the number of classes of binary quadratics of determinant $-n$, reckoning as $\frac{1}{2}$ each solution in which one of the unknowns is zero, and counting each class $(k,\ 0,\ k)$, $(2k,\ k,\ 2k)$ as $\frac{1}{2}$ and $\frac{1}{3}$ respectively.

An elementary demonstration of these results may be derived from the consideration that the equation is equivalent to

$$(x+y)(x+z) = x^2+n.$$

If *e.g.* $z \nless y \nless x$, then $x+z \nless x+y \nless 2x$, and therefore $(x+y,\ \pm x,\ x+z)$ is a reduced form.

There are thus generally six solutions derived by permutation, corresponding with two classes. If, however, one unknown is zero, there is only one class for the six solutions. If two of the unknowns are equal and the third not equal to them or to zero, there are three permutations and one class, since the forms $(2a,\ \pm a,\ c)$ are equivalent, as also the forms $(a,\ \pm b,\ a)$; but if the third unknown is zero there are three solutions with one corresponding class. Finally, if $x = y = z$, there is one solution with a corresponding class, the forms $(2a,\ \pm a,\ 2a)$ being equivalent.

On Gauss's Quadratic Identity

W. E. H. BERWICK.

The polynomials $Y(x)$, $Z(x)$, with which Mr. Upadhyaya's note* is concerned, are to be found tabulated as far as $p = 101$ in Dr. Hermann Teege's *Inaugural-Dissertation* "Über die $\frac{1}{2}(p-1)$-gliedrigen Gaussischen Perioden," Kiel, P. Peters, 1900. When Mr. Upadhyaya's result for $p = 37$ was communicated to *Nature* (June 9th, 1921), in a letter by Dr. G. B. Mathews, similar results for $p = 41, 43, 47, 53, 59, 61$ were calculated by Dr. H. C. Pocklington and Father J. Cullen, who were evidently unfamiliar with Dr. Teege's work, and printed in *Nature* of July 7th. I drew attention to Dr. Teege's work in *Nature* of July 21st, and a further letter by Mr. R. F. Whitehead appeared in the issue of October 13th, 1921.

Thursday, April 20th, 1922.

Mr. H. W. RICHMOND, President, in the Chair.

Present ten members.

Messrs. W. Cowan, H. Cramér, H. A. Hazden, J. B. Homer, H. C. Plummer, and L. Wootton were elected members of the Society.

Prof. Dr. L. Fejér (Budapest), Prof. Dr. E. Lindelöf (Helsingfors), Prof. Dr. F. Riesz (Szeged), and Prof. Dr. W. Sierpinski (Warsaw) were elected under By-law 20B.

Dr. S. Beatty, Mr. W. L. Ferrar, Miss L. E. Hardcastle, and Miss M. Stimson were nominated for election.

Prof. Dr. C. Runge (Göttingen) was nominated under By-law 20B.

Messrs. R. G. Cooke and W. P. D. MacMahon were admitted into the Society.

The President referred to the loss sustained by the Society in the death of Prof. G. B. Mathews.

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* See *Records of Proceedings at Meetings* for January 12th, 1922.

Prof. Hardy communicated a paper by Prof. H. Bohr, "Another Proof of Kronecker's Theorem."*

Prof. Hilton made an informal communication "On a Lamé Family of Surfaces."

Lt.-Col. Cunningham announced the publication at Nancy of an extensive table of indices.

The following papers were communicated by title from the chair :—

*On Water Waves due to Disturbance beneath the Surface : H. Lamb.

Cauchy's Theorem on the Integral of a Function between Imaginary Limits : G. Mittag-Leffler.

ABSTRACT.

On a Lamé Family of Surfaces

HAROLD HILTON.

Suppose u, v, w are functions of the rectangular Cartesian coordinates x, y, z of any point, and that

$$dx^2+dy^2+dz^2 \equiv a\,du^2+b\,dv^2+c\,dw^2+2f\,dv\,dw+2g\,dw\,du+2h\,du\,dv,$$

so that $\qquad a \equiv x_1^2+y_1^2+z_1^2, \quad f \equiv x_2x_3+y_2y_3+z_2z_3, \quad \&c.$

where suffixes 1, 2, 3 denote partial differentiation with respect to u, v, w respectively.

Write α for $(bg-fh)/(ab-h^2)$, β for $(af-gh)/(ab-h^2)$. Then $w =$ *constant* is a Lamé family, forming a triply orthogonal system with two other singly infinite families of surfaces, if

$$\begin{vmatrix} a & h & b \\ a' & h' & b' \\ a'' & h'' & b'' \end{vmatrix} = 0,$$

where

$$a' \equiv a_3-a_1\alpha-a_2\beta-a\alpha_1+a\beta_2-2h\beta_1,$$

$$h' \equiv h_3-h_1\alpha-h_2\beta-a\alpha_2-b\beta_1,$$

$$b' \equiv b_3-b_1\alpha-b_2\beta+b\alpha_1-b\beta_2-2h\alpha_2\,;$$

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* Printed in this volume.

and a'', h'', b'' are derived from a', h', b', a, β in the same way as a', h', b' are derived from a, h, b, a, β.

Thursday, May 11th, 1922.

Mr. H. W. RICHMOND, President, in the Chair.

Present thirty-three members and five visitors.

Dr. S. Beatty, Mr. W. L. Ferrar, Miss L. E. Hardcastle, and Miss M. Stimson were elected members of the Society.

Prof. Dr. C. Runge (Göttingen) was elected under By-law 20B.

Mr. E. C. Francis was nominated for election.

Prof. Dr. E. Schmidt (Berlin) was nominated under By-law 20B.

The President referred to the loss sustained by the Society in the death of Sir A. B. Kempe, formerly Treasurer (1883–1891) and President (1892–1893).

Prof. G. H. Hardy delivered a lecture "The Elements of the Analytic Theory of Numbers."

The following papers were communicated by title from the Chair:—

On the Displacement of Tensors: P. Dienes.

Some Limiting Forms in the Theory of Integral Equations: A. C. Dixon.

On Harmonic Equations and in particular the Equations associated with Parabolic and Circular Boundary Problems: E. L. Ince.

*Sextactic Cones and Tritangent Planes of the same System of a Quadri-Cubic Curve: W. P. Milne.

*On the Integer Solutions of the Equation $y^2 = ax^3+bx^2+cx+d$: L. J. Mordell.

*On the Manner of Divergence of the Legendre Series of an Integrable Function: B. M. Wilson.

On the Fundamental Theorem of Integration for Multiple Integrals: W. H. Young.

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* Printed in this volume.

ABSTRACTS.

Some Limiting Forms in the Theory of Integral Equations

A. C. DIXON.

1. For the integral equation (Kellogg, *Math. Ann.*, Vol. 58)

$$\phi(x)-\lambda\int_0^\pi \phi(t)\{\cot(t-x)+a\}\,dt=f(x),$$

the special values of λ are $\pm 1/\iota\pi$, $1/a\pi$, and the special functions are

$$\begin{array}{lll} \text{for} & 1/\iota\pi, & e^{2n\iota x} \quad (n=1,2,3,\ldots), \\ \text{for} & -1/\iota\pi, & e^{-2n\iota x} \quad (n=1,2,3,\ldots), \\ \text{for} & 1/a\pi, & 1. \end{array}$$

2. The equation $\phi(x)-\lambda\int_0^\infty \phi(t)\dfrac{dt}{t+x}=f(x)$,

is satisfied by $\phi(x)=f(x)+\dfrac{\tan b\pi}{\pi}\,x\displaystyle\int_0^\infty f(t)\,\dfrac{\left(\dfrac{x}{t}\right)^b-\left(\dfrac{x}{t}\right)^{1-b}}{x^2-t^2}\,dt$,

if $\sin b\pi=\lambda\pi$ and the real part of b is between 0 and 1; the index $1-b$ is to be changed into $-1-b$ if the real part of b is between 0 and -1. The solution fails if λ is a pure imaginary.

3. Some other types of functional equation may be treated as limiting forms of integral equations.

For instance, take $\phi(x)-\lambda\phi(x+a)=f(x)$, (3 . 1)

where ϕ, f are supposed to have the period π, and a is real. Since

$$\phi(x+a)=\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\int_0^\pi e^{2n\iota(x+a-t)}\phi(t)\,dt,$$

the equation has the form of an integral equation.

If $f(x)$ is expanded in a Fourier series $\Sigma A_n e^{2n\iota x}$, $\phi(x)$ is

$$\Sigma A_n e^{2n\iota x}/(1-\lambda e^{2n\iota a}).$$

The solution fails when $\lambda=e^{-2n\iota a}$, n being integral, unless $f(x)$ is such that

$$\int_0^\pi e^{-2n\iota a} f(x)\,dx=0,$$

and when this condition is satisfied the coefficient of $e^{2n\iota a}$ in $\phi(x)$ is arbitrary.

If $\psi(x)$ is another function with period π,

$$\int_0^\pi \{\phi(x)-\lambda\phi(x+a)\}\,\psi(x)\,dx = \int_0^\pi \{\psi(x)-\lambda\psi(x-a)\}\,\phi(x)\,dx,$$

and thus the associated equation is the similar one

$$\psi(x)-\lambda\psi(x-a) = g(x). \tag{3 . 2}$$

The special values of λ for which the solution fails are all on the circle $|\lambda| = 1$, whereas in ordinary cases the sequence of them tends to infinity.

Similarly take the equation

$$\phi(x)-\lambda\phi'(x+a) = f(x), \tag{3 . 3}$$

where ϕ, f are still periodic and $\phi'(x+a)$ may be written

$$\frac{1}{\pi}\sum_{-\infty}^{\infty}\int_0^\pi 2n\iota\, e^{2n\iota(a+x-t)}\,\phi(t)\,dt.$$

The solution here is, if $f(x) = \Sigma A_n e^{2n\iota x}$,

$$\phi(x) = \Sigma A_n e^{2n\iota x}/(1-2n\iota\lambda\, e^{2n\iota a}). \tag{3 . 31}$$

The special values of λ are $e^{-2n\iota a}/2n\iota$, n being integral; for this special value the solution fails unless

$$\int_0^\pi e^{-2n\iota x} f(x)\,dx = 0,$$

and if this condition is fulfilled the coefficient of $e^{2n\iota x}$ in $\phi(x)$ is arbitrary. The associated equation is the similar one

$$\psi(x)+\lambda\psi'(x-a) = g(x). \tag{3 . 4}$$

When $a = 0$, the equation (3 . 3) becomes the differential equation

$$\phi(x)-\lambda\phi'(x) = f(x). \tag{3 . 5}$$

The value of $\phi(x)$ given by (3 . 5) contains a complementary function

$$C\,e^{x/\lambda},$$

in addition to the terms of (3 . 31).

The complementary function has not, however, the period π, unless λ has one of the special values included in the form

$$1/2n\iota,$$

and thus the two results are reconciled.

The special values of λ in this case form two sequences tending to zero.

On the Integer Solutions of the Equation $y^2 = ax^3+bx^2+cx+d$

L. J. MORDELL.

In a paper appearing shortly in the *Messenger of Mathematics*, I have given a very simple proof that this equation has only a finite number of solutions in integers. The present paper contains another proof which I hoped would enable me to prove the obvious extension to the quartic. It led me, however, to a proof of a far more important proposition—namely, that the method of infinite descent applies to the indeterminate equations (of genus unity)

$$z^2 = ax^4+bx^3y+cx^2y^2+dxy^3+ey^4,$$

or to

$$0 = f(x, y, z),$$

where f is a ternary cubic in x, y, z; that is to say, that if either of these equations has an infinite number of integral solutions, all of them can be expressed rationally in terms of a *finite* number. So all the rational solutions of

$$y^2 = 4x^3-g_2x-g_3$$

are given by

$$x = \wp(u), \quad y = \wp'(u),$$

with

$$u = m_1u_1+m_2u_2+\ldots+m_nu_n,$$

for a *finite* number of values of u_1, u_2, ..., where m_1, m_2, ... take all integer values, positive, negative, and zero. An account of this will be given in a paper "On the Solution of the Indeterminate Equations of the Third and Fourth Degrees," to be presented shortly to the Cambridge Philosophical Society.

Thursday, June 8th, 1922.

Mr. H. W. RICHMOND, President, in the Chair.

Present twenty members.

Mr. E. C. Francis was elected a member of the Society.

Prof. Dr. E. Schmidt (Berlin) was elected under By-law 20B.

Mr. L. C. Piggott was nominated for election.

Mr. R. F. Budden, Mr. C. E. Wright, and Miss M. Stimson were admitted into the Society.

The President spoke briefly concerning M. Camille Jordan, late honorary member of the Society.

It was announced that the Annual General Meeting was fixed for November 9th, 1922.

Mr. L. J. Mordell communicated two papers (1) "On Gauss's Sums and the Law of Quadratic Reciprocity in any Field," (2) "On the Integral Solutions of the Equation $ey^2 = ax^3+bx^2+cx+d$."*

Mr. J. E. Campbell read a paper "The Deduction of the Ground-Form of Einstein's Statical Gravitational Field from Gauss's Expression for the Ground-Form in Ordinary Two-Way Space."*

Prof. G. H. Hardy read a paper (written in collaboration with Mr. J. E. Littlewood) "Fourier's Series and Power Series."

Lt.-Col. A. Cunningham read a paper "On Pellian Chains."

The following papers were communicated by title from the chair :—

*Commutative Ordinary Differential Operators: J. L. Burchnall and T. W. Chaundy.

The Displacement of Tensors: P. Dienes.

*On the General Invariant Theory of Quadrics: H. W. Turnbull.

Surface Waves on Limited Sheets of Water: J. Vint.

The Euclidean Geometry of Angle: D. K. Picken.

*A Relation between Bertrand's and Kelvin's Theorems on Impulses: G. I. Taylor.

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* Printed in this volume.

ABSTRACTS.

On Gauss's Sums and the Law of Quadratic Reciprocity in any Field

L. J. MORDELL.

In a recent number of these *Proceedings* I have proved very simply Hecke's reciprocity formulæ for the Gauss's sums in a quadratic field. I notice now that the proof becomes much simpler and more symmetrical by the use of a contour integral involving two variables of integration instead of one. Moreover, the new method gives the corresponding formulæ for any algebraic field, so that an easy and elementary proof of the law of quadratic reciprocity in the field can be deduced.

Hecke, it may be noted, has proved the reciprocity formulæ very elegantly, by assuming the law of Quadratic Reciprocity in the field, and also his functional equation for the Dedekind Zeta-function.

On Pellian Chains

Lt.-Col. ALLAN CUNNINGHAM, R.E.

This paper develops the *chain-relations* among the elements (τ', υ'), (τ, υ) of the Pellian equations

$$\tau'^2 - D\upsilon'^2 = \mp k^*, \qquad \tau^2 - D\upsilon^2 = +1.$$

A *chain-series* is a series of *composites* $N_1 = L_1 M_1$, $N_2 = L_2 M_2$, &c., which are *similar functions* of two elements x_r, y_r, such that

$$M_{r-1} = L_r, \quad M_r = L_{r+1} \quad \text{(for all values of } r\text{)}.$$

By the substitutions

$$\tau_1' = \sqrt{k} \cdot \tfrac{1}{2}(y^{\frac{1}{2}} \mp y^{-\frac{1}{2}}), \qquad \upsilon_1' = \sqrt{k} \cdot \tfrac{1}{2}(y^{\frac{1}{2}} \pm y^{-\frac{1}{2}})/\sqrt{D},$$

$$\tau_1 = \tfrac{1}{2}(y + y^{-1}), \qquad \upsilon_1 = \tfrac{1}{2}(y - y^{-1})/\sqrt{D},$$

---

* The only values of k used are such as are the middle terms of the series of partial quotients in the expansion of $\sqrt{D}$ as a continued fraction: they are the last entries in the second lines of each D in Degen's tables.

it is shown that the following numbers (N) are *chain-series* :—

$$N'_{r,\,s} = \tau_r'^2 + D\upsilon_s'^2, \qquad N_{r,\,s} = \tau_r^s + D\upsilon_s^2,$$

$$N''_{r,\,s} = k\tau_r'^2 + D\upsilon_s'^2, \quad N''_{r,\,s} = k\tau_r'^2 - D\upsilon_s'^2.$$

Further that the following Aurifeuillians are *chain-series* :—

$$N_r = \tau_r'^4 + 4\upsilon_r'^4, \quad N_r = \tau_r^4 + 4\upsilon_r^4,$$

$$N_r = (\tau_r'^6 + 3^3\upsilon_r'^6)/(\tau_r'^2 + 3\upsilon_r'^2), \quad N_r = (\tau_r^6 + 3^3\upsilon_r^6)/(\tau_r^2 + 3\upsilon_r^2).$$

On the General Invariant Theory of Quadrics

H. W. TURNBULL.

The property of equivalence which gives rise to a prepared system of symbolic factors in terms of which the concomitants of three quadrics can be expressed (see *Proc. London Math. Soc.*, Ser. 2, Vol. 20, p. 472) holds for any number of quadrics. With the same notation a_x^2, $(Ap)^2$, u_α^2 to denote a quadric f in point, line and plane coordinates, and with similar sets for an indefinite number of other quadrics $f' = b_x^2$, $f'' = c_x^2$, $f''' = d_x^2$, ..., it may be shown that the necessary types of factor for a prepared system are:—

$$a_x,\ u_\alpha,\ (abcd),\ (\alpha\beta\gamma\delta),\ (abcu),\ (\alpha\beta\gamma x),$$

$$\begin{array}{cc} (A_1A_2), & (\theta A_1\phi), \\ (A_1A_2A_3A_4A_5A_6), & (\theta A_1A_2\phi), \\ (A_1A_2 \ldots A_{10}), & \ldots \quad \ldots \\ & (\theta A_1A_2 \ldots A_8\phi), \end{array}$$

where A_i denotes one of the symbols $p, A, B, \ldots$; θ, ϕ denote two of $u, x, a, \alpha, b, \beta, \ldots$; and where certain simple restrictions hold.

Thus the total degree of a bracket is a multiple of four, reckoning u, a of degree one; p, A of degree two; x, α of degree three. Again, no two symbols in one bracket may be alike, except θ and ϕ. Also when three or more $A_1A_2A_3$ occur they may be interchanged as in a determinant.

These brackets may be readily expressed in terms of the original symbolic factors. For example, if $A = (aa')$, $C = (cc')$, $E = (ee')$, then

$$(ABCDEF) = \Sigma(aBc)(c'De)(e'Fa'),$$

the sum extending to eight terms obtained by permuting aa', cc', ee'

separately. Similarly, to illustrate the structure of a (θ, ϕ) bracket,

$$(aBCDE\rho) = \Sigma(aBc)(c'De)\,e'_{\rho},$$

a four-term series.

Finally, this prepared system shows clearly the principle of duality, answering to geometrical reciprocation. The three brackets on the left are self-reciprocal. Those on the right are reciprocal in pairs when θ, ϕ are both of type a or both of type α; but they are self-reciprocal when an even number of A symbols occur.

The Euclidean Geometry of Angle

D. K. PICKEN.

The paper is based on the figure consisting of two intersecting straight lines, intermediate between straight line and (complete) triangle. This figure is called in the paper "the complete angle," and for it the notations (l_1, l_2) or (AB, CD), or, simply AOB—equivalent to (OA, OB)—are used. Congruence of complete angles, expressed by $AOB \equiv A'O'B'$, is shown to be the proper characteristic angle relation of elementary geometry; by use of it geometrical theory is both simplified and made more general. The following propositions, *e.g.*, are quite general :—

$ABC \equiv ABD$ implies that B, C, D are collinear,
$ABC \equiv DCB$,, AB is parallel to CD,
$ABC \equiv ADC$,, A, B, C, D are concyclic,
$ABC \equiv DAC$,, AD is tangential to the circle ABC,
$ABC \equiv CDA$,, the circles ABC, ACD are congruent ;

and every congruence $PQR \equiv UVW$, obtained by using 3-permutations of A, B, C, D, can be given immediate general geometrical interpretation. These facts provide a most powerful elementary method in geometry, of wide applicability.

Fourier Series and Power Series

G. H. HARDY *and* J. E. LITTLEWOOD.

The "convergence problem" for Fourier's series is still unsolved: no one has succeeded in stating any simple system of conditions both necessary and sufficient for the convergence of the Fourier series of a given function $f(x)$. The same is true of the problem of the summability of the series by Cesàro's means, so long as the order of the means considered is fixed. Thus continuity is a sufficient condition for summability $(C, 1)$, but not a necessary condition; and, in general, none of the known sufficient conditions is necessary, and none of the necessary conditions sufficient.

There is, however, a problem which does admit a simple and complete solution. We may ask, not "when is the series summable $(C, 1)$, or summable (C, k), for some definite k?" but "when is it summable by *some* Cesàro mean?" The solution of this problem is contained in the following theorem.

THEOREM 1.—*In order that the Fourier series of $f(x)$, for $x=a$, should be summable by Cesàro's means, to sum s, it is necessary and sufficient that*

$$\phi_k(x) = \frac{1}{x}\int_0^x \frac{dx_1}{x_1}\int_0^{x_1}\frac{dx_2}{x_2}\cdots\int_0^{x_{k-2}}\frac{dx_{k-1}}{x_{k-1}}\int_0^{x_{k-1}}\phi(x_k)\,dx_k,$$

where $$\phi(x) = \tfrac{1}{2}\{f(a+x)+f(a-x)-2s\},$$

should, for some value of k, tend to zero with x.

We may express this more tersely by saying that *it is necessary and sufficient that $\phi(x)$ should be continuous on the average for $x=0$.*

There is a similar theorem for power series $\Sigma a_n x^n$ on the circle of convergence.

THEOREM 2.—*In order that Σa_n should be summable by Cesàro's means, to sum s, it is necessary and sufficient that*

(1) *$f(x) = \Sigma a_n x^n$ should be of finite order for $|x| < 1$, that is to say that*

$$f(x) = O(1-|x|)^{-K}$$

for some value of K;

$$f_k(x) = \frac{1}{1-x}\int_x^1 \frac{dx_1}{x_1}\int_{x_1}^1 \frac{dx_2}{x_2} \dots \int_{x_{k-2}}^1 \frac{dx_{k-1}}{x_{k-1}}\int_{x_{k-1}}^1 f(x_k)\,dx_k \tag{2}$$

should tend to s when x tends to 1.

The integrations are rectilinear; but x may tend to 1 *in any manner from inside the circle.*

The condition (1) is equivalent to

$$a_n = O(n^L) \tag{3}$$

for some L.

These theorems may recall Bohr's work in the theory of Dirichlet's series $\Sigma a_n n^{-s}$, where a similar phenomenon may be observed. The "convergence problem" defies solution, and so does that of summability (C, k) for any particular k; but the problem of summability by *some* mean has a definite solution. The series is summable for $\sigma > \sigma_0$, by some Cesàro mean, if and only if the function $f(\sigma+it) = f(s) = \Sigma a_n n^{-s}$ is regular, and of finite order in t, for $\sigma > \sigma_0$.

LIBRARY

Presents.

Between December 31st, 1921, and December 31st, 1922, the following presents were made to the Library by their respective authors and publishers:—

Boussinesq, J.—Cours de Physique Mathématique de la Faculté des Sciences, Compléments au Tome 3.
Cunningham, Allan.—Haupt-Exponents, Residue Indices, Primitive Roots, and Standard Congruences.
Freund, Margarete.—Über das Potential mehrfach belegter Flächen.
Gutzmer, A.—Zum Jubilaum der Logarithmen.
Juvet, G.—Introduction au Calcul Tensoriel et au Calcul Différentiel Absolu.
Lévy, Paul.—Leçons d'Analyse Fonctionnelle.
Numair, V.—Calculating Prodigies, and a Clue to the Evolution of Arithmetic.
See, T. J. J.—New Theory of the Aether, 5th and 6th papers.

Åbo Academy: Acta Math. et Phys., no. 1.
Brussels: Académie Royale de Belgique, Bulletin de la Classe des Sciences, tome 7, nos. 11, 12; tome 8, nos. 1–7. Annuaire, 1922.
Calcutta: Institution of Engineers (India), Journal, vol. 2, April. List of Members.
L'Enseignement Mathématique, 22me année, nos. 1–4.
Erlangen: Physikalisch-medizinische Sozietät, Sitzungsberichte, bände 52, 53.
Fundamenta Mathematicae, tome 3.
Göttingen: K. Gesellschaft der Wissenschaften, Nachrichten, Philologisch-historische Klasse, 1921, heft 1. Geschaftliche Mitteilungen, 1921.
Internationale Mathematische Unterrichtskommission, Berichte und Mitteilungen, erste folge, heft 12.
Kansas University: Science Bulletin, vol. 13, nos. 1–9.
Kyoto: Imperial University, College of Science, Memoirs, vol. 4, nos. 7, 8; vol. 5, nos. 1–6.
Masaryk University: Faculté des Sciences, Publications, nos. 8, 11–15.
Mathematical Gazette, vol. 11, nos. 156, 157.
Nation and Athenæum, vol. 30, nos. 15–26; vol. 31, nos. 1–27; vol. 32, nos. 1–11.
Nautical Almanac, 1924.
Sendai: Tôhoku Imperial University, Science Reports, vol. 10, nos. 5, 6; vol. 11, nos. 1–3.
Sendai: Tôhoku Mathematical Journal, vol. 20, nos. 1–4; vol. 21, nos. 3, 4.
Tokyo: Physico-Mathematical Society of Japan, Proceedings, vol. 3, no. 11; vol. 4, nos. 1–10.
Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht aller Schulgattungen: Sonderabruck, Jahrgang 46, hefte 1, 5.

Exchanges.

American Journal of Mathematics, vol. 44, no. 1.
Bologna: Reale Accademia delle Scienze, Rendiconti, vols. 24, 25.
Bologna: Unione Matematica Italiana, Bollettino, anno 1, no. 1.

Boston (Mass.): American Academy of Arts and Sciences, Proceedings, vol. 57, nos. 11–15.
Bulletin des Sciences Mathématiques, vol. 46, nos. 1–11.
Cambridge Philosophical Society, Proceedings, vol. 21, pts. 1–3.
Copenhagen: K. Danske Videnskabernes Selskabs, Oversigt, June 1919, May 1920; Mathematisk-fysiske Meddelelser, vol. 2, nos. 1, 2, 3, 5, 12; vol. 3, nos. 1–11.
Dublin: Royal Irish Academy, Proceedings, vol. 36, Section A, nos. 1–4.
Edinburgh: Royal Society, Proceedings, vol. 41, pt. 2; vol. 42, pts. 1, 2.
Firenze: Bibliotica Nazionale Centrale, Bollettino, nos. 246–257.
Hamburg University: Abhandlungen aus dem Mathematischen Seminar, baud 1, hefte 2–4.
Jahrbuch über die Fortschritte der Mathematik, band 45, heft 3.
Journal für Mathematik, band 152, hefte 1, 2.
Lancaster, Pa.: American Mathematical Society, Bulletin, vol. 27, nos. 9, 10; vol. 28, nos. 1–6. Transactions, vol. 23, no. 1.
La Plata: Universidad Nacional, Contribución al Estudios de las Ciencias físicas y matemáticas, no. 52.
London: Physical Society, Proceedings, vol. 34, nos. 1–5.
London: Royal Astronomical Society, Monthly Notices, vol. 82, nos. 1–9.
London: Royal Society, Philosophical Transactions, vol. 222, nos. 598–609. Proceedings, vol. 100, nos. 705–707; vol. 101, nos. 714–716.
Madras: Indian Mathematical Society, Journal, vol. 13, no. 6; vol. 14, nos. 1–4.
Massachusetts: Institute of Technology, Journal of Mathematics and Physics, vol. 1, no. 2.
Mathematische Annalen, bände 85, 86, 87.
Mathematische Zeitschrift, band 11, hefte 3, 4; band 12, hefte 1, 2; band 13, hefte 1–4; band 14, hefte 1–4; band 15, hefte 1, 2.
Milan: Reale Instituto Lombardo, Rendiconti, vol. 54, fasc. 2–20; vol. 55, fasc. 1–15.
Naples: Accademia delle Scienze Fisiche e Matematiche, Rendiconti, vol. 26, fasc. 7-12; vol. 27, fasc. 1–12; vol. 28, fasc. 1–8.
National Physical Laboratory: Collected Researches, vol. 16; Report for 1921.
Nature: vol. 109, nos. 2723–2747; vol. 110, nos. 2748–2772.
Oporto: Academia Polytecnica, Annaes Scientificos, vol. 13, no. 4; vol. 14, nos. 1–3.
Palermo: Rendiconti del Circolo Matematico, tomo 46, fasc. 1.
Paris: École Polytechnique, Journal, cahier 21.
Paris: Société Mathématique de France, Bulletin, tome 49, fasc. 1–4; tome 50, fasc. 1, 2.
Philadelphia: American Philosophical Society, Proceedings, vol. 60, nos. 2–4; vol. 61, nos. 1–3; List of Members.
Prague: Casopis pro pestováni Mathematisky a Fysiky, Rocnik 44–48, 50, 51.
Rome: Reale Accademia dei Lincei, Atti, Rendiconti, vol. 31 (1re sem.), nos. 1–12; vol. 31 (2me sem.), nos. 1, 2.
South African Journal of Science, vol. 18, nos. 1–4.
Toronto: Royal Canadian Institute, Transactions, vol. 13, pt. 2; vol. 14, pt. 1.
Toulouse University: Faculté des Sciences, Annales, série 3, tomes 11, 12.
Upsala: Royal Society of Sciences, Nova Acta, vol. 5, fasc. 1.
Washington: National Academy of Sciences, Proceedings, vol. 1, no. 12; vol. 3, no. 6; vol. 5, no. 10; vol. 8, nos. 1–11.
Washington: U.S. Naval Observatory, American Ephemeris and Nautical Almanac, 1924.
Zeitschrift für angewandte Mathematik und Mechanik, band 2, hefte 3–5.
Zürich: Naturforschende Gesellschaft, Vierteljahrsschrift, jahrgang 66, hefte 3, 4; jahrgang 67, hefte 1, 2.

Purchased.

Mathematische Annalen, band 83, hefte 3, 4; band 84, 85.
Messenger of Mathematics, vol. 50, nos. 4–12; vol. 51, nos. 1–10; vol. 52, nos. 1, 2.
Quarterly Journal of Mathematics, vol. 49, nos, 2, 3.

LIST OF MEMBERS ELECTED DURING THE SESSION 1921–1922.

† indicates that the Member has paid the life-composition, and is a Member for life.

Anthony, Rev. Herbert Douglas, M.A.; *The College, Richmond, Surrey.*
Beard, W. F., M.A.; *The Grammar School, Wakefield.*
Beatty, Samuel, M.A., Ph.D.; *The University, Toronto, Canada.*
Bohr, Dr. Harald; *St. Hans Torv* 24, *Copenhagen, Denmark.*
Budden, Mary T.; *Newnham College, Cambridge.*
Budden, Raymond Frederick, B.A.; 21 *Elmwood Gardens, Acton Hill, London, W.* 3.
†Burkill, J. C.; 43 *Defoe Avenue, Kew, Surrey.*
Buxton, Arnold, M.A.; *The Technical College, Cardiff.*
Castelain, Jean Joseph; *Avenue Labourdonnais, Quatres Bornes, Mauritius.*
†Church, Albert Edgar Robert, M.A.; *Sir John Cass Technical Institute, Jewry Street, London, E.C.* 3.
Cowan, William, B.Sc.; *The Technical College, Bradford, Yorkshire.*
Cramér, Harald; *Mörby, Stocksund, Sweden.*
Davis, Harold James, B.A.; 37 *Bradshaw Lane, Holmfield, Halifax.*
Dé, Rai Sahib K. P.; *University College, Rangoon, Burma.*
Dean, William Reginald, B.A.; 47 *Kitto Road, New Cross, London, S.E.* 14.
Fejér, Prof. Dr. Leopold; *Falk-Miksa-utca* 15, *Budapest V, Hungary.*
Ferrar, William Leonard, M.A.; *University College, Bangor, North Wales.*
Francis, E. C., B.A.; *Trinity College, Cambridge.*
Hardcastle, Lucy Emma, M.Sc.; *Green Rock House, Blakenall, Walsall.*
Hayden, Harold A., B.Sc.; *University College of South Wales, Cardiff.*
Homer, J. B., B.Sc.; 48 *Stretton Road, The Fosse, Leicester.*
Keynes, John Maynard, M.A., C.B.; *King's College, Cambridge.*
Laws, Bernard Courtney, D.Sc., A.R.C.Sc.; *Claverton, Cheam Road, Sutton, Surrey.*
Lindelöf, Prof. Dr. Ernst; *Sandrikskajan* 15, *Helsingfors, Finland.*
Lowry, Hugh Vernon, B.A.; 7 *Madeira Road, Bournemouth.*
MacMahon, William Percy Dartrey, Ph.B.; *Shafto House, Bilton Road, Rugby.*
Marr, William Law, D.Sc.; *High School for Girls, Aberdeen.*
Navarro, Justo López; *P.O. Box* 1220, *Manila, Philippine Islands.*
Plummer, Henry Crozier, M.A., F.R.S.; 41 *Half Moon Street, London, W.* 1.
Prescott, John, D.Sc.; *College of Technology, Manchester.*
Richardson, William Norman; 4 *Bath Island Road, Karachi, India.*
Riesz, Prof. Dr. Friedrich; *University of Szeged, Hungary.*
Runge, Prof. Dr. C.; *University of Göttingen.*
Sadd, Grace Dorothy, B.A.; *Bedford College, Regent's Park, London, N.W.* 1.
Schmidt, Prof. Dr. Erhard; *Altonaerstr.* 30, *Berlin, N.W.* 23.
Sierpinski, Prof. Dr. Walther; *Uniwersytet Warszawski, Varsovie, Poland.*
Smeal, Glenny, B.Sc.; 51 *Clarendon Road, Leeds.*
Steen, Stourton William Peile; 23 *De Freville Avenue, Cambridge.*
Steward, George Cotan, B.A.; *The University, Leeds.*
†Stewart, Caleb Andrew, M.A., B.Sc. *The University, Western Bank, Sheffield.*
Stimson, Margaret, M.A.; 30 *Gunter Grove, Chelsea, London, S.W.* 10.
Thompson, Thomas, B.A.; 61 *Meldon Terrace, Heaton, Newcastle-upon-Tyne.*
Wirtinger, Prof. Dr. Wilhelm; *IX Strudlhofgasse* 4, *Vienna, Austria.*
Wootton, David, B.Sc.; *The Secondary School, Weymouth.*
Wright, Charles Edward, M.Sc.; 7 *Carlton Villas, Belle Grove, Welling, Kent.*

The following changes of address have been notified:—

BENNETT, AGNES ELIZABETH; 18 *Willoughby Road, Hampstead, London, N.W.* 3.
BRILL, JOHN; *Connaught Club*, 75 *Seymour Street, London, W.* 2.
CULLEN, Rev. JAMES, S.J.; *Wimbledon College, S.W.* 19.
DALLAS, ROBERT JOHN, M.A.; 197 *Brecknock Road, Tufnell Park, N.* 19.
DARLING, H. B. C.; *The Ridge, Redlynch, Wilts.*
HICKS, WILLIAM MITCHINSON; *The Crowhurst Hotel, Crowhurst, Surrey.*
LONG, MARJORIE; *c.o. The Librarian, Bedford College, Regent's Park, London, N.W.* 1.
PROUDMAN, JOSEPH, M.A., D.Sc., F.R.A.S.; *Stapley, Grassendale Park, Liverpool.*
ROSE-INNES, J.; *Royal Institution, W.* 1.
TURNBULL, H. W., M.A., F.R.S.E.; 2 *Queen's Terrace, St. Andrews.*

Members deceased during the Session 1921–1922:—

ERNEST CARPMAEL.
CAMILLE JORDAN (honorary member).
ALFRED BRAY KEMPE.
JAMES MCMAHON.
GEORGE BALLARD MATHEWS.
MAX NOETHER (honorary member).

CORRIGENDA.

The list of members dated November, 1921, should be corrected as follows:—

Page 17. Professor MILLER's address is 1203 *W. Illinois Street, Urbana, Illinois, U.S.A.*
Page 20. The name of Professor ROSEVEARE is spelt incorrectly.
Page 21. Professor STEGGALL should be described as "Professor," not as "late Professor."

OBITUARY NOTICES

MAX NOETHER.

Max Noether passed away on December 13th, 1921, at Erlangen, where he had been Professor of Mathematics for nearly fifty years. He was born at Mannheim on September 24th, 1844, and took his doctorate at Heidelberg in 1868. He was a corresponding or honorary member of many learned societies, including the Academies of Berlin, Göttingen, Munich, Budapest, Copenhagen, Turin, the Royale Accademia dei Lincei, and the Institut de France. He was elected an honorary member of the London Mathematical Society in 1913 at the same time as Adolf Hurwitz, Paul Painlevé, Corrado Segre, and Woldemar Voigt. He seldom, if ever, visited England, but expressed his regret that he was unable to attend the International Congress of Mathematics at Cambridge in 1912, as he had reached an age when he was averse from travelling.

Although not so well known to fame out of his own country as some other famous mathematicians, he was deservedly held in the highest esteem, and after the death of Clebsch became the recognised leader of the algebraic-geometric school in Germany. He brought himself early to the front by his celebrated and fundamental "Satz aus der Theorie der algebraischen Functionen" (*Math. Annalen*, Vol. 6), which he followed in the next volume with the classical memoir "Ueber die algebraischen Functionen und ihre Anwendung in der Geometrie," written in collaboration with A. Brill. This memoir formed the solid foundation, and gave rise to the name, of "Geometry on a Curve," which has been developed and extended, especially in Italy, into a geometry of greater depth and power than anything preceding it. How remarkable these developments have been is clearly shown in H. F. Baker's fascinating review "Some Advances in the Theory of Algebraic Surfaces," published in our *Proceedings*, Ser. 2, Vol. 12, pp. 1–40.

Noether's mind was naturally intuitive, but he mistrusted intuition, and was apt to let anything suggested by it pass out of his thought. It seemed as if this cramped his powers to some extent in his later years. He found perhaps in his subject that intuition was often liable to mislead.

He was of course never content without algebraic or arithmetic proof, but had sometimes to be satisfied with an incomplete proof. Although naturally impatient he would take infinite pains to understand the thought of others, and to give them abundant help out of his own ample resources. There are many, including the writer of this note, who are grateful to him for his help. He had searching and peculiar methods of his own for testing the truth of things.

His name will always be intimately associated with the greatest of mathematical journals, the *Mathematische Annalen.* Scarcely a year passed from the appearance of the 2nd volume in 1870 till the 83rd in 1921 in which he did not make an important contribution to its pages. With the publication of the 42nd volume in 1893 he formally joined the editorial staff, and an obituary notice at the beginning of the 84th volume bears eloquent testimony to the great value of the services he rendered it. He contributed greatly to the advancement of mathematical science in three distinct ways: by the new and fruitful ideas contained in his original researches, by the patient investigation and encouragement he gave to other writers, and by his acutely critical and detailed historical work. The last consisted of the great work which he and A. Brill compiled in 1894, "Entwicklung der Theorie der algebraischen Functionen in älterer und neuerer Zeit," for the third volume of the *Jahresbericht der Deutschen Mathematiker Vereinigung*; and elaborate memorial surveys of the work of many famous mathematicians, beginning with Cayley and Sylvester and ending with Zeuthen, which he contributed to the pages of the *Math. Ann.* The "Entwicklung, etc." may be regarded as the completion and rounding off of his algebraic-geometric work, and in it the Noetherian Theorem comes fully into its own.

Of Noether's original (as distinguished from critical and historical) work the most interesting, because of its far reaching developments, is that which was published in Vols. 6 and 7 of the *Math. Ann.*, to which reference has already been made. This work is still so little known in England, notwithstanding that it is fully reproduced in Clebsch's "Geometrie," that there seems good reason to give an account of what it contains. In his first communication to the *Math. Ann.* (Vol. 2, p. 293) he makes use of a theorem which, up till then, had been taken for granted, viz. that a plane curve f which passed through all the points of intersection of two given curves ϕ, ψ had an equation of the form

$$f \equiv A\phi + B\psi = 0.$$

The real question was whether these purely geometrical conditions were sufficient, and he there gave a formal proof that they were sufficient in the case in which all the points were simple and finitely separated. He was thus led to consider the more general and fundamental problem as to what were the simplest and sufficient conditions which would require a curve f to take the form $A\phi+B\psi$, when ϕ, ψ had common multiple points with contact of any degree of complexity. To this question he gave the answer in Vol. 6, p. 351 (which had been previously published in the *Göttingen Nachrichten*, 1872, p. 490), in a form which soon came to be known as Noether's fundamental theorem. He recognised intuitively that the problem was one of approximation, and that the answer to it could be formulated thus:—The necessary and sufficient conditions that f must satisfy in order that it may be of the form $A\phi+B\psi$, are that f should be capable of being written in the form $A'\phi+B'\psi$ when the origin is moved in succession to each and every point of intersection of ϕ, ψ; where A', B' are ordinary power series which are, for each common point of ϕ, ψ, quite independent of what they may be for all the other points. This approximation of f with $A'\phi+B'\psi$ will have been carried far enough when, on equating coefficients of the terms in f and $A'\phi+B'\psi$ of a sufficiently high degree, the undetermined coefficients of A', B' cannot all be eliminated so as to give a *new* relation among the coefficients of f alone, *i.e.* a relation which is independent of all the similar relations obtainable from the equating of coefficients of terms of all *lower* degrees. It will be noticed that no conditions have to be satisfied at infinity.

To the proof of this he added a simple rider, of fundamental importance for its applications, that if ϕ, ψ have no common tangent at a common multiple point, of order i for ϕ and j for ψ, then all the relations for the coefficients of f alone corresponding to that intersection of ϕ, ψ come from the equating of coefficients up to degree $i+j-2$, so that it is *sufficient* (but of course not necessary) for f to have a multiple point of order $i+j-1$ at that point. At a later time he extended this, the so-called "simple" case, to the case where ϕ, ψ have higher singularities at a common point, which can be resolved by repeated transformation of type

$$x_1' : x_2' : x_3' = \frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3}$$

into separate common multiple points coming under the "simple" case. Though much has been written about the theorem these are the only important additions that have been made to it, beyond its extension in various ways to surfaces and hyperspaces. Of these the Lasker-Noether (named by Lasker the Noether-Dedekind) theorem is the most general and fundamental (*Math. Ann.*, Vol. 60, p. 51; and *Cambridge Tracts in Maths., etc.*, No. 19, p. 61).

Brill and Noether's classical memoir "Ueber die algebraischen Functionen, etc.," in Vol. 7 of the *Math. Ann.*, is founded essentially on the investigation of the properties of point-groups $g_R^{(r)}$ on a given curve $f=0$ cut out by a linear system of curves $\lambda_0\phi_0+\lambda_1\phi_1+\ldots+\lambda_r\phi_r=0$, where $\phi_0, \phi_1, \ldots, \phi_r$ are given linearly independent curves of the same order. Any fixed points common to $f, \phi_0, \phi_1, \ldots, \phi_r$ are left out of account, and R denotes the number of simple points in each group, while r is the *freedom* of the linear series $g_R^{(r)}$, *i.e.* the number of the R points which can be chosen at will to determine a particular group G_R of the series $g_R^{(r)}$. Thus the curve f of order n can be birationally transformed to a curve F of order R by means of the transformation equations

$$x_1' : x_2' : x_3' = \phi_0 : \phi_1 : \phi_2 ;$$

and this is possible so long as $r \geqslant 2$. (The only exceptions are curves for which the genus $p=0, 1, 2$, which require separate treatment; and hyperelliptic curves, or curves on which exist a series $g_2^{(1)}$, since in this case the transformation is not birational.) Hence one important problem is to find the least value of R for which a series $g_R^{(2)}$ exists, as this value of R gives the lowest value of the order of a curve F into which f can be birationally transformed.

It was shown that the series $g_R^{(r)}$ on f which were of the greatest interest and importance were those which were cut out by a linear system $(\phi_0, \phi_1, \ldots, \phi_r)$ *adjoined* to f, especially when the curves $\phi_0, \phi_1, \ldots, \phi_r$ were of order $n-3$. A curve ϕ adjoined to f is a curve having a multiple point of order $i-1$ at each ordinary multiple point of order i on f, and a corresponding higher singularity at each point of higher singularity on f. If G_R is any given group of R points on f, the series $g_R^{(r)}$ of greatest freedom r, having G_R as one of its groups, can be found by drawing any adjoined curve through G_R to cut f again in a group G_Q (apart from the multiple points of f). Then the whole system of adjoined curves through G_Q of the order of ϕ will cut out on f the required series $g_R^{(r)}$ of greatest freedom. *This series $g_R^{(r)}$ is a fixed series, independent of the particular group G_Q by means of which it is derived, i.e. it is a series uniquely dependent on any one of its groups G_R.* This is one way of stating the Brill-Noether theorem of residuation, and is proved by the help of the rider to Noether's theorem mentioned above. This series $g_R^{(r)}$ is called a *complete coresidual* series (every one of its groups being *residual* to any group G_Q to which any one of its groups G_R is residual) or the linear series of *unrestricted freedom* which any given group G_R determines.

We will suppose now that any series denoted by $g_R^{(r)}$ is a series of unrestricted freedom r unless the contrary is stated. In any group G_R of $g_R^{(r)}$ r points can be chosen freely, and the remaining $R-r$ points are dependent on the r points. Also $R-r \leqslant p$, where p is the genus of the curve f. *Further, if* $R-r<p$, *the group* G_R *lies on an adjoined curve of order* $n-3$. (This is still more true if the freedom r is restricted, for, on removing the restriction, r would increase and $R-r$ diminish; but with restricted freedom we may have $R-r \geqslant p$ and yet G_R on an adjoined curve of order $n-3$.) Now, taking $R-r<p$, and drawing all the adjoined $(n-3)$-ics which pass through a fixed group G_R of $g_R^{(r)}$, we determine the series $g_Q^{(q)}$, of freedom q, which is residual to the series $g_R^{(r)}$, where $Q+R = 2p-2$. The two mutually residual series $g_Q^{(q)}$, $g_R^{(r)}$ may be considered together; or $g_R^{(r)}$ may be considered alone, in which case q must be regarded as a number not less intimately associated with the series $g_R^{(r)}$ than r is; for r is the freedom of the series $g_R^{(r)}$ on f, while q is the freedom of the whole system of adjoined $(n-3)$-ics through any fixed group G_R of $g_R^{(r)}$. *The numbers* R, r, q, p *are connected by the relation*

$$R = p-q+r-1.$$

This theorem is called the Riemann-Roch theorem for plane curves, and is also known under the more definitive title of the Brill-Noether reciprocity theorem. Other ways of stating the theorem are

$$\text{(i)}\quad Q-R = 2(q-r), \quad Q+R = 2(p-1),$$

and

$$\text{(ii)}\quad R-2r = Q-2q = p-q-r-1 \geqslant 0.$$

In seeking the number of relations that must exist between the points of a group G_R on f, in order that the unrestricted series $g_R^{(r)}$ determined by G_R may be of freedom r, Brill and Noether obtained the formula $(q+1)\,r$, or $r\,(p+r-R)$. This was got by counting equations of condition, and is only valid under stringent limitations, as also the inequality

$$p \geqslant (q+1)(r+1)$$

derived from it. It is possibly correct for curves f of given genus which are *general of their kind*, and in the memoir it is only applied to such curves, but it requires a strict proof. Assuming the correctness of the formula, then, in fixing the group G_R on f there are $r(p+r-R)$ points dependent on the rest by the equations of condition and $R-r(p+r-R)$ which can be chosen freely. This cannot fall below r, since a group of $g_R^{(r)}$ has r free points even after G_R has been fixed. Hence

$$R-r\,(p+r-R) \geqslant r.$$

This inequality determines the least value R can have for a given value of r in a series $g_R^{(r)}$. When $r = 2$, it becomes $R \geqslant \frac{2}{3}(p+3)$. Hence a *general* curve f of given genus p can be birationally transformed into a curve F of order R equal to the integral part of $\frac{2}{3}(p+4)$. A curve of less order with given genus p, such as a quintic with $p = 5, 6$ or an n-ic with less than $\frac{1}{2}(n-2)(n-4)$ double points, is not a *general* curve of genus p.

The above result is not only interesting, and even surprising, in itself, but gains additional interest from its relation to Riemann's theorem that a curve f of genus p (general of its kind) can be birationally transformed to a curve F whose coefficients depend on $3p-3$ parameters or moduli. A curve f of order n and genus p has $\frac{1}{2}n(n+3)$ coefficients, which can be reduced to $\frac{1}{2}n(n+3)-8$ by a collinear transformation; these satisfy a number of independent relations equal to the number of the double points, viz. $\frac{1}{2}(n-1)(n-2)-p$, leaving $3n+p-9$ effective constants for f. The curve f can be transformed to a curve F of order R as above, and the process of transformation admits of imposing 0, 1, 2 additional geometrical conditions on F (such as requiring it to have 0, 1, 2 points of undulation) according as $p = 0, 1, 2 \bmod 3$. The number of effective constants left for F are therefore $3R+p-9$, $3R+p-10$, $3R+p-11$, *i.e.* $3p-3$ in all three cases, since $3R = 2p+6$, $2p+7$, $2p+8$ when $p = 0, 1, 2 \bmod 3$. This really only shows that the number of moduli can be brought as low as $3p-3$, not that it cannot be brought lower; but other proofs of the result are given in the memoir.

We have dwelt at length on these two memoirs because of their fundamental and historical importance. The greater part of Noether's original work may be said to have been a natural continuation of them. He contributed much besides to the theory of elimination, higher singularities, the geometry of hyperspace, and Abelian and Theta functions.

F. S. M.

CAMILLE JORDAN.

CAMILLE JORDAN was born at Lyon on January 5th, 1838. His father, Alexander Jordan, was an engineer of bridges and roads, and deputy of the National Assembly. His mother, *née* Joséphine Puvis de Chavannes, was a sister of the well known painter. His great-uncle Camille was the famous orator and politician. As a boy he studied at the Collège d'Oullins and Lycée de Lyon. He entered the École Polytechnique in 1855, and was a mining engineer for some years at Privas and then at Paris. He married Isabelle Munet in 1862. He had six sons, two daughters, and numerous grandchildren. His latter years were clouded by bereavement. He lost a daughter in 1912, and his wife in 1918. His sons Charles and Pierre fell in 1914, while a third son was badly wounded about the same time and fell in 1915. The eldest child of his son, Prof. Jordan of the Sorbonne, was killed in 1916.

Camille Jordan made his mark early as an able mathematician, and his career was full of distinction. He obtained his doctorate in 1861, was examiner at the École Polytechnique in 1873 and professor there 1876 to 1917, besides being titular professor at the Collège de France, 1883 to 1912. He succeeded Chasles as Membre de l'Institut in 1881; and was vice-président of the Académie des Sciences in 1915, and président in 1916. He was vice-président of the Fourth International Congress of Mathematicians at Rome in 1908, and président d'honneur at the recent Congress at Strasbourg. His fame was recognised by the conferment of several honours by universities and learned societies in other lands.

In 1885 he succeeded Résal as editor of the *Journal de Mathématiques*, and till his death earned the gratitude of mathematicians for his invaluable work in that capacity. His published work was almost entirely mathematical, but he retained an interest in classical studies and was no narrow specialist. He died of sudden heart-failure on January 20th, 1922.

His earliest published papers seem to have been four on topography in 1866. In these he extended the definition of symmetrical polyhedra to those whose general arrangement of edges and angles, apart from their magnitudes, appears the same to an observer, no matter at what vertex he may be stationed. Topography was a subject to which he returned at intervals, and his studies on "Jordan curves," dividing the plane into two distinct portions, are celebrated. It was fitting that his last paper also should have been on the same subject; in the *Journal de Mathématiques* for 1920 he discusses the number of paths joining given points, no one of which crosses itself.

For the next few years Jordan was a most prolific writer. Except for the papers on Analysis Situs above mentioned, and two in 1868 and 1875 on the oscillations of a body about its position of equilibrium, his work seems to have consisted entirely of investigations on group-theory leading up to his *Traité des Substitutions*, published by Gauthier-Villars in 1870; followed by a series of articles extending and amplifying the results contained in that book.

Jordan ascribes his interest in Algebra to Serret's *Cours d'Algèbre supérieure.* The object of his book was "to develop the methods of Galois and to create a theory of them by showing how easily they permit us to solve all the principal problems of the Theory of Equations." The *Traité*, modestly described as "a commentary on Galois' works," was of first-rate importance and high originality, It was divided into four books, the first on congruences in the theory of numbers, the second on permutation-groups and the properties of linear substitutions, the third on irrationals with applications to elliptic functions and geometry of curves and surfaces, and the fourth on the solution of algebraic equations.

Among papers not subsequently incorporated into the *Traité* the most important was that in the *Annali di Matematica* for 1869, on "Groups of Movements," which formed the basis of all subsequent work on the theory of crystal structure as developed by Sohncke, Fedorow, Schoenflies, Barlow, &c., and which has received recently important applications in "X-ray analysis" of crystalline media. Jordan's paper was written solely from the point of view of the pure mathematician, but his use of technical crystallographic terms shows that he realised to some extent the important application which might be made of his results.

As years passed on, Jordan's interest in Algebra seems to have developed in the direction of abstract groups and linear substitutions; for after 1875 he wrote little on permutation-groups or theory of equations. Noteworthy are papers on the canonical shape of bilinear and quadratic forms. His first attempt in 1914 at obtaining all polynomials invariant under a given linear substitution proved incorrect, but he returned to the problem shortly after with success. His proof that groups of order $p^m q^2$ are composite, marked an important advance in the solution of a problem which was finally solved by Prof. Burnside, who showed that the same was true for any group of order $p^m q^n$.

Early in his career Jordan was led by his interest in the theory of substitutions to consider applications to the solution of differential equations and the properties of hyperelliptic and doubly periodic functions. This, and his continued interest in problems of Analysis Situs, bore fruit later in the work by which he is now probably most widely known, his famous *Cours d'Analyse*, which has run through three editions and has played an important part in the education of every analyst of the present day. This work, first published in 1882–7, is, like other French works of a somewhat similar kind, no ordinary textbook, but contains the fruit of a mass of original research, which as time went on absorbed Jordan's attention more and more. It was the first extensive work which contained a serious account of the theory of aggregates, a really rigorous discussion of the fundamental double limit problems of the integral calculus, and an account, in the modern spirit, of the theory of Fourier's series. A good deal of it has now been superseded by the work of Borel and Lebesgue, but it remains a source of inspiration for every student.

Two parts of the book stand out, perhaps, for originality and power. The first is the part which deals with Jordan curves and the problems of Analysis Situs to which they give rise. "Jordan's theorem," already referred to, that a simple closed Jordan curve divides the plane into two distinct connected regions, is one of the most justly famous in modern mathematics. In the discussion of questions of this and a similar kind, connected with areas and lengths of curves, Jordan was led to formulate the concept of a "function of bounded variation," which has played a fundamental part in the later theory of Fourier's series and allied developments. The concept is a generalisation of that of a steadily increasing function; a function is of bounded variation if, and only if, it is the difference of two increasing functions. But the generalisation is one of those which bring wholly new light and unity into a theory, for functions of bounded variation form a group for the elementary operations, whereas monotonic functions do not, and it would be difficult to exaggerate the importance, in all the modern theory of functions of a real variable, of Jordan's illuminating idea.

To the end of his life he was a researcher, though naturally the labour involved in his editorship of an important periodical prevented his output being quite as rapid of late as it was in his earlier years.

He was elected an honorary member of the London Mathematical Society in 1907, but never contributed to the *Proceedings*.

H. H.

GEORGE BALLARD MATHEWS, 1861–1922.

George Ballard Mathews was born in London on February 23rd, 1861. His father, George Mathews, was the son of a Herefordshire yeoman farmer, and engaged for many years with a London business house. His mother's maiden name was Harriet Ballard, daughter of George Ballard, Secretary to the Board of Inland Revenue, and a remarkable man, from whom Mathews inherited much of his personality and intellect. From his mother he inherited his literary and musical abilities, his interest in general subjects, and his powers of calculation.

The family returned to Herefordshire in 1866, and Mathews, in 1872, entered Ludlow Grammar School, where his brilliant intellect soon asserted itself. Records of 1875 state that he was first in the top form with prizes in Mathematics, French, Drawing, Form Prize, and Parents' Prize. During the four years, 1874–78, he was captain of the school. From the Rev. W. C. Sparrow, one of the most successful Head Masters of the school, he received instruction in Hebrew as well as in Greek and Latin.

There followed a year at University College, London, of which body he afterwards became a Fellow, during which he attended Henrici's lectures on geometry and began to study Sanskrit. In 1879 St. John's College, Cambridge, offered him the senior scholarship of his year in either Mathematics or Classics. Carrying out his intention of reading for the Mathematical Tripos, he became a private pupil of Mr. W. H. Besant of St. John's. The keen competition for leading places in the Tripos of this period had brought fame to Mr. E. J. Routh as a coach. All the abler candidates went to Routh as a matter of course, for Routh had a long series of Senior Wranglers to his credit. However, Mathews' name was read out first in the list of 1883, this being the only break in a succession of about thirty consecutive seniors trained by Routh.

In 1884, Mathews was appointed to the Chair of Mathematics in the newly constituted University College of North Wales at Bangor, and in the same year was elected to a Fellowship at St. John's. His colleagues at Bangor were all of the same generation as himself, and included such men as Profs. Andrew Gray, James Dobbie, and the late Henry Jones, under the leadership of Principal Harry R. Reichel (the last three named have all since been knighted). Writing in *Nature* (June 3rd, 1922) of the period which followed, Prof. Gray says: "[Mathews] came full of eager enthusiasm for the teaching of mathematics and for original mathematical work, and for ten years laboured hard in the hope of founding something like a school of mathematical study in North Wales. But, alas! these hopes were dashed. Perhaps he was a little impatient, and I certainly did my best to counsel him to wait, and to find out the effect of the new Welsh University on the studies of the place, but without effect. The best of the Welsh students were at that time attracted by the Neo-Hegelian philosophy, and some of them, as seems to be the way with such students, seemed not a little proud that their mental tendencies were not mathematical. To this curious type of intellectual pride Mathews referred eloquently in the posthumous paper published in *Nature* of April 22nd. In that paper, he lamented the revival of the fallacious arguments for the supremacy of the Latin-Greek classics as an educational instrument; but he in no way under-valued classical culture, only he thought that, to an Englishman, the inheritor of a copious and flexible language, and of a literature unparalleled in the past, a training in Latin and Greek was far from indispensable and might have its disadvantages."

The Bangor Chair was resigned in 1896, and shortly followed (1897) by Mathews' election into the Royal Society, on the Council of which he served for a year. He also served on the Council of our Society from 1897 to 1904. Having returned to Cambridge he was first appointed to be Mathematical Lecturer at St. John's and afterwards to be University Lecturer, and also acted for a year (1904–5) as Mathematical Secretary to the Philosophical Society. The Cambridge appointment, like the earlier one in Bangor, was resigned on account of temperament, and he returned to live in Bangor. Since 1911 he held a special Lectureship in the North Wales College, and again acted as Professor there during the two Sessions 1917–19. The honorary degree of LL.D. was conferred by Glasgow University in 1915.

While thoroughly familiar with all branches of pure mathematics, Mathews' main interests were in the theory of numbers and projective geometry. The theory of numbers, which, in its widest sense, is the theory of discrete, as opposed to continuous, magnitude, has passed through four well-defined stages of development. First there came the period of Diophantine Analysis proper, of which the greatest exponents, after Diophantos among the ancient Greeks, were Fermat and Euler. In this, the general problem is to determine all the solutions in rational numbers of a system of $m\,(< n)$ algebraic equations

$$R_i(x_1, x_2, \ldots, x_n) = 0 \quad (i = 1, 2, \ldots, m).$$

Next came the discovery of the law of quadratic reciprocity which rendered possible a discussion of quadratic arithmetical forms, so ably expounded by Gauss in the *Disquisitiones Arithmeticæ.* Such writers as Lejeune-Dirichlet, Eisenstein, and Stephen Smith added much to what Gauss had done, and a scholarly introduction to the whole theory was given by Mathews in his *Theory of Numbers* of 1892.

The third stage was marked by Dedekind's discovery of his theory of ideal numbers, which completely restore to a system of algebraic numbers certain factorisation properties of ordinary integers that appear at first to be lost. Mathews' was probably the first mind in England to realise the far-reaching effect of Dedekind's discovery, two papers by him on the subject appearing in the London Mathematical Society's *Proceedings* of 1892.

The last stage is the analytical theory, which is intimately bound up with certain transcendental functions of a complex variable. This theory had little attraction for Mathews, though his book contains an account of Riemann's classical memoir on the distribution of primes, Unfortunately Mathews wrote in 1892, four years before the proof of the prime-number theorem. Riemann had actually proved but little, beyond some formal properties of the ζ-function, and the modern analytical theory, as developed by Messrs. Hadamard, de la Vallée Poussin, Landau, Hardy and Littlewood, has diverged entirely from his original line of attack.

The tract *Algebraic Equations* on a kindred topic, written fifteen years later, contains a masterly exposition of Galois' theory, completed by Jordan and others, showing how the different types of irrationality which can be defined by an algebraic equation are associated with different types of group.

Written in collaboration with Prof. Andrew Gray and mainly concerned with physical applications, the *Treatise on Bessel Functions* (1894) is still a standard work. The *Projective Geometry* (1914), inspired by Henrici's lectures in London many years before, contains two unusual features: first, an exposition of the logical groundwork of the subject; and, secondly, an account of Staudt's theory of complex elements (whereby a real involution defines a complex point or line). Mathews also brought out a new edition (1904) of R. F. Scott's *Determinants*, and contributed articles on "Number" and "Universal Algebra" to the 1910 edition of the *Encyclopædia Britannica.*

Most of Mathews' technical papers appeared in the London Mathematical Society's *Proceedings* or in the *Messenger of Mathematics.* Some of the earliest, beginning in 1886, are connected with Fermat's Last Theorem, a question which absorbed much of his attention at intervals during the rest of his life. Being firmly convinced of the veracity of Fermat's statement, he tried to approach the problem by such methods as might have been devised in Fermat's day. His last contribution to the subject appeared in a review published in *Nature* of January 5th, 1922.

A few of the earlier papers deal with geometrical subjects, such as twisted quartic curves, geometry on a quadric surface, and porisms. Nearly all the rest have an arithmetical bearing. Several papers discuss points connected with the Gaussian theory of quadratic forms, the subject of his book. In 1892 he wrote on quadratic forms with complex coefficients, a subject to which he returned twenty years later. In early days he corresponded freely with Cayley and Sylvester.

Among the specialised branches of the higher arithmetic Mathews was mainly interested in the complex multiplication of elliptic functions. Being an assiduous disciple of Dedekind and Klein, he fully realised the fundamental character of the absolute invariant $j(\omega)$ and of the type of irrationality involved in it. Papers written in the nineties, when Sir George Greenhill and Mr. Russell were working at the subject too, give special detailed properties of the lemniscate functions. He also showed the significance of Klein's principal moduli and calculated some new class-invariants. Returning to the subject in later years, he produced a fuller manuscript on the lemniscate functions: its publication has been delayed by the war and his subsequent illness.

Ever since the mid-eighties Mathews was a frequent contributor to *Nature* on mathematical topics. His articles and reviews, most of which appeared over the initials "G. B. M.," were always written in a careful and scholarly style; they contained his considered opinion on the book or point concerned. His keen yet kindly criticism was undoubtedly of the greatest service to the many writers whose work passed through his hands. In conversation with the present writer he once expressed the opinion that some of his best work had appeared in *Nature* reviews.

A man of simple tastes and naturally retiring by disposition, Mathews expressed sound judgment on both men and affairs. Some of his views were those of an idealist and hardly feasible in the domain of practical politics. His capacity for maturely grasping everything with which his mind came into contact made him unique in the experience of his friends. Only one or two sides of so versatile a man's brilliant intellect really appealed to most people. When he was appointed Professor of Mathematics at Bangor, at the age of twenty-three, it was manifest that he could equally well fill four or more chairs in the College. Again, to quote Prof. Gray: "Mathews had a knowledge of Latin and Greek as minute and accurate as that generally possessed by professional classical scholars.

He wrote pure and elegant Latin. I remember his amusing himself by turning into Latin prose an original philosophical dissertation which had happened to come into his hands and had arrested his attention. I remember also some Latin verses which he published anonymously, and which were much praised by a very eminent scholar. He wrote also charming English essays in the style of Charles Lamb, of whom he was a great admirer. These, I fear, are lost, but one of them, 'On a Cockloft,' was a perfect gem, a charming piece of the most natural and simple prose."

A humorous controversy in Latin Elegiacs between Profs. Arnold and Mathews, in which Mathews did not come off second best, inspired Principal Reichel's epigram, addressed, in the College Magazine, to a mathematical student who had been reading Mathews' verses :—

Versibus an numeris noster praestantior anceps
  Mattius ; attonito cur puer ore siles ?
Qui numeros dicit vult idem dicere versus,
  Testibus innumeris, o numerose puer.

In the early days at Bangor Mathews began to learn Arabic, and spent much time of later years in reading and translating Arabic poetry. His translations are full of the spirit of the original lines ; it is to be hoped that some of them will be published later.

Mathews' knowledge of music, again, was fully as advanced as that of most professional musicians. His copies of Gauss and Bach were placed together on the same shelf, and he considered some of Sir Edward Elgar's compositions to be as fine as the work of Beethoven and Handel.

Undoubtedly, the strength of his numerous interests prevented him from making the most of his mathematical work. His name will be mainly remembered in connection with the higher arithmetic.

The last three years of his life were clouded by a series of operations for cancer, which finally gained the mastery. He died, unmarried, in a Liverpool nursing home on March 19th, 1922.

W. E. H. B.

MATHEMATICS
DEPARTMENT

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The London Mathematical Society.

RECORDS OF PROCEEDINGS AT MEETINGS.

Thursday, March 9th, 1922.

Mr. H. W. RICHMOND, President, in the Chair.

Present twenty-one members.

Messrs. R. F. Budden, A. E. R. Church, H. J. Davis, H. V. Lowry, W. F. D. MacMahon, J. L. Navarro, S. W. P. Steen, and T. Thompson were elected members of the Society.

Prof. Dr. W. Wirtinger was elected under Bye-law 20*b*.

Messrs. W. Cowan, H. Cramér, H. A. Hazden, J. B. Homer, H. C. Plummer, and D. Wootton were nominated for election.

Prof. Dr. L. Fejér of Budapest, Prof. Dr. E. Lindelöf of Helsingfors, Prof. Dr. F. Riesz of Szeged, and Prof. Dr. W. Sierpinski of Warsaw were nominated under Bye-law 20*b*.

The President stated that the date of Prof. Hardy's lecture would be May 11th.

Prof. W. H. Young read a paper "On the Theory of Functions of Two Complex Variables."

Col. R. L. Hippisley read a paper "The Nodes of the Three-Bar Sextic."

Lt.-Col. A. Cunningham read a paper "On Least Primitive Roots."

Mr. R. F. Whitehead made a communication "The Number of Solutions in Positive Integers of the Equation $yz+zx+xy = n$."

The following papers were communicated by title from the Chair :—

The Determination of the Criterion to prevent "Hunting" in Hartnell's Governor : T. Stuart.

The Average Value of a Functional : N. Wiener.

ABSTRACTS.

The Nodes of the Three-Bar Sextic

Col. R. L. HIPPISLEY, C.B.

A proof is given of the theorem that the line joining the other extremities of the principal chords passing through a node passes through the other two nodes. From this the sum of the arguments of the elliptic functions by which the coordinates of the nodes can be expressed is determined. A relation is also given between the two arguments for a node. A relation between the areas of the parallelograms of the triple generation theorem when the tracing point is at a node is given, and certain theorems regarding the Simson lines of the nodes.

On Least Primitive Roots

Lt.-Col. ALLAN CUNNINGHAM, R.E.; H. J. WOODALL; and T. G. CREAK.

The paper contains a table of the least primitive roots, both positive and negative, of the 2800 odd primes < 25410.

The table shows two important results—

(1) The *maximum* least primitive roots for the 2800 primes are $+31$ (occurs *twice* only) and -39 (*twice*).

(2) The small numbers $\not> 12$ supply 2637 of the positive roots and 2688 of the negative roots out of the totals of 2800.

As the use of primitive roots in computation involves very frequent multiplication by the root itself, these results are of great practical importance. These results seem new, inasmuch as Jacobi used 29 primitive roots >100 in computing the *Canon Arithmeticus*, which extends only to primes < 1000, for which ± 21 is the highest really required: and Desmarest in a table giving *one* primitive root for all primes < 10000 gives 36 roots > 1000.

The process used by the joint authors for finding the least primitive roots is described: it is far shorter than that described in Mathews's *Theory of Numbers*, Vol. 1, Art. 20.

On the Number of Solutions in Positive Integers of the Equation $yz+zx+xy=n$.

R. F. WHITEHEAD.

In an abstract of a forthcoming paper, which appeared in the *Records of Proceedings* for November 17th, 1921, Mr. Mordell has stated that the number of solutions is equal to three times the number of classes of binary quadratics of determinant $-n$, reckoning as $\frac{1}{2}$ each solution in which one of the unknowns is zero, and counting each class $(k, 0, k)$, $(2k, k, 2k)$ as $\frac{1}{2}$ and $\frac{1}{3}$ respectively.

An elementary demonstration of these results may be derived from the consideration that the equation is equivalent to

$$(x+y)(x+z) = x^2+n.$$

If *e.g.* $z \nless y \nless x$, then $x+z \nless x+y \nless 2x$, and therefore $(x+y, \pm x, x+z)$ is a reduced form.

There are thus generally six solutions derived by permutation, corresponding with two classes. If, however, one unknown is zero, there is only one class for the six solutions. If two of the unknowns are equal and the third not equal to them or to zero, there are three permutations and one class, since the forms $(2a, \pm a, c)$ are equivalent, as also the forms $(a, \pm b, a)$; but if the third unknown is zero there are three solutions with one corresponding class. Finally, if $x=y=z$, there is one solution with a corresponding class, the forms $(2a, \pm a, 2a)$ being equivalent.

On Gauss's Quadratic Identity

W. E. H. BERWICK.

The polynomials $Y(x)$, $Z(x)$, with which Mr. Upadhyaya's note* is concerned, are to be found tabulated as far as $p=101$ in Dr. Hermann Teege's *Inaugural-Dissertation* "Über die $\frac{1}{2}(p-1)$-gliedrigen Gaussischen Perioden," Kiel, P. Peters, 1900. When Mr. Upadhyaya's result for

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* See *Records of Proceedings at Meetings* for November 17th, 1921.

$p = 37$ was communicated to *Nature* (June 9th, 1921), in a letter by Dr. G. B. Mathews, similar results for $p = 41, 43, 47, 53, 59, 61$ were calculated by Dr. H. C. Pocklington and Father J. Cullen, who were evidently unfamiliar with Dr. Teege's work, and printed in *Nature* of July 7th. I drew attention to Dr. Teege's work in *Nature* of July 21st, and a further letter by Mr. R. F. Whitehead appeared in the issue of October 13th, 1921.

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The London Mathematical Society.

RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1921–JUNE, 1922.

Thursday, April 20th, 1922.

Mr. H. W. RICHMOND, President, in the Chair.

Present ten members.

Messrs. W. Cowan, H. Cramér, H. A. Hazden, J. B. Homer, H. C. Plummer, and L. Wootton were elected members of the Society.

Prof. Dr. L. Fejér (Budapest), Prof. Dr. E. Lindelöf (Helsingfors), Prof. Dr. F. Riesz (Szeged), and Prof. Dr. W. Sierpinski (Warsaw) were elected under By-law 20B.

Dr. S. Beatty, Mr. W. L. Ferrar, Miss L. E. Hardcastle, and Miss M. Stimson were nominated for election.

Prof. Dr. C. Runge (Göttingen) was nominated under By-law 20B.

Messrs. R. G. Cooke and W. P. D. MacMahon were admitted into the Society.

The President referred to the loss sustained by the Society in the death of Prof. G. B. Mathews.

Prof. Hardy communicated a paper by Prof. H. Bohr, "Another Proof of Kronecker's Theorem."

Prof. Hilton made an informal communication "On a Lamé Family of Surfaces."

Lt.-Col. Cunningham announced the publication at Nancy of an extensive table of indices.

The following papers were communicated by title from the chair :—

On Water Waves due to Disturbance beneath the Surface : H. Lamb.

Cauchy's Theorem on the Integral of a Function between Imaginary Limits : G. Mittag-Leffler.

ABSTRACT.

On a Lamé Family of Surfaces

Prof. HAROLD HILTON.

Suppose u, v, w are functions of the rectangular Cartesian coordinates x, y, z of any point, and that

$$dx^2+dy^2+dz^2 \equiv a\,du^2+b\,dv^2+c\,dw^2+2f\,dv\,dw+2g\,dw\,du+2h\,du\,dv,$$

so that $$a \equiv x_1^2+y_1^2+z_1^2, \quad f \equiv x_2x_3+y_2y_3+z_2z_3, \quad \&c.$$

where suffixes 1, 2, 3 denote partial differentiation with respect to u, v, w respectively.

Write α for $(bg-fh)/(ab-h^2)$, β for $(af-gh)/(ab-h^2)$. Then $w =$ *constant* is a Lamé family, forming a triply orthogonal system with two other singly infinite families of surfaces, if

$$\begin{vmatrix} a & h & b \\ a' & h' & b' \\ a'' & h'' & b'' \end{vmatrix} = 0,$$

where
$$\begin{aligned} a' &\equiv a_3-a_1\alpha-a_2\beta-a\alpha_1+a\beta_2-2h\beta_1, \\ h' &\equiv h_3-h_1\alpha-h_2\beta-a\alpha_2-b\beta_1, \\ b' &\equiv b_3-b_1\alpha-b_2\beta+b\alpha_1-b\beta_2-2h\alpha_2; \end{aligned}$$

and a'', h'', b'' are derived from a', h', b', α, β in the same way as a', h', b' are derived from a, h, b, α, β.

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MATHEMATICS
DEPARTMENT

The London Mathematical Society.

RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1921–JUNE, 1922.

Thursday, May 11*th*, 1922.

Mr. H. W. RICHMOND, President, in the Chair.

Present thirty-three members and five visitors.

Dr. S. Beatty, Mr. W. L. Ferrar, Miss L. E. Hardcastle, and Miss M. Stimson were elected members of the Society.

Prof. Dr. C. Runge (Göttingen) was elected under By-law 20B.

Mr. E. C. Francis was nominated for election.

Prof. Dr. E. Schmidt (Berlin) was nominated under By-law 20B.

The President referred to the loss sustained by the Society in the death of Sir A. B. Kempe, formerly Treasurer (1883–1891) and President (1892–1893).

Prof. G. H. Hardy delivered a lecture "The Elements of the Analytic Theory of Numbers."

The following papers were communicated by title from the Chair:—

On the Displacement of Tensors: P. Dienes.

Some Limiting Forms in the Theory of Integral Equations: A. C. Dixon.

On Harmonic Equations and in particular the Equations associated with Parabolic and Circular Boundary Problems: E. L. Ince.

Sextactic Cones and Tritangent Planes of the same System of a Quadri-Cubic Curve: W. P. Milne.

On the Integer Solutions of the Equation $y^2 = ax^3+bx^2+cx+d$: L. J. Mordell.

On the Manner of Divergence of the Legendre Series of an Integrable Function: B. M. Wilson.

On the Fundamental Theorem of Integration for Multiple Integrals: W. H. Young.

ABSTRACTS.

Some Limiting Forms in the Theory of Integral Equations

A. C. DIXON.

1. For the integral equation (Kellogg, *Math. Ann.*, Vol. 58)

$$\phi(x)-\lambda\int_0^\pi \phi(t)\,\{\cot(t-x)+a\}\,dt=f(x),$$

the special values of λ are $\pm 1/\iota\pi$, $1/a\pi$, and the special functions are

$$\begin{array}{lll} \text{for} & 1/\iota\pi, & e^{2n\iota x}\quad (n=1,2,3,\ldots), \\ \text{for} & -1/\iota\pi, & e^{-2n\iota x}\quad (n=1,2,3,\ldots), \\ \text{for} & 1/a\pi, & 1. \end{array}$$

2. The equation $\displaystyle \phi(x)-\lambda\int_0^\infty \phi(t)\,\frac{dt}{t+x}=f(x),$

is satisfied by $\displaystyle \phi(x)=f(x)+\frac{\tan b\pi}{\pi}\,x\int_0^\infty f(t)\,\frac{\left(\frac{x}{t}\right)^b-\left(\frac{x}{t}\right)^{1-b}}{x^2-t^2}\,dt,$

if $\sin b\pi=\lambda\pi$ and the real part of b is between 0 and 1; the index $1-b$ is to be changed into $-1-b$ if the real part of b is between 0 and -1. The solution fails if λ is a pure imaginary.

3. Some other types of functional equation may be treated as limiting forms of integral equations.

For instance, take $\phi(x)-\lambda\phi(x+a)=f(x)$, (3 . 1)

where ϕ, f are supposed to have the period π, and a is real. Since

$$\phi(x+a)=\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\int_0^\pi e^{2n\iota(x+a-t)}\,\phi(t)\,dt,$$

the equation has the form of an integral equation.

If $f(x)$ is expanded in a Fourier series $\Sigma A_n e^{2n\iota x}$, $\phi(x)$ is

$$\Sigma A_n e^{2n\iota x}/(1-\lambda e^{2n\iota a}).$$

The solution fails when $\lambda=e^{-2n\iota a}$, n being integral, unless $f(x)$ is such that

$$\int_0^\pi e^{-2n\iota a} f(x)\,dx=0,$$

and when this condition is satisfied the coefficient of $e^{2n\iota a}$ in $\phi(x)$ is arbitrary.

If $\psi(x)$ is another function with period π,

$$\int_0^\pi \{\phi(x)-\lambda\phi(x+a)\}\,\psi(x)\,dx = \int_0^\pi \{\psi(x)-\lambda\psi(x-a)\}\,\phi(x)\,dx,$$

and thus the associated equation is the similar one

$$\psi(x)-\lambda\psi(x-a) = g(x). \tag{3.2}$$

The special values of λ for which the solution fails are all on the circle $|\lambda| = 1$, whereas in ordinary cases the sequence of them tends to infinity.

Similarly take the equation

$$\phi(x)-\lambda\phi'(x+a) = f(x), \tag{3.3}$$

where ϕ, f are still periodic and $\phi'(x+a)$ may be written

$$\frac{1}{\pi}\sum_{-\infty}^{\infty}\int_0^\pi 2n\iota\, e^{2n\iota(a+x-t)}\,\phi(t)\,dt.$$

The solution here is, if $f(x) = \Sigma A_n e^{2n\iota x}$,

$$\phi(x) = \Sigma A_n e^{2n\iota x}/(1-2n\iota\lambda\, e^{2n\iota a}). \tag{3.31}$$

The special values of λ are $e^{-2n\iota a}/2n\iota$, n being integral; for this special value the solution fails unless

$$\int_0^\pi e^{-2n\iota x} f(x)\,dx = 0,$$

and if this condition is fulfilled the coefficient of $e^{2n\iota x}$ in $\phi(x)$ is arbitrary. The associated equation is the similar one

$$\psi(x)+\lambda\psi'(x-a) = g(x). \tag{3.4}$$

When $a = 0$, the equation (3.3) becomes the differential equation

$$\phi(x)-\lambda\phi'(x) = f(x). \tag{3.5}$$

The value of $\phi(x)$ given by (3.5) contains a complementary function

$$C\,e^{x/\lambda},$$

in addition to the terms of (3.31).

The complementary function has not, however, the period π, unless λ has one of the special values included in the form

$$1/2n,$$

and thus the two results are reconciled.

The special values of λ in this case form two sequences tending to zero.

On the Integer Solutions of the Equation $y^2 = ax^3+bx^2+cx+d$

L. J. MORDELL.

In a paper appearing shortly in the *Messenger of Mathematics*, I have given a very simple proof that this equation has only a finite number of solutions in integers. The present paper contains another proof which I hoped would enable me to prove the obvious extension to the quartic. It led me, however, to a proof of a far more important proposition—namely, that the method of infinite descent applies to the indeterminate equations (of genus unity)

$$z^2 = ax^4+bx^3y+cx^2y^2+dxy^3+ey^4,$$

or to

$$0 = f(x, y, z),$$

where f is a ternary cubic in x, y, z; that is to say, that if either of these equations has an infinite number of integral solutions, all of them can be expressed rationally in terms of a *finite* number. So all the rational solutions of

$$y^2 = 4x^3-g_2x-g_3$$

are given by

$$x = \wp(u), \quad y = \wp'(u),$$

with

$$u = m_1u_1+m_2u_2+\ldots+m_nu_n,$$

for a *finite* number of values of $u_1, u_2, \ldots$, where $m_1, m_2, \ldots$ take all integer values, positive, negative, and zero. An account of this will be given in a paper "On the Solution of the Indeterminate Equations of the Third and Fourth Degrees," to be presented shortly to the Cambridge Philosophical Society.

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The London Mathematical Society.

RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1921–JUNE, 1922.

Thursday, June 8th, 1922.

Mr. H. W. RICHMOND, President, in the Chair.

Present twenty members.

Mr. E. C. Francis was elected a member of the Society.

Prof. Dr. E. Schmidt (Berlin) was elected under By-law 20B.

Mr. L. C. Piggott was nominated for election.

Mr. R. F. Budden, Mr. C. E. Wright, and Miss M. Stimson were admitted into the Society.

The President spoke briefly concerning M. Camille Jordan, late honorary member of the Society.

It was announced that the Annual General Meeting was fixed for November 9th, 1922.

Mr. L. J. Mordell communicated two papers (1) "On Gauss's Sums and the Law of Quadratic Reciprocity in any Field," (2) "On the Integral Solutions of the Equation $ey^2 = ax^3+bx^2+cx+d$."

Mr. J. E. Campbell read a paper "The Deduction of the Ground-Form of Einstein's Statical Gravitational Field from Gauss's Expression for the Ground-Form in Ordinary Two-Way Space."

Prof. G. H. Hardy read a paper (written in collaboration with Mr. J. E. Littlewood) "Fourier's Series and Power Series."

Lt.-Col. A. Cunningham read a paper "On Pellian Chains."

The following papers were communicated by title from the chair:—

Commutative Ordinary Differential Operators: J. L. Burchnall and T. W. Chaundy.

The Displacement of Tensors: P. Dienes.

On the General Invariant Theory of Quadrics: H. W. Turnbull.

Surface Waves on Limited Sheets of Water: J. Vint.

The Euclidean Geometry of Angle: D. K. Picken.

A Relation between Bertrand's and Kelvin's Theorems on Impulses: G. I. Taylor.

ABSTRACTS.

On Gauss's Sums and the Law of Quadratic Reciprocity in any Field

Mr. L. J. MORDELL.

In a recent number of these *Proceedings* I have proved very simply Hecke's reciprocity formulæ for the Gauss's sums in a quadratic field. I notice now that the proof becomes much simpler and more symmetrical by the use of a contour integral involving two variables of integration instead of one. Moreover, the new method gives the corresponding formulæ for any algebraic field, so that an easy and elementary proof of the law of quadratic reciprocity in the field can be deduced.

Hecke, it may be noted, has proved the reciprocity formulæ very elegantly, by assuming the law of Quadratic Reciprocity in the field, and also his functional equation for the Dedekind Zeta-function.

On Pellian Chains

Lt.-Col. ALLAN CUNNINGHAM, R.E.

This paper develops the *chain-relations* among the elements (τ', υ'), (τ, υ) of the Pellian equations

$\tau'^2-D\upsilon'^2=\mp k$[*], $\quad \tau^2-D\upsilon^2=+1.$

A *chain-series* is a series of *composites* $N_1=L_1M_1$, $N_2=L_2M_2$, &c., which are *similar functions* of two elements x_r, y_r, such that

$$M_{r-1}=L_r, \quad M_r=L_{r+1} \quad \text{(for all values of } r\text{)}.$$

By the substitutions

$$\tau_1'=\sqrt{k}\,.\,\tfrac{1}{2}(y^{\frac{1}{2}}\mp y^{-\frac{1}{2}}), \qquad \upsilon_1'=\sqrt{k}\,.\,\tfrac{1}{2}(y^{\frac{1}{2}}\pm y^{-\frac{1}{2}})/\sqrt{D},$$

$$\tau_1=\tfrac{1}{2}(y+y^{-1}), \qquad \upsilon_1=\tfrac{1}{2}(y-y^{-1})/\sqrt{D},$$

---

[*] The only values of k used are such as are the middle terms of the series of partial quotients in the expansion of $\sqrt{D}$ as a continued fraction: they are the last entries in the second lines of each D in Degen's tables.

it is shown that the following numbers (N) are *chain-series* :—

$$N'_{r,s} = \tau_r'^2 + Dv_s'^2, \qquad N_{r,s} = \tau_r^s + Dv_s^2,$$

$$N''_{r,s} = k\tau_r'^2 + Dv_s'^2, \quad N''_{r,s} = k\tau_r'^2 - Dv_s'^2.$$

Further that the following Aurifeuillians are *chain-series* :—

$$N'_r = \tau_r'^4 + 4v_r'^4, \quad N_r = \tau_r^4 + 4v_r^4,$$

$$N'_r = (\tau_r'^6 + 3^3 v_r'^6)/(\tau_r'^2 + 3v_r'^2), \quad N_r = (\tau_r^6 + 3^3 v_r^6)/(\tau_r^2 + 3v_r^2).$$

On the General Invariant Theory of Quadrics

Prof. H. W. TURNBULL.

The property of equivalence which gives rise to a prepared system of symbolic factors in terms of which the concomitants of three quadrics can be expressed (see *Proc. London Math. Soc.*, Ser. 2, Vol. 20, p. 472) holds for any number of quadrics. With the same notation a_x^2, $(Ap)^2$, u_α^2 to denote a quadric f in point, line and plane coordinates, and with similar sets for an indefinite number of other quadrics $f' = b_x^2$, $f'' = c_x^2$, $f''' = d_x^2$, ..., it may be shown that the necessary types of factor for a prepared system are:—

$$a_x, \ u_\alpha, \ (abcd), \ (\alpha\beta\gamma\delta), \ (abcu), \ (\alpha\beta\gamma x),$$

$$\begin{array}{cc}
(A_1 A_2), & (\theta A_1 \phi), \\
(A_1 A_2 A_3 A_4 A_5 A_6), & (\theta A_1 A_2 \phi), \\
(A_1 A_2 \ldots A_{10}), & \ldots \quad \ldots \\
 & (\theta A_1 A_2 \ldots A_8 \phi),
\end{array}$$

where A_i denotes one of the symbols $p, A, B, \ldots$; θ, ϕ denote two of $u, x, a, \alpha, b, \beta, \ldots$; and where certain simple restrictions hold.

Thus the total degree of a bracket is a multiple of four, reckoning u, a of degree one; p, A of degree two; x, α of degree three. Again, no two symbols in one bracket may be alike, except θ and ϕ. Also when three or more $A_1 A_2 A_3$ occur they may be interchanged as in a determinant.

These brackets may be readily expressed in terms of the original symbolic factors. For example, if $A = (aa')$, $C = (cc')$, $E = (ee')$, then

$$(ABCDEF) = \Sigma(aBc)(c'De)(e'Fa'),$$

the sum extending to eight terms obtained by permuting aa', cc', ee'

separately. Similarly, to illustrate the structure of a (θ, ϕ) bracket,

$$(aBCDE\rho) = \Sigma(aBc)(c'De)\,e'_{\rho},$$

a four-term series.

Finally, this prepared system shows clearly the principle of duality, answering to geometrical reciprocation. The three brackets on the left are self-reciprocal. Those on the right are reciprocal in pairs when θ, ϕ are both of type a or both of type α; but they are self-reciprocal when an even number of A symbols occur.

The Euclidean Geometry of Angle

Prof. D. K. PICKEN.

The paper is based on the figure consisting of two intersecting straight lines, intermediate between straight line and (complete) triangle. This figure is called in the paper "the complete angle," and for it the notations (l_1, l_2) or (AB, CD), or, simply AOB—equivalent to (OA, OB)—are used. Congruence of complete angles, expressed by $AOB \equiv A'O'B'$, is shown to be the proper characteristic angle relation of elementary geometry; by use of it geometrical theory is both simplified and made more general.

The following propositions, *e.g.*, are quite general :—

$ABC \equiv ABD$ implies that B, C, D are collinear,
$ABC \equiv DCB$,, AB is parallel to CD,
$ABC \equiv ADC$,, A, B, C, D are concyclic,
$ABC \equiv DAC$,, AD is tangential to the circle ABC,
$ABC \equiv CDA$,, the circles ABC, ACD are congruent ;

and every congruence $PQR \equiv UVW$, obtained by using 3-permutations of A, B, C, D, can be given immediate general geometrical interpretation. These facts provide a most powerful elementary method in geometry, of wide applicability.

Fourier Series and Power Series

Prof. G. H. HARDY *and* Mr. J. E. LITTLEWOOD.

The "convergence problem" for Fourier's series is still unsolved: no one has succeeded in stating any simple system of conditions both necessary and sufficient for the convergence of the Fourier series of a given function $f(x)$. The same is true of the problem of the summability of the series by Cesàro's means, so long as the order of the means considered is fixed. Thus continuity is a sufficient condition for summability $(C, 1)$, but not a necessary condition; and, in general, none of the known sufficient conditions is necessary, and none of the necessary conditions sufficient.

There is, however, a problem which does admit a simple and complete solution. We may ask, not "when is the series summable $(C, 1)$, or summable (C, k), for some definite k?" but "when is it summable by *some* Cesàro mean?" The solution of this problem is contained in the following theorem.

THEOREM 1.—*In order that the Fourier series of $f(x)$, for $x=a$, should be summable by Cesàro's means, to sum s, it is necessary and sufficient that*

$$\phi_k(x) = \frac{1}{x}\int_0^x \frac{dx_1}{x_1}\int_0^{x_1}\frac{dx_2}{x_2}\cdots\int_0^{x_{k-2}}\frac{dx_{k-1}}{x_{k-1}}\int_0^{x_{k-1}}\phi(x_k)\,dx_k,$$

where $$\phi(x) = \tfrac{1}{2}\{f(a+x)+f(a-x)-2s\},$$

should, for some value of k, tend to zero with x.

We may express this more tersely by saying that *it is necessary and sufficient that $\phi(x)$ should be continuous on the average for $x=0$.*

There is a similar theorem for power series $\Sigma a_n x^n$ on the circle of convergence.

THEOREM 2.—*In order that Σa_n should be summable by Cesàro's means, to sum s, it is necessary and sufficient that*

(1) $f(x) = \Sigma a_n x^n$ *should be of finite order for $|x| < 1$, that is to say that*

$$f(x) = O(1-|x|)^{-K}$$

for some value of K;

$$(2) \qquad f_k(x) = \frac{1}{1-x}\int_x^1 \frac{dx_1}{x_1}\int_{x_1}^1 \frac{dx_2}{x_2} \cdots \int_{x_{k-2}}^1 \frac{dx_{k-1}}{x_{k-1}}\int_{x_{k-1}}^1 f(x_k)\,dx_k$$

should tend to s when x tends to 1.

The integrations are rectilinear; but x may tend to 1 in any manner from inside the circle.

The condition (1) is equivalent to

$$(3) \qquad a_n = O(n^L)$$

for some L.

These theorems may recall Bohr's work in the theory of Dirichlet's series $\Sigma a_n n^{-s}$, where a similar phenomenon may be observed. The "convergence problem" defies solution, and so does that of summability (C, k) for any particular k; but the problem of summability by *some* mean has a definite solution. The series is summable for $\sigma > \sigma_0$, by some Cesàro mean, if and only if the function $f(\sigma + it) = f(s) = \Sigma a_n n^{-s}$ is regular, and of finite order in t, for $\sigma > \sigma_0$.

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The London Mathematical Society.

RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1922–JUNE, 1923.

Annual General Meeting.

Thursday, November 9th, 1922.

Mr. H. W. RICHMOND, President, in the Chair, and later
Prof. W. H. YOUNG, President, in the Chair.

Present thirty members and two visitors.

The President announced that the Manchester Mathematical Society had made a donation of £10 to the Society.

The President announced that a reciprocal agreement had been made with the American Mathematical Society for a reduction of subscription, as follows :—

1.—Members of the London Mathematical Society not residing in the United States or Canada will be admitted to full membership in the American Mathematical Society by payment of the entrance fee of \$5 and one-half of the full annual dues (at present \$6), such membership carrying with it the *Bulletin* without extra charge. If any such member desires the *Transactions* in place of the *Bulletin*, he shall pay annual dues of \$5 ; if he desires both the *Bulletin* and *Transactions*, he shall pay half the full annual dues plus \$5.

2.—Members of the London Mathematical Society not residing in the United States or Canada, who are life members of the American Mathematical Society, may obtain the *Transactions* by payment of \$5 annually.

3.—Members of the American Mathematical Society not residing in Great Britain and Ireland will be admitted to full membership in the London Mathematical Society by payment of the entrance fee of £2. 2*s*. and one-half of the full annual subscription (at present £2. 2*s*.).

4.—Any person, who is a member of both Societies, shall not be entitled to the reductions above mentioned as regards both Societies, but shall pay full dues or subscriptions to that one of the two Societies of which he first became a member.

5.—The above arrangements take effect as regards the London Mathematical Society for the year beginning November 1st, 1921, and as regards the American Mathematical Society for the year beginning January 1st, 1922.

The Treasurer of the London Mathematical Society will, so far as practicable, give effect to these arrangements when he next applies for subscriptions due. But any member who is entitled to reduction in the London Mathematical Society's subscription, and who may be asked to pay the full subscription, is requested to write to the Treasurer, as cases will probably arise where the Treasurer will be unaware of rights to reduced subscription.

Members of the London Mathematical Society not residing in the United States or Canada, and who are at present subscribing members of the American Mathematical Society, are requested to communicate direct with the latter Society at 501 West 116th Street, New York, U.S.A., with regard to their subscriptions, and also as regards their options to receive the *Transactions* or the *Bulletin*, or both.

The Treasurer presented his Report for the Session 1921–22, which was duly received.

Lt.-Col. Cunningham was appointed Auditor.

Mr. L. C. Piggott was elected a member of the Society.

Messrs. W. E. Baker, P. N. Dé, A. E. Ingham, R. E. C. Kennedy, H. Thomas, E. C. Titchmarsh, A. G. Willox, Sir John Heaton, and Mrs. E. B. Veitch were nominated for election.

Prof. Dr. A. Korn, of Berlin, was nominated for election under By-law 20B.

Mr. W. L. Ferrar was admitted into the Society.

The meeting proceeded to the election of a Council and Officers. The list is as follows:—President, W. H. Young; Vice-Presidents, A. L. Dixon, A. E. Jolliffe, H. W. Richmond; Treasurer, A. E. Western; Secretaries, G. H. Hardy, G. N. Watson; other members of the Council, J. E. Campbell, L. N. G. Filon, H. Hilton, Miss H. P. Hudson, J. E. Littlewood, A. E. H. Love, E. A. Milne, L. J. Mordell, F. B. Pidduck.

The retiring President then vacated the Chair, which was taken by Prof. Young, and delivered an address on "The Mathematical Problems of Shell-Flight."

The following papers were communicated by title from the Chair:—

Asymptotic Expansions and Generalized Hypergeometric Functions: W. R. Burwell.

Determinants whose Elements are Determinants: W. L. Ferrar.

Transformation of Variables in a Multiple Integral: A. Kerekjártó.

(i) The Properties of certain Numbers, (ii) Contribution to the Evaluation of Persymmetric Determinants: C. Krishnamachari and M. Bhimasena Rao.

On Trigonometric Series involving Algebraic Numbers: L. J. Mordell.

Analogues of Waring's Problem for Rational Numbers: H. W. Richmond.

Weber's Integral Theorem: E. C. Titchmarsh.

ABSTRACTS.

Determinants whose Elements are Determinants

W. L. FERRAR.

If Δ denote the determinant formed by taking the first n rows of an array of n columns and $n+m$ rows (columns $a, b, \ldots, k$),

A_r the minor of a_r in Δ,

$\Delta_{1,p}$ the determinant obtained by replacing in Δ the p-th row by the $(n+1)$-th row of the array,

$A_r^{1,p}$ the minor of a_r in this determinant, with similar notation for other determinants.

The paper uses the relation

$$A_p^{t,s}\Delta \equiv A_p\Delta_{t,s} - A_s\Delta_{t,p}, \qquad (1)$$

and its analogues to express a determinant such as

$$|A_x, \ldots, D_w, E_p^{r,s}, \ldots, K_q^{v,t}|$$

as a product of the Δ's.

Theorem C of Prof. E. T. Whittaker's paper (1918) (*Proc. Edin. Math. Soc.*, Vol. 36, pp. 107–115) gives a particular case.

The relation (1) is also used to give a short proof of a theorem given by Cayley (without proof).*

In conclusion, an extension of Bazin's theorem is obtained by use of previous results, and Theorem B of the paper quoted above shown to follow easily from the mode of proof here used.

Commutative Ordinary Differential Operators †

J. L. BURCHNALL *and* T. W. CHAUNDY.

We consider the problem of the form of two operators $\phi(D)$, $\psi(D)$ polynomial in D, such that their alternant vanishes identically. We solve the problem for the case of $\phi(D)$ a quadratic operator, hyperelliptic functions being in general necessary; when, in addition, $\psi(D)$ is cubic, then $\phi(D)\,y = 0$ is effectively Lamé's equation.

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* Muir, *History of Determinants*, Vol. 2, p. 15.

† Read at the Meeting of June 8th, 1922.

It is also shown that any two such commutative operators are linked by an algebraic identity $f\{\phi(D), \psi(D)\} \equiv 0$, and that from $\phi(D)$, $\psi(D)$ can be developed a group of commutative pairs obeying the same identity. The close connection of the theory with the theory of adjoint operators is exhibited.

Weber's Integral Theorem

E. C. TITCHMARSH.

This paper contains a proof of the following theorem :—

If $x > a > 0$ and $\sqrt{t}\,f(t)$ is summable in the infinite interval (a, ∞), and $f(t)$ is of bounded variation in a neighbourhood of $t = x$, then

$$\int_0^\infty \frac{C_\nu(xu, au)}{J_\nu^2(au)+Y_\nu^2(au)}\,u\,du \int_a^\infty C_\nu(tu, au)\,t\,f(t)\,dt = \tfrac{1}{2}\{f(x+0)+f(x-0)\},$$

when J_ν and Y_ν are the Bessel functions of the first and second kinds of order ν, and

$$C_\nu(xu, au) = J_\nu(xu)\,Y_\nu(au) - Y_\nu(xu)\,J_\nu(au).$$

A Theorem concerning Summable Series

G. H. HARDY *and* J. E. LITTLEWOOD.

The results concerning power series and Fourier series, stated in a former communication to the Society,* depend on the following general arithmetic theorem, which has many interesting applications.

The necessary and sufficient condition that the series Σa_n *should be summable* (C, r) *is that the series* Σb_n, *where*

$$b_n = \frac{a_n}{n+1} + \frac{a_{n+1}}{n+2} + \dots \; (C, r-1)$$

should be summable $(C, r-1)$.

This holds for all positive integral values of r; and, if a series Σa_n is said to be summable $(C, -1)$ when it converges and $na_n = o(1)$, then it holds for $r = 0$.

* See *Records*, June 8th, 1922.

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The London Mathematical Society.

RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1922–JUNE, 1923.

Thursday, January 18th, 1923.

Prof. W. H. YOUNG, President, in the Chair.

Present thirty members.

The minutes of the last meeting were read and confirmed.

Messrs. J. P. Gabbatt, S. F. Grace, and C. G. F. James were elected members of the Society.

Prof. G. A. Bliss and Prof. O. E. Glenn were nominated for election.

Messrs. W. E. Baker, P. N. Dé, Sir John Heaton, A. E. Ingham, E. C. Titchmarsh, F. P. White, and R. F. Whitehead, were admitted into the Society.

Mr. L. J. Mordell delivered a lecture on "An Introduction to the Arithmetic Theory of Algebraic Numbers and its Modern Developments."

The following papers were communicated by title from the Chair:—

The Rational Linear Algebras of Maximum and Minimum Ranks: L. E. Dickson.

The Zeros of Mathieu Functions: Einar Hille.

Plane Homaloidal Families of General Degree: Miss H. P. Hudson.

Abel's Theorem and its Converse (II): G. H. Hardy and J. E. Littlewood.

A Linear Differential Equation with Periodic Coefficients: E. L. Ince.

A Property of the 24 Inflexional Tangents of the Non-Singular Plane Quartic: A. E. Jolliffe.

The Algebraic Two-Two Correspondence: R. Vaidyanathaswami.

An Application of Pfeiffer's Method of a Problem of Hardy and Littlewood: B. M. Wilson.

On Successions with Sub-Sequences Converging to an Integral: W. H. Young.

ABSTRACTS.

The Zeros of Mathieu Functions

EINAR HILLE.

A Mathieu function is defined as a solution of the differential equation

$$w''+(a+b\cos 2z)\,w = 0.$$

The main purpose of the paper is to describe the distribution in the complex plane of the zeros of these functions when a and b are real. Assume $b<0$, and divide up the plane in simply infinite parallel strips by means of the lines $y=0$ and $x=(2k+1)\frac{1}{2}\pi$, where k takes on all integral values. Then the general solution has infinitely many zeros in each strip; the zeros forming a point-set of simple structure. For each strip there are two linearly independent solutions which are free from zeros in the strip. These solutions have interesting asymptotic properties. The periodic solutions which include the *functions of the elliptic cylinder* are treated at some length.

The investigation is based partly upon considering a transformed equation, asymptotic to a sine-equation in the sense of Boutroux, partly upon applying integral equalities of Sturmian nature.

Note on a Theorem concerning Fourier Series

G. H. HARDY *and* J. E. LITTLEWOOD.

Dr. Marcel Riesz has indicated to us a material simplification in the proof of Theorem X of our paper "Abel's Theorem and its Converse" in Vol. 18 (1918) of the *Proceedings* (see in particular pp. 229–231).

It is a question of proving that, *if*

$$A_n = O\left(\frac{1}{n}\right),$$

and ΣA_n *converges to* A, *then*

$$\Phi(\alpha) = \Sigma A_n \frac{\sin n\alpha}{n\alpha} \to A,$$

when $\alpha \to 0$.

We may suppose $A = 0$ and $|nA_n| < 1$ for all values of n. Given δ, then, we can choose μ so that

(1) $$\left|\sum_1^\mu A_n\right| < \delta$$

and

(2) $$\left|\sum_\nu^{\nu'} A_n\right| < \delta,$$

if $\mu \leqslant \nu < \nu'$. We write

$$\Phi = \sum_1^n + \sum_{\mu+1}^m + \sum_{m+1}^\infty = \Phi_1 + \Phi_2 + \Phi_3,$$

where $$m = \left[\frac{K}{a}\right].$$

We have first

(3) $$|\Phi_3| < \frac{1}{a}\sum_{m+1}^\infty \frac{1}{n^2} < \frac{1}{ma} < \frac{2}{K} < \epsilon,$$

by choice of K. When K has been fixed, the function $(\sin na)/na$ has, in the range of summation covered by Φ_2, a finite number $k(K)$ of maxima and minima, all numerically less than unity. Hence $|\Phi_2|$ is less than the sum of $k(K)$ terms of the type (2); *i.e.*

(4) $$|\Phi_2| < \delta k(K) < \epsilon,$$

by choice of δ (*i.e.* of μ). We may obviously suppose $\delta < \epsilon$.

Finally, when μ is fixed, we can choose a_0 so that

(5) $$|\Phi_1| \leqslant \left|\sum_1^\mu A_n\right| + \epsilon < \delta + \epsilon < 2\epsilon,$$

for $0 < a \leqslant a_0$, by (1). From (3), (4), and (5) our conclusion follows.

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The London Mathematical Society.

RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1922–JUNE, 1923.

Thursday, February 8th, 1923.

Prof. W. H. YOUNG, President, in the Chair.

Present sixteen members.

The minutes of the last meeting were read and confirmed.

Prof. G. A. Bliss and Prof. O. E. Glenn were elected members of the Society.

The President announced that Sir A. B. Kempe had left a legacy of £200 to the Society.

Mr. T. A. Brown read a paper on "A Class of Factorial Series."

Miss H. P. Hudson read a paper on "Plane Homaloidal Families of General Degree."

Prof. A. E. Jolliffe read a paper on "A Property of the 24 Inflexional Tangents of the Non-Singular Plane Quartic."

Prof. G. H. Hardy read a paper (written in collaboration with Mr. J. E. Littlewood) on "Abel's Theorem and its Converse (II)."

Prof. G. N. Watson gave an account of a paper by Mr. T. A. Lumsden on "A certain Fourier-Bessel Series."

The following papers were communicated by title from the Chair:—

The Hypergeometric Function with k Denominators: D. M. Wrinch.

On Successions with Sub-Sequences converging to an Integral: W. H. Young.

A Note on the Riesz-Fischer Theorem in the Theory of Trigonometrical Series: E. C. Titchmarsh.

ABSTRACTS.

On a Class of Factorial Series

T. ARNOLD BROWN.

Necessary and sufficient conditions are obtained that an analytic function $f(z)$ may be expressible in the form

$$e_0+e_1 z+e_2\frac{z(z-1)}{2!}+e_3\frac{(z+1)\,z(z-1)}{3!}+\dots.$$

A certain interpolation formula of de la Vallée Poussin is generalised and its relation to the above series is exhibited.

Runge's solution of a certain interpolation problem is also extended.

The methods employed depend upon Carlson's researches on Taylor's series. Carlson has discussed the factorial series

$$\Sigma e_\nu \frac{z(z-1)\ldots(z-\nu+1)}{\nu!}$$

in a similar way. It is shown that, in certain respects, the two series differ widely in their behaviour.

A Note on the Riesz-Fischer Theorem in the Theory of Trigonometrical Series

E. C. TITCHMARSH.

The Riesz-Fischer theorem states that if a sequence (a_n) is such that $\Sigma\,|\,a_n\,|^2$ is convergent, then there is a function of summable square of which the given numbers are the Fourier coefficients. It would be natural to suppose that, if $\Sigma\,|\,a_n\,|^\alpha$, where $\alpha > 2$, is convergent, the series is necessarily the Fourier series of a function of which some lower power is summable. But Hardy and Littlewood* have proved that this is untrue, by showing that the series

$$\Sigma n^{-\alpha}\cos(n^2\pi x), \quad \Sigma n^{-\alpha}\sin(n^2\pi x) \quad (\alpha \leqslant \tfrac{1}{2}),$$

are not Fourier series, or even "generalised" Fourier series, since they are not convergent or summable by any of Cesàro's means for any irrational value of x. They are merely formal series with which no sum has been associated.

I have constructed a trigonometrical series which proves the same point, but in a quite different way. My series is everywhere convergent, and is in fact the "generalised" Fourier series of its sum. Further, the convergence exponent of the sequence of coefficients is 2. But the series is not a Fourier series, since its sum is not summable.

---

* G. H. Hardy and J. E. Littlewood, "Some Problems of Diophantine Approximation", *Acta Mathematica*, Vol. 37, pp. 232–238.

The analysis necessary to prove these points is direct and elementary, whereas that of Hardy and Littlewood rests on the theory of the linear transformation of the Theta-functions.

The function $f(t)$ is defined in $(-\pi, \pi)$ as follows :—

$$f(t) = \log p \cos(tp^2 \log p)$$

for

$$\pi/\log(p+1) < t \leqslant \pi/\log p \quad (p = 3, 4, 5, \ldots),$$

$f(t)$ being zero elsewhere, and $f(-t) = -f(t)$. The series

$$\sum_{p=3}^{\infty} \log p \int_{\pi/\log(p+1)}^{\pi/\log p} |\cos(tp^2 \log p)|\, dt$$

is divergent, so that $f(t)$ is not absolutely integrable; but

$$\sum_{p=3}^{\infty} \int_{\pi/\log(p+1)}^{\pi/\log p} f(t)\, dt = \Sigma \log p \int_{\pi/\log(p+1)}^{\pi/\log p} \cos(tp^2 \log p)\, dt = \Sigma O\left(\frac{1}{p^2}\right)$$

is convergent, so that

$$\lim_{\eta \to 0} \int_{\eta}^{\pi} f(t)\, dt$$

exists, and so do the generalised Fourier coefficients

$$b_n = \frac{2}{\pi} \lim_{\eta \to 0} \int_{\eta}^{\pi} f(t) \sin nt\, dt.$$

Finally, it may be proved that

$$b_n = O\{n^{-\frac{1}{2}}(\log n)^2\},$$

so that $\Sigma |b_n|^{2+\epsilon}$ is convergent if $\epsilon > 0$.

The Hypergeometric Function with k Denominators

D. M. WRINCH.

In this paper, the hypergeometric function with one denominator $f(z, p)$ is defined by the contour integral

$$f(z, p) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^t t^{-p-1} e^{z/t}\, dt,$$

and it is deduced that

$$f(z, p) = \sum_{s=0}^{\infty} \frac{z^s}{\Gamma(s+1)\Gamma(p+s+1)}.$$

Evidently $$f\left(-\frac{z^2}{4}, p\right) = \left(\frac{z}{2}\right)^{-p} J_p(z).$$

Various properties of the function are obtained, some of which yield new results for Bessel functions.

The hypergeometric function with k denominators $f(z, p_1, p_2, \ldots, p_k)$ is defined by a contour integral having the generalised hypergeometric function with $k-1$ denominators in the integrand. Various properties of the function $f(z, p_1, p_2, \ldots, p_k)$ follow by induction from the properties of the function $f(z, p)$, including the recurrence formulæ and the fact that

$$f(z, p_1, p_2, \ldots, p_k) = \sum_{s=0}^{\infty} z^s / \{\Gamma(s+1)\,\Gamma(p_1+s+1) \ldots \Gamma(p_k+s+1)\}.$$

Some of the properties of the function when one or more of the parameters $p_1, p_2, \ldots, p_k$ is zero or a positive or negative integer are discussed. It is established that when $p_1, p_2, \ldots, p_k$ are severally equal either to zero or a positive or negative integer, the function $f(z, p_1, p_2, \ldots, p_k)$ is the coefficient of $t_1^{p_1} t_2^{p_2} \ldots t_k^{p_k}$ in the expansion in a k-multiple series in powers of $t_1, t_2, \ldots, t_k$ of the function

$$e^{t_1+t_2+\ldots+t_k+z/(t_1 t_2 \ldots t_k)}.$$

Solution of a certain Differential Equation

T. C. HUDSON.

The solution of the equation

$$d/d\theta\,[d^2\lambda/d\theta^2+\lambda(1+4E\cos\theta)/(1+E\cos\theta)] = d\xi/d\theta+2\eta,$$

in which E is constant, may be written in the form

$$\begin{aligned}\lambda = {} & S(1+EC)\,[\lambda_1+\textstyle\int C(1+EC)^{-1}\,\xi\,d\theta] \\ & +C(1+EC)\lambda_2/2E+C^2(2+EC)(1+EC)^{-1}\textstyle\int \eta\,d\theta \\ & -C(1+EC)^{-1}[\lambda_3+\textstyle\int \{C(2+EC)\eta+S(1+EC)\xi\}\,d\theta] \\ & +2S(1+EC)\textstyle\int C(1+EC)^{-3}[E\lambda_3+\textstyle\int \{(1+EC)^2\,\eta+ES(1+EC)\,\xi\}\,d\theta]\,d\theta,\end{aligned}$$

in which $C = \cos\theta$ and $S = \sin\theta$, and λ_1, λ_2, λ_3 are constants.

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The London Mathematical Society.

RECORDS OF PROCEEDINGS AT MEETINGS.

SESSION NOVEMBER, 1922–JUNE, 1923.

Thursday, March 8th, 1923.

Prof. W. H. YOUNG, President, and later Mr. H. W. RICHMOND, Ex-President, in the Chair.

Present twenty members.

Dr. Einar Hille, Mr. H. O. Newboult, and Miss Enid M. Turner were nominated for election.

Prof. A. E. H. Love read a paper on "Some Electrostatic Distributions in Two Dimensions."

Prof. G. H. Hardy read a paper, written in collaboration with Mr. E. C. Titchmarsh, on "Solutions of some Integral Equations considered by Bateman, Kapteyn, Littlewood, and Milne."

Prof. W. H. Young read a paper on "Stokes's Theorem."

The following papers were communicated by title from the Chair:—

(i) Functions of Intervals, (ii) The Problem of Area: J. C. Burkill.

On some Integral Equations connected with the Elliptic Cylinder Functions: S. C. Dhar.

A Hyperspatial Analogue of Feuerbach's Theorem: J. P. Gabbatt.

Génération et étude d'une surface du troisième ordre particulière: M. Merker.

The Order of Magnitude of the Coefficients in a Generalised Fourier Series: E. C. Titchmarsh.

The General Symbolic Notation for the Principle of Duality, and its Application to Determinants: H. W. Turnbull.

On Lattice-Points in Regions of Two or Three Dimensions: I. Vinogradow.

ABSTRACTS.

The General Symbolic Notation for the Principle of Duality, and its Application to Determinants

H. W. TURNBULL

This paper generalises results well known in the symbolic treatment of ternary forms, and includes a proof that, for n variables, all the fundamental symbolic identities may be reciprocated. The notation of a previous investigation[*] is employed thus:—

If symbols a, b, c, a_1, &c., have currency one[†], the simplest identity between $n+1$ such symbols is

$$(a_1 a_2 \dots a_n) a_x = (a a_2 \dots a_n) a_{1x} + (a_1 a a_3 \dots a_n) a_{2x} + \dots \text{ to } n \text{ terms.}$$

The dual identity is

$$(\alpha_1 \alpha_2 \dots \alpha_n) u_\alpha = (\alpha \alpha_2 \dots \alpha_n) u_{\alpha_1} + \dots,$$

where, for example, u_α^2 is the reciprocal of the quadratic a_x^2. By introducing suitable types of bracket factors analogous to the ternary factor $(ab \,.\, \rho\sigma)$ defined by

$$(ab \,.\, \rho\sigma) = a_\rho b_\sigma - a_\sigma b_\rho$$

it is shown that *any* identity or syzygy may be reciprocated letter for letter. A symbol of currency r reciprocates into one of currency $n-r$.

The notation readily adapts itself to the theory of determinants; for the bracket factors are determinants of orders n, $2n$, $3n$, ... In the general identities the rows or columns of these determinants are arbitrary. By giving special values to the symbols involved many well known theorems emerge as special cases of something more general.

Some Electrostatic Distributions in Two Dimensions

A. E. H. LOVE

The condenser problem is solved for all cases of two thin plates, with straight parallel edges and of infinite length, provided that their planes are parallel, whether the rectilinear segments in which the plates cut a transverse plane are equal or unequal, and whether the line which bisects one segment at right angles bisects the other or not. Formulae are given for the correction to the capacity on account of displacement. The problem of influence for one such plate, in presence of a charged line parallel to its edges, is solved, and the solution for a free charge on a circular cylindrical sheet with edges along two generators is given. A general theorem is obtained to the effect that the problem of given free charges on two cylindrical conductors, external to each other, can always be solved when the condenser problem can be solved for the same two conductors, and the example of two circular cylinders is worked out.

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[*] "The irreducible concomitants of two quadratics in n variables", *Camb. Phil. Trans.*, Vol. 21, pp. 197–209.

[†] Cf. *Proc. London Math. Soc.*, Ser. 2, Vol. 21, p. 381.

The Order of Magnitude of the Coefficients in a Generalised Fourier Series

E. C. TITCHMARSH

If a function $f(t)$ is integrable (in the sense of Lebesgue) over the range $(\epsilon, 2\pi)$ for all positive values of ϵ, and the limit

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{2\pi} f(t)\,dt$$

exists, then we may define the generalised Fourier coefficients of the function by the formulæ

$$\pi a_n = \lim_{\epsilon \to 0} \int_{\epsilon}^{2\pi} f(t) \cos nt\,dt, \quad \pi b_n = \lim_{\epsilon \to 0} \int_{\epsilon}^{2\pi} f(t) \sin nt\,dt.$$

It is easily seen on integrating by parts that $a_n = o(n)$, $b_n = o(n)$: it is indeed well known that this is true under very much more general conditions, for example whenever the integrals defining the coefficients are of the general Denjoy type.

It is the object of the present note to show that no more than this is generally true. More precisely:—

Let the function $\lambda(n)$ *be positive and decrease steadily to zero as* $n \to \infty$. *However slowly* $\lambda(n) \to \infty$, *there is a function* $f(t)$ *of the type described such that*

$$a_n \neq o\{n\lambda(n)\}, \quad b_n \neq o\{n\lambda(n)\}.$$

The method will be sufficiently illustrated by giving an example in which

$$J_n = \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi} \sin nt f(t)\,dt \sim \tfrac{1}{2}\pi n \lambda(n),$$

when $n \to \infty$ through a particular sequence of values. Let

$$f(t) = p^2 n_p \lambda(n_p) \sin n_p t, \quad \pi/(p+1) < t \leqslant \pi/p,$$

where $p = 1, 2, 3, \ldots$ and n_p is an integer depending on p, which tends to infinity so rapidly that $n_p > 2n_{p-1}$ and $\Sigma p^2 \lambda(n_p)$ is convergent. We have

$$J_{n_q} = \sum_{p=1}^{\infty} p^2 n_p \lambda(n_p) \int_{\pi/(p+1)}^{\pi/p} \sin n_p t \sin n_q t\,dt.$$

It is easily seen that the qth term of this sum is

$$\tfrac{1}{2}\pi n_q \lambda(n_q) + o(1)$$

and the modulus of the remainder is less than

$$\sum_{p=1}^{\infty} \frac{p^2 n_p \lambda(n_p)}{n_p + n_q} + \sum_{p=1}^{q-1} \frac{p^2 n_p \lambda(n_p)}{n_q - n_p} + \sum_{p=q+1}^{\infty} \frac{p^2 n_p \lambda(n_p)}{n_p - n_q}.$$

Each of these sums tends to zero, so that the result follows.

Solutions of some Integral Equations considered by Bateman, Kapteyn, Littlewood, and Milne

G. H. HARDY *and* E. C. TITCHMARSH

The typical equations are

$$\text{(K)} \quad f'(x) = \tfrac{1}{2} \int_0^{\infty} \frac{J_1(t)}{t} \{f(x+t) - f(x-t)\}\, dt,$$

$$\text{(M)} \quad f'(x) = \tfrac{1}{2} \int_0^{\infty} \frac{e^{-t}}{t} \{f(x+t) - f(x-t)\}\, dt.$$

It is shown that the latter equation has no solution, other than the obvious solution

$$f(x) = ax^2 + 2bx + c,$$

of less rapid increase than an exponential of the type $e^{A|x|}$ $(A < 1)$. The equation (K), on the other hand, has a multitude of solutions. These may be characterised roughly (*a*) as functions defined by definite integrals of the type

$$\int_0^1 \{\lambda(u) \cos xu + \mu(u) \sin xu\}\, du,$$

or (*b*) as integral functions of the complex variable x subject to certain inequalities.

Functions of Intervals

J. C. BURKILL

A *function of intervals* is defined by a set of rules correlating a number $g(I)$ with each interval I of an aggregate of intervals. The only functions of intervals which have hitherto been studied have been supposed to be *additive;* this restriction is not made in the present paper.

We define the *integral* $\int_R g(I)$ by subdividing R into intervals $I_1, \ldots I_n$, and taking the limit of $\sum_{i=1}^{i=n} g(I_i)$ as the dimensions of I_i tend to zero. If $g(I)$ is *absolutely continuous*, we prove that its integral may be taken over a measurable set; this forms a generalisation of the classical theory of measure.

The *derivative* of $g(I)$ at a point x is the limit of $g(I)$ divided by the measure of I, for intervals of a *regular* family approaching x. The main result is that the integral over a measurable set of an absolutely continuous function of intervals is equal to the Lebesgue integral of its derivative. This includes as a special case the fundamental theorem of Lebesgue integration.

The Problem of Area

J. C. BURKILL

Given a surface defined by

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

we seek an expression for its area as a double integral analogous to the integral formula for the length of a curve.

The formula in terms of the ordinary Jacobian J holds only when the derivatives $\partial x/\partial u \ldots$ satisfy certain conditions. We introduce here a *modified Jacobian.*

We first define the *increment of* (x, y) *over a rectangle* R in the (u, v) plane as

$$\Delta(x, y) = \tfrac{1}{2}(x_1 y_2 - x_2 y_1 + x_2 y_3 - x_3 y_2 + x_3 y_4 - x_4 y_3 + x_4 y_1 - x_1 y_4),$$

where 1, 2, 3, 4 denote the corners of R in anti-clockwise order, and x_1 is the value of $x(u, v)$ at the point 1, &c.

The modified Jacobian of (x, y) with respect to (u, v) is then defined as the limit of the ratio of the increment of (x, y) over a rectangle in the (u, v) plane to the area of the rectangle as the sides of the rectangle tend to zero.

We obtain the integral formula for the area of a surface in terms of modified Jacobians under the conditions that $\Delta(y, z)$, $\Delta(z, x)$, $\Delta(x, y)$ are absolutely continuous functions of rectangles. These conditions are analogous to those for the validity of the formula for the length of a curve.

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List of Members elected during the Session 1921-1922.

† indicates that the Member has paid the life-composition, and is a Member for life.

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MARR, WILLIAM LAW, D.Sc.; *High School for Girls, Aberdeen.*
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†STEWART, CALEB ANDREW, M.A., B.Sc.; *The University, Western Bank, Sheffield.*
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THOMPSON, THOMAS, B.A.; 61 *Meldon Terrace, Heaton, Newcastle-upon-Tyne.*
WIRTINGER, Prof. Dr. WILHELM; *IX Strudlhofgasse* 4, *Vienna, Austria.*
WOOTTON, DAVID, B.Sc.; *The Secondary School, Weymouth.*
WRIGHT, CHARLES EDWARD, M.Sc.; 7 *Carlton Villas, Belle Grove, Welling, Kent.*

[P.T.O.

The following changes of address have been notified :—

BENNETT, AGNES ELIZABETH; 18 *Willoughby Road, Hampstead, London, N.W.* 3.
BRILL, JOHN; *Connaught Club,* 75 *Seymour Street, London, W.* 2.
CULLEN, Rev JAMES, S.J.; *Wimbledon College, S.W.* 19.
DALLAS, ROBERT JOHN, M.A.; 197 *Brecknock Road, Tufnell Park, N.* 19.
DARLING, H. B. C.; *The Ridge, Redlynch, Wilts.*
HICKS, WILLIAM MITCHINSON; *The Crowhurst Hotel, Crowhurst, Surrey.*
LONG, MARJORIE; *c.o. The Librarian, Bedford College, Regent's Park, London, N.W.* 1.
PROUDMAN, JOSEPH, M.A., D.Sc., F.R.A.S.; *Stapley, Grassendale Park, Liverpool.*
ROSE-INNES, J.; *Royal Institution, W.* 1.
TURNBULL, H. W., M.A., F.R.S.E.; 2 *Queen's Terrace, St. Andrews.*

Members deceased during the Session 1921–1922 :—

ERNEST CARPMAEL.
CAMILLE JORDAN (honorary member).
ALFRED BRAY KEMPE.
JAMES MCMAHON.
GEORGE BALLARD MATHEWS.
MAX NOETHER (honorary member).

CORRIGENDA.

The list of members dated November, 1921, should be corrected as follows :—

Page 17. Professor MILLER's address is 1203 *W. Illinois Street, Urbana, Illinois, U.S.A.*
Page 20. The name of Professor ROSEVEARE is spelt incorrectly.
Page 21. Professor STEGGALL should be described as "Professor," not as "late Professor."

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